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A UNIQUE DECOMPOSITION THEOREM FOR TIGHT CONTACT 3-MANIFOLDS

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ABSTRACT. It has been shown by V. Colin that every tight contact 3-manifold can be written as a connected sum of prime manifolds. Here we prove that the summands in this decomposition are unique up to order and contactomorphism.

1. INTRODUCTION

All 3-manifolds in this note are understood to be smooth and oriented. Except for the more 'local' statements in Sections 2 and 3, we assume in addition that all manifolds are closed and connected. A 3-manifold is called *non-trivial* if it is not diffeomorphic to S^3. A non-trivial 3-manifold P is said to be *prime* if in every connected sum decomposition $P = P_0 \# P_1$ one of the summands P_0, P_1 is S^3. It is known that every non-trivial 3-manifold M admits a prime decomposition, i.e., M can be written as a connected sum of finitely many prime manifolds. The main step in the proof of this fact is due to H. Kneser [11], cf. [8]. Moreover, as shown by J. Milnor [12], the summands in this prime decomposition are unique up to order and diffeomorphism.

The purpose of the present note is to prove the analogous result for tight contact 3-manifolds; see the following section for a summary of the contact geometric notions used in this paper. The basis for the argument is a connected sum construction for such manifolds, due to V. Colin [1] and reproved by K. Honda [9]. Given a fixed connected sum decomposition $M = M_0 \# M_1$ of a 3-manifold M, Colin's result says that tight contact structures ξ_i on M_i, $i = 0, 1$, give rise to a tight contact structure $\xi_0 \# \xi_1$ on M, uniquely defined up to isotopy. Conversely, for any tight contact structure ξ on M there are — up to isotopy — unique tight contact structures ξ_i on M_i, $i = 0, 1$, such that $\xi_0 \# \xi_1$ is the given contact structure ξ. The prime decomposition theorem for tight contact 3-manifolds is an immediate consequence.

Although Colin's result goes a long way, it is not quite strong enough to prove the unique decomposition theorem for tight contact 3-manifolds. This is due to the fact that the system of 2-spheres in a given manifold M defining the prime decomposition of M is not, in general, unique up to isotopy, as can be seen by a simple example. Take two lens spaces P, P' with fundamental groups $\mathbf{Z}_p, \mathbf{Z}_{p'}$, where p and p' are distinct prime numbers (or any other pair of prime 3-manifolds with distinct non-trivial fundamental groups). Let $P \# P'$ be their connected sum and $S \subset P \# P'$ the 2-sphere defining the prime decomposition. Now form the connected sum with a further copy of P by removing a 3-disc each from $P \# P'$ and P and identifying boundaries. The diffeomorphism type of the resulting 3-manifold $P \# P \# P'$ does not depend on the choice of these discs. In particular, we may assume that the 3-disc in $P \# P'$ was chosen in the complement of the separating 2-sphere S. Up to isotopy there are two such choices. Correspondingly, there are two pairs of 2-spheres S, S_1 and S, S_2 in $P \# P \# P'$ defining a prime decomposition. The complement of S_i is $\left(P \setminus D^3\right) \sqcup \left(P \# P' \setminus D^3\right)$ for both $i = 1$ or 2. But the complement of S is either $\left(P \# P \setminus D^3\right) \sqcup \left(P' \setminus D^3\right)$ or $\left(P \setminus D^3\right) \sqcup \left(P' \# P \setminus D^3\right)$. So the two systems of 2-spheres $S \sqcup S_1$ and $S \sqcup S_2$ cannot be isotopic.

The argument for the unique decomposition of tight contact 3-manifolds given here closely follows the variant of Milnor's argument given in J. Hempel's book [8].

2. Basic notions of contact geometry

A contact structure ξ on a 3-manifold M is a totally non-integrable 2-plane field. Our contact structures are understood to be cooriented and positive. This means that they can be defined as $\xi = \ker \alpha$ with a globally defined 1-form α satisfying the non-integrability condition that the 3-form $\alpha \wedge d\alpha$ be a positive volume form.

