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VECTOR FIELDS IN THE PRESENCE OF A CONTACT STRUCTURE

by Valentin OVSIENKO

ABSTRACT. We consider the Lie algebra of all vector fields on a contact manifold as a module over the Lie subalgebra of contact vector fields. This module is split into a direct sum of two submodules: the contact algebra itself and the space of tangent vector fields. We study the geometric nature of these two modules.

1. INTRODUCTION

Let M be a smooth manifold and $\text{Vect}(M)$ the Lie algebra of all smooth vector fields on M . We consider the case when M is $(2n + 1)$ -dimensional and can be equipped with a contact structure. For instance, if $\dim M = 3$, and M is compact and orientable, then a famous theorem of 3-dimensional topology states that there is always a contact structure on M .

Let $\text{CVect}(M)$ be the Lie algebra of smooth vector fields on M preserving the contact structure. This Lie algebra naturally acts on $\text{Vect}(M)$ (by Lie bracket). We will study the structure of $\text{Vect}(M)$ as a $\text{CVect}(M)$ -module. First, we observe that $\text{Vect}(M)$ is split, as a $\text{CVect}(M)$ -module, into a direct sum of two submodules:

$$\text{Vect}(M) \cong \text{CVect}(M) \oplus \text{TVect}(M),$$

where $\text{TVect}(M)$ is the space of vector fields tangent to the contact distribution. Note that the latter space is a $\text{CVect}(M)$ -module but not a Lie subalgebra of $\text{Vect}(M)$.

The main purpose of this paper is to study the above two spaces geometrically. The most important notion for us is that of *invariance*. All the maps and isomorphisms we consider are invariant with respect to the group of contact diffeomorphisms of M . Since we consider only local maps, this is equivalent to invariance with respect to the action of the Lie algebra $\text{CVect}(M)$.

It is known, see [5, 6], that the adjoint action of $\text{CVect}(M)$ has the following geometric interpretation:

$$\text{CVect}(M) \cong \mathcal{F}_{-\frac{1}{n+1}}(M),$$

where $\mathcal{F}_{-\frac{1}{n+1}}(M)$ is the space of tensor densities of degree $-\frac{1}{n+1}$ on M , that is, of sections of the line bundle

$$\left| \bigwedge^{2n+1} T^*M \right|^{-\frac{1}{n+1}} \rightarrow M.$$

In particular, this provides the existence of a nonlinear invariant functional on $\text{CVect}(M)$ defined on the contact vector fields with nowhere vanishing contact Hamiltonians.

The analogous interpretation of $\text{TVect}(M)$ is more complicated:

$$\text{TVect}(M) \cong \Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M),$$

where $\Omega_0^2(M)$ is the space of 2-forms on M vanishing on the contact distribution. Here and below the tensor products are defined over $C^\infty(M)$.

We study the relations between $\text{TVect}(M)$ and $\text{CVect}(M)$. We prove the existence of a non-degenerate skew-symmetric invariant bilinear map

$$\mathcal{B}: \text{TVect}(M) \wedge \text{TVect}(M) \rightarrow \text{CVect}(M)$$

that measures the non-integrability, i.e., the failure of the Lie bracket of two tangent vector fields to remain tangent.

In order to provide explicit formulæ, we introduce a notion of Heisenberg structure on M . Usually, to write explicit formulæ in contact geometry, one uses the Darboux coordinates. However, this is not the best way to proceed (as already noticed in [4]). The Heisenberg structure provides a universal expression for a contact vector field and its actions.

2. CONTACT AND TANGENT VECTOR FIELDS

In this section we recall the basic definitions of contact geometry. We then prove our first statement on a decomposition of the Lie algebra of all smooth vector fields viewed as a module over the Lie algebra of contact vector fields.

2.1 MAIN DEFINITIONS

Let M be a $(2n + 1)$ -dimensional manifold. A contact structure on M is a codimension 1 distribution ξ which is completely non-integrable. The distribution ξ can be defined (locally) as the kernel of a differential 1-form α defined up to multiplication by a nowhere vanishing function. Complete non-integrability means that

$$(1) \quad \Omega := \alpha \wedge (d\alpha)^n \neq 0$$

everywhere on M . The above condition is also equivalent to the fact that the restriction $d\alpha|_{\xi}$ to any contact hyperplane is a non-degenerate 2-form. In particular, $\ker d\alpha$ is one-dimensional. Note that if M is orientable and the contact structure is coorientable, then the form α can be globally defined on M ; the form Ω is then a volume form.

