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TILE HOMOTOPY GROUPS

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ABSTRACT. The technique of using checkerboard colorings to show the impossibility of some tiling problems is well-known. Conway and Lagarias have introduced a new technique using boundary words. They show that their method is at least as strong as any generalized coloring argument. They successfully apply their technique, which involves some understanding of specific finitely presented groups, to two tiling problems. Partly because of the difficulty in working with finitely presented groups, their technique has only been applied in a handful of cases.

We present a slightly different approach to the Conway-Lagarias technique, which we hope provides further insight. We also give a strategy for working with the finitely presented groups that arise, and we are able to apply it in a number of cases.

1. INTRODUCTION

A classical problem is the following (see [3, pp. 142, 394], [7]).

Remove two diagonally opposite corners from a checkerboard. Dominoes are placed on the board, each covering exactly two (vertically or horizontally) adjacent squares. Can all 62 *squares be covered by* 31 *dominoes ?*

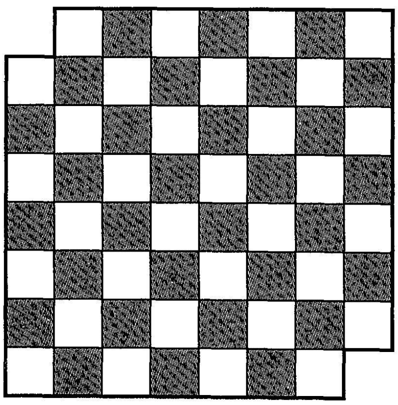


FIGURE 1.1
Mutilated checkerboard

The key to the solution is to note that each domino covers one black square and one red square, whereas the “mutilated checkerboard” has 32 squares of one color and 30 of the other color. Therefore we see that it cannot be tiled.

A smaller version of this problem uses a mutilated 4×4 checkerboard. For this problem, exhaustive analysis is easy; there are two ways to cover the marked square. In the first case, this forces the location of the next 3 dominoes, and isolates a square that cannot be covered. In the second case, the next 5 dominoes are forced, again isolating a square that cannot be covered.

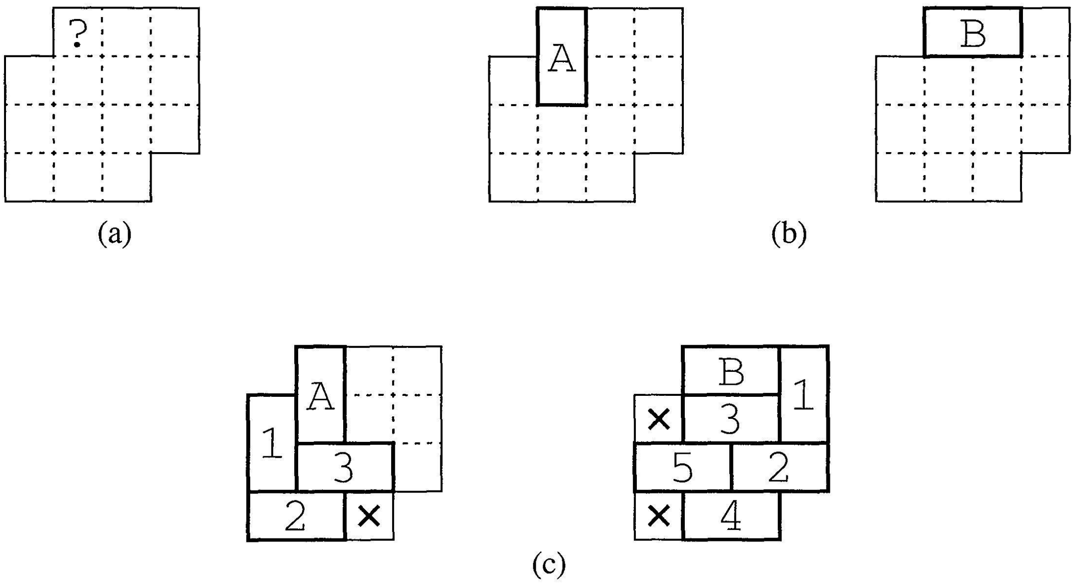


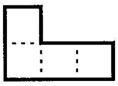
FIGURE 1.2

Analysis of mutilated 4×4 checkerboard:
(a) First cell to cover. (b) Two ways to cover it. (c) Both cases force a contradiction.

A similar exhaustive analysis can be applied to the mutilated 8×8 checkerboard, but it is dramatically more cumbersome. The elegance of this approach may be questionable, but its validity is fine.

This is the type of problem we will consider in this paper. We will have a finite set $\mathcal{T}$ of polyomino prototiles, and a finite region we are trying to tile with $\mathcal{T}$. There is no restriction on the use of tiles in $\mathcal{T}$; we may use any tile repeatedly, or we may fail to utilize any given tile. We will be interested in negative results, where we can show that the region cannot be tiled. In light of the remarks above concerning exhaustive search, we will be especially interested in techniques that can prove that infinitely many such regions are untileable. (Although the example of the mutilated checkerboard is only a single shape, it is clear that the same technique applies to infinitely many regions.)

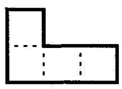
To fix ideas, we will mainly focus on the following type of tiling problem. Our protoset will be a small set of polyominoes, and we'll be interested in tiling rectangles with the set. The same techniques work with little modification for protosets consisting of "polyiamonds" or "polyhexes".

Another typical example is the following. Can 25 copies of the shape  cover a 10×10 square? (The tiles may be rotated and/or reflected.) Again, the answer is "no". Label the squares in alternate rows by 1 and 5, as shown.

1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5
1	1	1	1	1	1	1	1	1	1
5	5	5	5	5	5	5	5	5	5

FIGURE 1.3
10×10 square

Then every placement of a tile covers either one 1 and three 5's, or one 5 and three 1's. In either case, the total it covers is a multiple of 8. However, the 10×10 square covers a total of 300, which is not a multiple of 8, so the square cannot be tiled.

Although the 10×10 square is a single shape, and thus can be exhaustively examined, this same numbering argument shows that  cannot tile any rectangle whose area is congruent to 4 modulo 8. See [8], [10], [11, pp. 42–43] for this example. We will show below (Proposition 2.10) that this type of argument can always be done by a suitable numbering of the squares.