A diffeomorphism $f\colon (M, \xi) \to (M', \xi')$ between contact manifolds is said to be a *contactomorphism* if its differential maps ξ to ξ' (preserving coorientations).

A contact structure ξ on a 3-manifold M is called *overtwisted* if there is an *embedded* 2-disc $\Delta \subset M$ tangent to ξ along the boundary, that is, with $T_p\Delta = \xi_p$ for all $p \in \partial\Delta$ (not just $T_p(\partial\Delta) \subset \xi_p$!). A disc with this property is referred to as an *overtwisted disc*. A contact structure ξ is called *tight* if it is not overtwisted.

A fundamental result of Ya. Eliashberg [2] says that the classification of overtwisted contact structures reduces to a homotopical problem: every homotopy class of cooriented 2-plane fields contains an overtwisted contact structure, and any two overtwisted contact structures that are homotopic as 2-plane fields are also homotopic as overtwisted contact structures (and hence, according to Gray stability, isotopic). The classification of tight contact structures, on the other hand, is a difficult problem having deep connections with 3-manifold topology. For instance, the standard contact structure

$$\xi_{st} := \ker(x\,dy - y\,dx + z\,dt - t\,dz)$$

on $S^3 \subset \mathbf{R}^4$ is the unique tight contact structure, up to isotopy, on S^3, while homotopy classes of 2-plane fields on S^3 (and hence isotopy classes of overtwisted contact structures) are in one-to-one correspondence with $\pi_3(S^2) \cong \mathbf{Z}$. We also write ξ_{st} for the standard contact structure

$$\ker(dz + x\,dy - y\,dx)$$

on $\mathbf{R}^3$; the contact manifold $(\mathbf{R}^3, \xi_{st})$ is in fact contactomorphic to the complement of a point in (S^3, ξ_{st}).

Given an embedded oriented surface S in a contact 3-manifold (M, ξ), the intersections $T_pS \cap \xi_p$, $p \in S$, define an oriented 1-dimensional foliation on S with singularities at the points p where the tangent plane T_pS coincides with ξ_p. This is called the *characteristic foliation* of S and is denoted by S_ξ. As shown by E. Giroux [7], the characteristic foliation S_ξ determines the germ of ξ near S. This allows one to glue contact manifolds along surfaces with diffeomorphic characteristic foliations.

The following fundamental theorem of Eliashberg lies behind all uniqueness statements in the results of Colin.

THEOREM 1 ([3], Theorem 2.1.3). *Two tight contact structures on the 3-disc D^3 which induce the same characteristic foliation on the boundary ∂D^3 are isotopic rel boundary.* □

A surface $S \subset (M, \xi)$ is called *convex* if there is a vector field transverse to S whose flow preserves ξ. It turns out that the characteristic foliation S_ξ being of Morse-Smale type is sufficient for such a flow to exist. This condition can always be guaranteed by a C^∞-small perturbation of any given surface S.

For more detailed introductions to contact geometry see [4], [10] and [5].

3. Colin's results

In this section we collect the results from [1] that we shall need. Given an embedding $f\colon S \to (M,\xi)$, we write $S_{f^*\xi}$ for the induced characteristic foliation on S, that is, the pull-back to S via f of the characteristic foliation $\big(f(S)\big)_\xi$.