A vector field X on M is a *contact vector field* if it preserves the contact distribution ξ . In terms of contact forms this means that for every contact form α , the Lie derivative of α with respect to X is proportional to α :

$$(2) \quad L_X \alpha = f_X \alpha,$$

where $f_X \in C^\infty(M)$. The space of all contact vector fields is a Lie algebra that we denote by $\text{CVect}(M)$.

Let us now fix a contact form α . A contact vector field X is called *strictly contact* if it preserves α , in other words, if $f_X = 0$ everywhere on M . Strictly contact vector fields form a Lie subalgebra of $\text{CVect}(M)$. There is one particular strictly contact vector field Z called the *Reeb field* (or characteristic vector field). It is defined by the following two properties:

$$Z \in \ker d\alpha, \quad \alpha(Z) \equiv 1.$$

We will also consider the space, $\text{TVect}(M)$, of vector fields *tangent* to the contact distribution. That this space is not a Lie subalgebra of $\text{Vect}(M)$ follows from non-integrability of the contact distribution.

2.2 THE DECOMPOSITION OF $\text{Vect}(M)$

Let $\text{Vect}(M)$ be the Lie algebra of all smooth vector fields on M . The Lie bracket defines a natural action of $\text{CVect}(M)$ on $\text{Vect}(M)$. In particular, the Lie bracket of a contact vector field with a tangent vector field is again a tangent vector field. Therefore, $\text{TVect}(M)$ is a module over $\text{CVect}(M)$.

PROPOSITION 2.1. *The space $\text{Vect}(M)$ is split into a direct sum of two $\text{CVect}(M)$ -modules :*

$$\text{Vect}(M) \cong \text{CVect}(M) \oplus \text{TVect}(M).$$

Proof. Both spaces on the right hand side are $\text{CVect}(M)$ -modules. It then remains to check that every vector field can be uniquely decomposed into a sum of a contact vector field and a tangent vector field.

Given a vector field X , there exists a tangent vector field Y such that $X - Y$ is contact. Indeed, consider a 1-form $\beta = L_X\alpha$ and its restriction $\beta|_\xi$ to a contact hyperplane ξ . If Y is a tangent vector field then $L_Y\alpha = i_Y(d\alpha)$. Since $d\alpha$ is non-degenerate on ξ , there exists for any 1-form β a tangent field Y such that $i_Y(d\alpha)|_\xi = \beta|_\xi$. This means that $X - Y$ is contact.

Furthermore, the intersection of $\text{CVect}(M)$ and $\text{TVect}(M)$ is zero. Indeed, let X be a non-zero vector field which is contact and tangent at the same time. Then $L_X\alpha = f\alpha$ for some function f and $L_X\alpha = i_X(d\alpha)$. Since $\ker f\alpha$ contains $\xi = \ker \alpha$ while the restriction $d\alpha|_\xi$ is non-degenerate, this is a contradiction. $\square$

3. THE ADJOINT REPRESENTATION OF $\text{CVect}(M)$

In this section we study the action of $\text{CVect}(M)$ on itself.

3.1 FIXING A CONTACT FORM: CONTACT HAMILTONIANS

Let M be orientable; fix a contact form α on M . Every contact vector field X is then characterized by a function

$$H = \alpha(X).$$

This is a one-to-one correspondence between $\text{CVect}(M)$ and the space $C^\infty(M)$ of smooth functions on M , see e.g. [1]. We can denote the contact vector field corresponding to H by X_H . The function H is called the contact Hamiltonian of X_H .

EXAMPLE 3.1. The contact Hamiltonian of the Reeb field Z is the constant function $H \equiv 1$. Note also that the function f_X in (2) is given by the derivative $f_{X_H} = Z(H)$.

The Lie algebra $\text{CVect}(M)$ is then identified with $C^\infty(M)$ equipped with the *Lagrange bracket* defined by $X_{\{H_1, H_2\}} := [X_{H_1}, X_{H_2}]$. One checks that

$$(3) \quad \{H_1, H_2\} = X_{H_1}(H_2) - Z(H_1)H_2.$$

The formula expresses the adjoint representation of $\text{CVect}(M)$ in terms of contact Hamiltonians. The second term on the right hand side shows that this action is different from the natural action of $\text{CVect}(M)$ on $C^\infty(M)$. Let us now clarify the geometric meaning of this action.

