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For $R$ a fixed set of representatives for $G/H$, the map

$$\begin{aligned}\varphi_R\colon \operatorname{Ind}_H^G(S_{\widetilde{D}}) &\to S_{\bar{D}}\\ f &\mapsto \{f(r)\}_{r\in R}\end{aligned}$$

is well-defined by $H$-equivariance of the elements of $S_{\widetilde{D}}$ and one checks that it defines a $G$-equivariant isometric bijection. Similarly for the adjoint operators.

The following example is a particular case of the previous lemma.

EXAMPLE 3.2. Let us look at the case $\widetilde{M} = M \times G$. A section $\widetilde{s} \in C_c^\infty(\widetilde{M}, \pi^*E)$ is an element $\widetilde{s} = \{s_g\}_{g\in G}$ where $s_g \in C^\infty(M,E)$ and $s_g = 0$ for all but finitely many $g$'s. Note that $L^2(\widetilde{M}, \pi^*E)$ can be identified with $\ell^2(G) \otimes L^2(M,E)$. Now

$$\widetilde{D}\widetilde{s} = \{Ds_g\}_{g\in G} \in C_c^\infty(\widetilde{M}, \pi^*F)$$

and hence $S_{\widetilde{D}}$ may be identified with $\ell^2(G) \otimes S_D \cong \ell^2(G)^d$, where $d = \dim_{\mathbb{C}}(S_D)$. In this identification the projection $P$ onto $S_{\widetilde{D}}$ becomes the identity in $M_d(\mathcal{N}(G))$ and thus

$$\dim_G(S_{\widetilde{D}}) = \sum_{i=1}^{d} \langle e, e\rangle = d = \dim_{\mathbb{C}}(S_D)\,.$$

A similar argument for $D^*$ shows that in this case not only does the $L^2$-Index of $\widetilde{D}$ coincide with the Index of $D$, but also the individual terms of the difference correspond to each other. This is not the case in general, see Example 2.2.

## 4. ON $K$-HOMOLOGY

Many ideas of this section go back to the seminal article by Baum and Connes [3], which has been circulating for many years and has only recently been published.

An elliptic pseudo-differential operator $D$ on the closed manifold $M$ can also be used to define an element $[D] \in K_0(M)$, the $K$-homology of $M$, and according to Baum and Douglas [4], all elements of $K_0(M)$ are of the form $[D]$. The index defined in Section 2 extends to a well-defined homomorphism (cf. [4])

$$\text{Index}\colon K_0(M) \to \mathbf{Z}\,,$$

such that $\text{Index}([D]) = \text{Index}(D)$. On the other hand, the projection $\text{pr}\colon M \to \{pt\}$ induces, after identifying $K_0(\{pt\})$ with $\mathbf{Z}$, a homomorphism

$$(*) \qquad \text{pr}_*\colon K_0(M) \to \mathbf{Z},$$

which, as explained in [4], satisfies

$$\text{pr}_*([D]) = \text{Index}([D])\,.$$

More generally (cf. [4]), for a not necessarily finite CW-complex $X$, every $x \in K_0(X)$ is of the form $f_*[D]$ for some $f\colon M \to X$, and $K_0(X)$ is obtained as a colimit over $K_0(M_\alpha)$, where the $M_\alpha$ form a directed system consisting of closed Riemannian manifolds (these homology groups $K_0(X)$ are naturally isomorphic to the ones defined using the Bott spectrum; sometimes, they are referred to as $K$-homology groups with *compact supports*). The index map from above extends to a homomorphism

$$\text{Index}\colon K_0(X) \to \mathbf{Z}\,,$$

such that $\text{Index}(x) = \text{Index}([D])$ if $x = f_*[D]$, with $f\colon M \to X$.

We now consider the case of $X = BG$, the classifying space of the discrete group $G$, and obtain thus for any $f\colon M \to BG$ a commutative diagram

$$\begin{array}{ccc} K_0(M) & \xrightarrow{\text{Index}} & \mathbf{Z} \\ {\scriptstyle f_*}\downarrow & & \| \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbf{Z}\,. \end{array}$$

Note that $(*)$ from above implies the following naturality property for the index homomorphism.

LEMMA 4.1. *For any homomorphism $\varphi\colon H \to G$ one has a commutative diagram*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\text{Index}} & \mathbf{Z} \\ {\scriptstyle (B\varphi)_*}\downarrow & & \| \\ K_0(BG) & \xrightarrow{\text{Index}} & \mathbf{Z}\,. \end{array} \qquad \square$$

We now turn to the $L^2$-index of Section 2. It extends to a homomorphism

$$\mathrm{Index}_G \colon K_0(BG) \to \mathbf{R}$$

as follows. Each $x \in K_0(BG)$ is of the form $f_*(y)$ for some $y = [D] \in K_0(M)$, $f \colon M \to BG$, $M$ a closed smooth manifold and $D$ an elliptic operator on $M$. Let $\widetilde{D}$ be the lifted operator to $\widetilde{M}$, the $G$-covering space induced by $f \colon M \to BG$. Then put

$$\mathrm{Index}_G(x) := \mathrm{Index}_G(\widetilde{D}) .$$

One checks that $\mathrm{Index}_G(x)$ is indeed well-defined, either by direct computation, or by identifying it with $\tau(x)$, where $\tau$ denotes the composite of the assembly map $K_0(BG) \to K_0(C^*_r G)$ with the natural trace $K_0(C^*_r G) \to \mathbf{R}$ (for this latter point of view, see Higson-Roe [10]; for a discussion of the assembly map see e.g. Kasparov [12], or Valette [14]). The following naturality property of this index map is a consequence of Lemma 3.1.

LEMMA 4.2. *For $H < G$ the following diagram commutes:*

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\mathrm{Index}_H} & \mathbf{R} \\ \downarrow & & \| \\ K_0(BG) & \xrightarrow{\mathrm{Index}_G} & \mathbf{R}. \end{array} \qquad \square$$

Atiyah's $L^2$-Index Theorem 2.1 for a given $G$ can now be expressed as the statement (as already observed in [10])

$$\mathrm{Index}_G = \mathrm{Index} \colon K_0(BG) \to \mathbf{R} .$$

## 5. ALGEBRAIC PROOF OF ATIYAH'S $L^2$-INDEX THEOREM

Recall that a group $A$ is said to be *acyclic* if $H_*(BA, \mathbf{Z}) = 0$ for $* > 0$. For $G$ a countable group, there exists an embedding $G \to A_G$ into a countable acyclic group $A_G$. There are many constructions of such a group $A_G$ available in the literature, see for instance Kan-Thurston [11, Proposition 3.5], Berrick-Varadarajan [5] or Berrick-Chatterji-Mislin [6]; these different constructions are to be compared in Berrick's forthcoming work [7]. It follows that the suspension $\Sigma BA_G$ is contractible, and therefore the inclusion $\{e\} \to A_G$