Lemma 2 ([1], Lemme 5). *Let (M,ξ) be a tight contact 3-manifold.*

(a) *Given an embedding $f\colon S^2 \to M$, there is an orientation-preserving embedding $g\colon D^3 \to \mathbf{R}^3$ such that $S^2_{g^*\xi_{st}} = S^2_{f^*\xi}$. The tight contact structure $g^*\xi_{st}$ on D^3 is uniquely determined, up to isotopy rel boundary, by the characteristic foliation $S^2_{g^*\xi_{st}}$ on the boundary.*

(b) *Given embeddings $f_0, f_1\colon S^2 \to M$, there is a tight contact structure η on $S^2 \times [0,1]$ such that the characteristic foliation $(S^2 \times \{i\})_\eta$ coincides with $S^2_{f_i^*\xi}$, $i = 0,1$. This contact structure η is unique up to isotopy rel boundary.* □

We can now define surgery of index 1 on a given tight contact 3-manifold (M,ξ) as follows; this includes the formation of a connected sum.

Equip the 3-disc D^3 with its standard orientation. Let $\phi_0, \phi_1\colon D^3 \to M$ be embeddings such that ϕ_0 reverses and ϕ_1 preserves orientation, and whose images $B_i := \phi_i(D^3) \subset M$ are disjoint. Let η be the contact structure on $S^2 \times [0,1]$, constructed in the preceding lemma, with the property that $(S^2 \times \{i\})_\eta = (\partial D^3)_{\phi_i^*\xi}$. Then set

$$(M',\xi') = \big(M \setminus \mathrm{Int}\,(B_0 \cup B_1), \xi\big) \cup_\partial \big(S^2 \times [0,1], \eta\big),$$

where 'Int' stands for interior, and $\cup_\partial$ denotes the obvious gluing along the boundary.

If $M = M_0 \sqcup M_1$ is the disjoint union of two connected tight contact 3-manifolds (M_0,ξ_0), (M_1,ξ_1), and $B_i \subset M_i$, $i = 0,1$, then M' is the connected sum $M_0 \# M_1$ of M_0 and M_1, and we write $\xi_0 \# \xi_1$ for the contact structure ξ' in this specific case. We also use the notation $(M_0,\xi_0) \# (M_1,\xi_1)$ for this connected sum of tight contact 3-manifolds. As in the topological case, this connected sum operation is commutative and associative; these are consequences of the discussion that follows. From Theorem 1 we deduce that (S^3,ξ_{st}) serves as the neutral element.

LEMMA 3 ([1], Corollaire 8). *Let* (M',ξ') *be a contact* 3*-manifold and* $f_t\colon S^2 \to M'$, $t \in [0,1]$, *an isotopy of embeddings. If the spheres* $S_i := f_i(S^2)$, $i=0,1$, *are convex with respect to* ξ', *and* $(M' \setminus S_0, \xi')$ *is tight, then so is* $(M' \setminus S_1, \xi')$. □

LEMMA 4 ([1], Proposition 9). *The manifold* (M',ξ') *obtained, in the way described above, via surgery of index* 1 *on a tight contact* 3*-manifold* (M,ξ), *is tight and only depends, up to contactomorphism, on the isotopy class of the embeddings* ϕ_0, ϕ_1. □

In particular, with notation as before, the contact structure $\xi_0 \# \xi_1$ on $M_0 \# M_1$ is tight and does not depend, up to contactomorphism, on the choice of embeddings $B_i \subset M_i$.

4. THE UNIQUE DECOMPOSITION THEOREM

We can now formulate the unique decomposition theorem for tight contact 3-manifolds.

THEOREM 5. *Every non-trivial tight contact* 3*-manifold* (M,ξ) *is contactomorphic to a connected sum*

$$(M_1,\xi_1) \# \cdots \# (M_k,\xi_k)$$

of finitely many prime tight contact 3*-manifolds. The summands* (M_i,ξ_i), $i=1,\ldots,k$, *are unique up to order and contactomorphism.*