3.2 SPACE OF TENSOR DENSITIES

Let V be a vector space of dimension d and λ an arbitrary real number. A λ -density on V is a function $\phi: \bigwedge^d V \setminus \{0\} \rightarrow \mathbf{R}$ homogeneous of degree λ , that is, such that

$$\phi(\kappa w) = |\kappa|^\lambda \phi(w)$$

for all $\kappa \in \mathbf{R} \setminus \{0\}$ and $w \in \bigwedge^d V \setminus \{0\}$. This is a one-dimensional vector space that we denote by $\mathbf{F}_\lambda(V)$.

Let M be a smooth manifold of dimension d ; consider the determinant bundle $\bigwedge^d TM \rightarrow M$.

DEFINITION 3.2. A *tensor density* of degree $\lambda \in \mathbf{R}$ on M is a smooth function on the complement of the zero section $\bigwedge^d TM \setminus M$, homogeneous of degree λ . The space of tensor densities of degree λ on M will be denoted by $\mathcal{F}_\lambda(M)$.

In other words, a tensor density of degree λ on M is a section of the line bundle $\mathbf{F}_\lambda(TM)$, i.e. a field of λ -densities on the tangent space. Equivalently, consider the line bundle $\bigwedge^d T^*M \rightarrow M$. Then, the line bundle $|\bigwedge^d T^*M|^\lambda$ is well defined for every $\lambda \in \mathbf{R}$ and naturally isomorphic to $\mathbf{F}_\lambda(TM)$. It worth noticing that the bundle $\mathbf{F}_\lambda(TM)$ is *trivial* for any M and all λ .

Every space $\mathcal{F}_\lambda(M)$ is naturally a module over the Lie algebra $\text{Vect}(M)$. Let us give here without proof the basic properties of these modules:

- The space $\mathcal{F}_0(M)$ is simply $C^\infty(M)$.
- The space $\mathcal{F}_1(M)$ is isomorphic as a $\text{Vect}(M)$ -module to the space $\Omega_d(M)$ of differential d -forms if and only if M is orientable. If Ω is a volume form on M , then one represents tensor densities in the form

$$\varphi = f \Omega^\lambda,$$

where f is a function.

- The $\text{Vect}(M)$ -modules $\mathcal{F}_\lambda(M)$ and $\mathcal{F}_\mu(M)$ are isomorphic if and only if $\lambda = \mu$.
- If M is compact then there is an invariant functional

$$(4) \quad \int_M : \mathcal{F}_1(M) \rightarrow \mathbf{R}.$$

More generally, there is an invariant pairing

$$\langle \mathcal{F}_\lambda(M), \mathcal{F}_{1-\lambda}(M) \rangle \rightarrow \mathbf{R}$$

given by the integration of the product of tensor densities.

To summarize, the 1-parameter family of spaces $\mathcal{F}_\lambda(M)$ can be viewed as a deformation of the natural $\text{Vect}(M)$ -action on the space of functions. We refer to [3] for more information on modules of tensor densities and invariant differential operators on these modules.

3.3 TENSOR DENSITIES ON A CONTACT MANIFOLD

Let now M be a contact manifold of dimension $d = 2n + 1$. In this case, there is one more way to define tensor densities. Consider the $(2n + 2)$ -dimensional submanifold S of the cotangent bundle $T^*M \setminus M$ that consists of all non-zero covectors vanishing on the contact distribution ξ . The restriction to S of the canonical symplectic structure on T^*M defines a symplectic structure on S . The manifold S is called the *symplectization* of M (cf. [1, 2]). Clearly S is a line bundle over M , its sections are the 1-forms on M vanishing on ξ . Note that, in the case where M is orientable, S is a trivial line bundle over M .

There is a natural lift of $\text{CVect}(M)$ to S . Indeed, a vector field X on M can be lifted to T^*M , and, if X is contact, then it preserves the subbundle S . The space of sections $\text{Sec}(S)$ is therefore a $\text{CVect}(M)$ -module.

The sections of the bundle S can be viewed as tensor densities of degree $\frac{1}{n+1}$ on M .

PROPOSITION 3.3. *There is a natural isomorphism of $\text{CVect}(M)$ -modules*

$$\text{Sec}(S) \cong \mathcal{F}_{\frac{1}{n+1}}(M).$$

Proof. A section of S is a 1-form on M vanishing on the contact distribution. For every contact vector field X and a volume form Ω as in (1) one has

$$L_X \Omega = (n+1)f_X \Omega.$$

The Lie derivative of a tensor density of degree λ is then given by

$$L_X(f \Omega^\lambda) = (X(f) + \lambda(n+1)f_X f) \Omega^\lambda.$$

The result follows from formula (2). $\square$

One can now represent tensor densities in terms of a contact form: $\varphi = f \alpha^{(n+1)\lambda}$.

3.4 CONTACT HAMILTONIAN AS A TENSOR DENSITY

In this section we identify the algebra $\text{CVect}(M)$ with a space of tensor densities of degree $-\frac{1}{n+1}$ on M ; the adjoint action is simply a Lie derivative on this space. The result of this section is known (see [5] and [6], Section 7.5) and given here for the sake of completeness.

Let us define a different version of contact Hamiltonian of a contact vector field X as a tensor density on M of degree $-\frac{1}{n+1}$:

$$\mathcal{H} := \alpha(X) \alpha^{-1}.$$

An important feature of this definition is that it is independent of the choice of α . Let us denote the corresponding contact vector field by $X_{\mathcal{H}}$.

The space $\mathcal{F}_{-\frac{1}{n+1}}(M)$ is now identified with $\text{CVect}(M)$. Moreover, the Lie bracket of contact vector fields corresponds to the Lie derivative.

PROPOSITION 3.4. *The adjoint representation of $\text{CVect}(M)$ is isomorphic to $\mathcal{F}_{-\frac{1}{n+1}}(M)$.*

Proof. The Lagrange bracket coincides with a Lie derivative:

$$(5) \quad \{\mathcal{H}_1, \mathcal{H}_2\} = L_{X_{\mathcal{H}_1}}(\mathcal{H}_2).$$

This formula is equivalent to (3). $\square$

Geometrically speaking, a contact Hamiltonian is not a function but rather a tensor density of degree $-\frac{1}{n+1}$.

3.5 INVARIANT FUNCTIONAL ON $\text{CVect}(M)$

Assume M is compact and orientable; fix a contact form α and the corresponding volume form $\Omega = \alpha \wedge d\alpha^n$. The geometric interpretation of the adjoint action of $\text{CVect}(M)$ implies the existence of an invariant (non-linear) functional on $\text{CVect}(M)$.

Let $\text{CVect}^*(M)$ be the set of contact vector fields with nowhere vanishing contact Hamiltonians (i.e., the corresponding contact Hamiltonian has no zeroes on M). This is an invariant open subset of $\text{CVect}(M)$.

COROLLARY 3.5. *The functional on $\text{CVect}^*(M)$ defined by*

$$\mathcal{I}: X_H \mapsto \int_M H^{-(n+1)} \Omega$$

is invariant. This functional is independent of the choice of the contact form.

Proof. Consider is a contact vector field X_F , then according to (3) one has

$$L_{X_F}(H^{-(n+1)}) = X_F(H^{-(n+1)}) + (n+1)Z(F)H^{-(n+1)},$$

so that the quantity $H^{-(n+1)} \Omega$ is a well defined element of the space $\mathcal{F}_1(M)$. The functional $\mathcal{I}$ is then given by the invariant functional (4).

Furthermore, choose a different contact form $\alpha' = f\alpha$ and the corresponding volume form $\Omega' = f^{n+1}\Omega$. The contact Hamiltonian of the vector field X_H with respect to the contact form α' is the function $H' = \alpha'(X_H) = fH$. Hence, $H'^{-(n+1)}\Omega' = H^{-(n+1)}\Omega$ so that the functional $\mathcal{I}$ is, indeed, independent of the choice of the contact form. $\square$

4. THE STRUCTURE OF $\text{TVect}(M)$

In this section we study the structure of the space of tangent vector fields $\text{TVect}(M)$ viewed as a $\text{CVect}(M)$ -module.

4.1 A GEOMETRIC REALIZATION

Let us start with a geometric realization of the $\text{CVect}(M)$ -module structure on $\text{TVect}(M)$ which is quite similar to that of Section 3.4.

Let $\Omega_0^2(M)$ be the space of 2-forms on M vanishing on the contact distribution. In other words, elements of $\Omega_0^2(M)$ are proportional to α :

$$\omega = \alpha \wedge \beta,$$

where β is an arbitrary 1-form.

The following statement is similar to Proposition 3.4.

THEOREM 4.1. *There is an isomorphism of $\text{CVect}(M)$ -modules*

$$\text{TVect}(M) \cong \Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M),$$

where the tensor product is defined over $C^\infty(M)$.

Proof. Let M be orientable; fix a contact form α on M . Consider a linear map from $\text{TVect}(M)$ to the space $\Omega_0^2(M)$ that associates to a tangent vector field X the 2-form

$$\langle X, \alpha \wedge d\alpha \rangle = -\alpha \wedge i_X d\alpha.$$

This map is bijective since the restriction $d\alpha|_\xi$ of the 2-form $d\alpha$ to the contact hyperplane ξ is non-degenerate.

However, the above map depends on the choice of the contact form and, therefore, cannot be $\text{CVect}(M)$ -invariant. In order to make this map independent of the choice of α , one defines the map

$$(6) \quad X \mapsto \langle X, \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}$$

with values in $\Omega_0^2(M) \otimes \mathcal{F}_{-\frac{2}{n+1}}(M)$. Note that the term α^{-2} on the right hand side is a well defined element of the space of tensor densities $\mathcal{F}_{-\frac{2}{n+1}}(M)$, see Section 3.3.

It remains to check the $\text{CVect}(M)$ -invariance of the map (6). Let X_H be a contact vector field; one has

$$\begin{aligned} L_{X_H} (\langle X, \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}) &= \langle [X_H, X], \alpha \wedge d\alpha \rangle \otimes \alpha^{-2} \\ &\quad + \langle X, f_X \alpha \wedge d\alpha + \alpha \wedge df_X \alpha \rangle \otimes \alpha^{-2} \\ &\quad - \langle X, \alpha \wedge d\alpha \rangle \otimes (2f_X \alpha^{-2}) \\ &= \langle [X_H, X], \alpha \wedge d\alpha \rangle \otimes \alpha^{-2}. \end{aligned}$$

Hence the result. $\square$

The isomorphism (6) identifies the $\text{CVect}(M)$ -action on $\text{TVect}(M)$ by Lie bracket with the usual Lie derivative. It is natural to say that this map defines an analog of contact Hamiltonian of a tangent vector field.

4.2 A SKEW-SYMMETRIC PAIRING ON $\text{TVect}(M)$ OVER $\text{CVect}(M)$

There exists an invariant skew-symmetric bilinear map from $\text{TVect}(M)$ to $\text{CVect}(M)$ that can be understood as a “symplectic structure” on the space $\text{TVect}(M)$ over $\text{CVect}(M)$.

THEOREM 4.2. *There exists a non-degenerate skew-symmetric invariant bilinear map*

$$\mathcal{B}: \text{TVect}(M) \wedge \text{TVect}(M) \rightarrow \text{CVect}(M),$$

where the $\wedge$ -product is defined over $C^\infty(M)$.

Proof. Assume first that M is orientable and fix the contact form α . Given 2 tangent vector fields X and Y , consider the function

$$H_{X,Y} = \langle X \wedge Y, d\alpha \rangle.$$

Define first a bilinear map B from $\text{TVect}(M)$ to $C^\infty(M)$ by

$$(7) \quad B_\alpha: X \wedge Y \mapsto H_{X,Y}.$$

The definition of the function $H_{X,Y}$ and thus of the map B_α depends on the choice of α . Our task is to understand it as a map with values in $\text{CVect}(M)$ which is independent of the choice of the contact form. This will, in particular, extend the definition to the case where M is not orientable.

It turns out that the above function $H_{X,Y}$ is a well defined contact Hamiltonian.

LEMMA 4.3. *Choose a different contact form $\alpha' = f\alpha$, then*

$$H'_{X,Y} = f H_{X,Y}.$$

Proof. By definition,

$$H'_{X,Y} = \langle X \wedge Y, d\alpha' \rangle = f \langle X \wedge Y, d\alpha \rangle + \langle X \wedge Y, df \wedge \alpha \rangle = f H_{X,Y}$$

since the second term vanishes. $\square$

We observe that the function $H_{X,Y}$ depends on the choice of α precisely in the same way as a contact Hamiltonian (cf. Section 3.1). It follows that the bilinear map

$$(8) \quad \mathcal{B}: X \wedge Y \mapsto H_{X,Y} \alpha^{-1}$$

with values in $\mathcal{F}_{-\frac{1}{n+1}} \cong \text{CVect}(M)$ (cf. Section 3.4) is well defined and independent of the choice of α .

It remains to check that the constructed map (8) is $\text{CVect}(M)$ -invariant. This can be done directly but also follows from

PROPOSITION 4.4. *The Lie bracket of two tangent vector fields $X, Y \in \text{TVect}(M)$ is of the form*

$$(9) \quad [X, Y] = \mathcal{B}(X, Y) + (\text{tangent vector field}).$$

Proof. Consider the decomposition from Proposition 2.1 applied to the Lie bracket $[X, Y]$. The “non-tangent” component of $[X, Y]$ is a contact vector field with contact Hamiltonian $\alpha([X, Y])$. One has

$$i_{[X, Y]}\alpha = (L_X i_Y - i_Y L_X)\alpha = -i_Y L_X \alpha = -i_Y i_X d\alpha = H_{X, Y}.$$

The result follows. $\square$

Theorem 4.2 is proved. $\square$

Proposition 4.4 is an alternative definition of $\mathcal{B}$: the map $\mathcal{B}$ measures the failure of the Lie bracket of two tangent vector fields to remain tangent.

5. HEISENBERG STRUCTURES

In order to investigate the structure of $\text{TVect}(M)$ as a $\text{CVect}(M)$ -module in more detail, we will write explicit formulæ for the $\text{CVect}(M)$ -action.

We assume that there is an action of the Heisenberg Lie algebra $\mathfrak{h}_n$ on M , such that the center acts by the Reeb field while the generators are tangent to the contact structure. We then say that M is equipped with the Heisenberg structure. Existence of a globally defined Heisenberg structure is a strong condition on M , however, locally such structure always exists.

5.1 DEFINITION OF A HEISENBERG STRUCTURE

Recall that the Heisenberg Lie algebra $\mathfrak{h}_n$ is a nilpotent Lie algebra of dimension $2n+1$ with the basis $\{a_1, \dots, a_n, b_1, \dots, b_n, z\}$ and the commutation relations

$$[a_i, b_j] = \delta_{ij} z, \quad [a_i, a_j] = [b_i, b_j] = [a_i, z] = [b_i, z] = 0, \quad i, j = 1, \dots, n.$$

The element z spans the one-dimensional center of $\mathfrak{h}_n$.

REMARK 5.1. The algebra $\mathfrak{h}_n$ appears naturally in the context of symplectic geometry as a Poisson algebra of linear functions on the standard $2n$ -dimensional symplectic space.

We say that M is equipped with a *Heisenberg structure* if one fixes a contact form α on M and a $\mathfrak{h}_n$ -action spanned by $2n + 1$ vector fields $\{A_1, \dots, A_n, B_1, \dots, B_n, Z\}$, such that the $2n$ vector fields A_i, B_j are independent at any point and tangent to the contact structure:

$$i_{A_i} \alpha = i_{B_j} \alpha = 0$$

and $[A_i, B_i] = Z$, where Z is the Reeb field, while the other Lie brackets are zero.

5.2 EXAMPLE: THE LOCAL HEISENBERG STRUCTURE

The Darboux theorem states that locally contact manifolds are diffeomorphic to each other. An effective way to formulate this theorem is to say that in a neighborhood of any point of M there is a system of local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ such that the contact structure ξ is given by the 1-form

$$\alpha = \sum_{i=1}^n \frac{x_i dy_i - y_i dx_i}{2} + dz.$$

These coordinates are called the Darboux coordinates.

PROPOSITION 5.2. *The vector fields*

$$(10) \quad A_i = \frac{\partial}{\partial x_i} + \frac{y_i}{2} \frac{\partial}{\partial z}, \quad B_i = -\frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial z}, \quad Z = \frac{\partial}{\partial z},$$

where $i = 1, \dots, n$, define a Heisenberg structure on $\mathbf{R}^{2n+1}$.

Proof. One readily checks that A_i, B_j are tangent and

$$[A_i, B_j] = \delta_{ij} Z$$

while other commutation relations are zero. The vector field Z is simply the Reeb field. $\square$

There is a well-known formula for a contact vector field in the Darboux coordinates (see e.g. [1, 2, 4]). We will not use this formula since the expression in terms of the Heisenberg structure is much simpler.

5.3 CONTACT VECTOR FIELDS AND HEISENBERG STRUCTURE

Assume that M is equipped with an arbitrary Heisenberg structure. It turns out that every contact vector field can be expressed in terms of the basis of the $\mathfrak{h}_n$ -action by a universal formula.

PROPOSITION 5.3. *Given an arbitrary Heisenberg structure on M , a contact vector field with a contact Hamiltonian H is given by the formula*

$$(11) \quad X_H = HZ - \sum_{i=1}^n (A_i(H)B_i - B_i(H)A_i) .$$

Proof. Let us first check that the vector field (11) is, indeed, contact. If X is as on the right hand side of (11), then the Lie derivative $L_X\alpha := (d \circ i_X + i_X \circ d)\alpha$ is given by

$$L_X\alpha = dH - \sum_{i=1}^n (A_i(H)i_{B_i} - B_i(H)i_{A_i})d\alpha .$$

To show that the 1-form $L_X\alpha$ is proportional to α , it suffices to check that

$$i_{A_i}(L_X\alpha) = i_{B_j}(L_X\alpha) = 0 \quad \text{for all } i, j = 1, \dots, n .$$

The first relation is a consequence of the formulæ $i_{A_i}(dH) = A_i(H)$ together with

$$(12) \quad \begin{aligned} i_{A_i}i_{B_j}d\alpha &= i_{A_i}(L_{B_j}\alpha) = i_{[A_i, B_j]}\alpha = \delta_{ij}i_Z\alpha = \delta_{ij} , \\ i_{A_i}i_{A_j}d\alpha &= i_{B_i}i_{B_j}d\alpha = 0 . \end{aligned}$$

The second one follows from the similar relations for i_{B_j} .

Secondly, observe that, if X is as in (11), then $i_X\alpha = H$. This means that the contact Hamiltonian of the contact vector field (11) is precisely H . $\square$

Note that a formula similar to (11) was used in [4] to define a contact structure.

5.4 THE ACTION OF $\text{CVect}(M)$ ON $\text{TVect}(M)$

Since $2n$ vector fields A_i and B_j are linearly independent at any point, they form a basis of $\text{TVect}(M)$ over $C^\infty(M)$. Therefore, an arbitrary tangent vector field X has a unique decomposition

$$(13) \quad X = \sum_{i=1}^n (F_i A_i + G_i B_i) ,$$

where (F_i, G_j) in a $2n$ -tuple of smooth functions on M . The space $\text{TVect}(M)$ is now identified with the direct sum

$$\text{TVect}(M) \cong \underbrace{C^\infty(M) \oplus \cdots \oplus C^\infty(M)}_{2n \text{ times}}.$$

Let us calculate explicitly the action of $\text{CVect}(M)$ on $\text{TVect}(M)$.

PROPOSITION 5.4. *The action of $\text{CVect}(M)$ on $\text{TVect}(M)$ is given by the first-order $(2n \times 2n)$ -matrix differential operator*

$$(14) \quad X_H \begin{pmatrix} F \\ G \end{pmatrix} = \left(X_H \cdot \mathbf{1} - \begin{pmatrix} AB(H) & BB(H) \\ -AA(H) & -BA(H) \end{pmatrix} \right) \begin{pmatrix} F \\ G \end{pmatrix},$$

where F and G are n -vector functions, $\mathbf{1}$ is the unit $(2n \times 2n)$ -matrix, $AA(H)$, $AB(H)$, $BA(H)$ and $BB(H)$ are $(n \times n)$ -matrices, namely

$$AA(H)_{ij} = A_i A_j(H),$$

and the three other expressions are similar.

Proof. Straightforward from (11) and (13). $\square$

PROPOSITION 5.5. *The bilinear map (7) has the following explicit expression:*

$$H_{X, \tilde{X}} = \sum_{i=1}^n \begin{vmatrix} F_i & \tilde{F}_i \\ G_i & \tilde{G}_i \end{vmatrix},$$

where $X = \sum_{i=1}^n (F_i A_i + G_i B_i)$, and $\tilde{X} = \sum_{j=1}^n (\tilde{F}_j A_j + \tilde{G}_j B_j)$.

Proof. This follows from definition (7) and formula (12). $\square$

Note that formula (14) implies that $H_{X, \tilde{X}}$ transforms as a contact Hamiltonian according to (3) since the partial traces of the $(2n \times 2n)$ -matrix in (14) are $A_i B_i(H) - B_i A_i(H) = Z(H)$.

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