The proof of this theorem requires a few preparations. First of all, we observe that there is a well-defined procedure for capping off a compact tight contact 3-manifold whose boundary consists of a collection of 2-spheres. Indeed, suppose that (M,ξ) is a tight contact 3-manifold with boundary $\partial M = S_1 \sqcup \cdots \sqcup S_k$, where each S_i is diffeomorphic to S^2. Choose orientation-reversing diffeomorphisms $f_i\colon \partial D^3 \to S_i$. By Lemma 2(a) one finds an orientation-preserving embedding $g_i\colon D^3 \to \mathbf{R}^3$ such that $S^2_{g_i^*\xi_{st}} = S^2_{f_i^*\xi}$. The tight contact structures $\eta_i := g_i^*\xi_{st}$, $i=1,\ldots,k$, on D^3 — each of them uniquely determined by the characteristic foliation it induces on the boundary — can then be used to form the closed contact manifold

$$(\widehat{M},\hat{\xi}) = (M,\xi) \cup_\partial \big((D^3,\eta_1) \cup \ldots \cup (D^3,\eta_k)\big),$$

where the gluing is defined by the embeddings f_i.

Eliashberg's theorem entails that we arrive at a contactomorphic manifold if instead of gluing discs along the S_i we first perturb the boundary spheres into convex spheres S_i' in the interior of (M,ξ), cut off the spherical shell between S_i and S_i', and then glue discs along the S_i'. The following is implicit in Colin's work [1].

LEMMA 6. *The contact manifold $(\widehat{M},\hat{\xi})$ is tight.*

Proof. It suffices to deal with a gluing $(M',\xi') := (M,\xi)\cup_{S_0}(D^3,\eta_0)$ along one boundary component of M, which by the above we may assume to be convex with respect to ξ. Given an embedded 2-disc $\Delta\subset M'$, there is an isotopy of S_0 in M' to a sphere S_1 disjoint from Δ. Since $(M'\setminus S_0,\xi')$ is tight, the same is true for $(M'\setminus S_1,\xi')$ by Lemma 3. So Δ cannot be an overtwisted disc. □

Notice that for the validity of this argument it is irrelevant whether one of the constituents of the boundary gluing was a disc.

Given an embedded 2-sphere $S\subset \mathrm{Int}(M)$, we can find a product neighbourhood $S\times[-1,1]\subset M$ of $S\equiv S\times\{0\}$. Set $M_S=M\setminus\big(S\times(-1,1)\big)$. Again by Theorem 1, the contactomorphism type of $(\widehat{M}_S,\hat{\xi})$ is independent of the choice of this product neighbourhood; this follows by comparing the resulting manifolds using two given product neighbourhoods with a third manifold constructed from a product neighbourhood contained in the first two. In particular, this justifies our notation $(\widehat{M}_S,\hat{\xi})$.

LEMMA 7. *If S_0 and S_1 are isotopic 2-spheres in* Int(M), *then $(\widehat{M}_{S_0},\hat{\xi})$ and $(\widehat{M}_{S_1},\hat{\xi})$ are contactomorphic.*

Proof. This is clear if S_1 is isotopic to S_0 inside a product neighbourhood $S_0\times(-1,1)$. The general case follows by breaking up the isotopy into smaller ones that move the sphere inside a product neighbourhood only. For details see the proof of [1, Corollaire 8]. □

Given a connected sum decomposition $M = M_0 \# M_1$ of a closed, connected 3-manifold with a tight contact structure ξ, let $S\subset M$ be an embedded sphere defining this connected sum, i.e. $\widehat{M}_S = M_0\sqcup M_1$. The described constructions imply that

$$(M,\xi)=(M_0,\hat{\xi}|_{M_0})\,\#\,(M_1,\hat{\xi}|_{M_1}).$$

So the topological prime decomposition of M also gives us a decomposition of (M,ξ) into prime tight contact 3-manifolds. The only remaining issue is the uniqueness of this decomposition up to contactomorphism of the summands.

A 3-manifold M is said to be *irreducible* if every embedded 2-sphere bounds a 3-disc in M. Clearly, irreducible 3-manifolds (except S^3) are prime. There is but one orientable prime 3-manifold that is not irreducible, namely, $S^2\times S^1$ [8, Lemma 3.13]. In a connected sum $M=M_0\,\#\,S^2\times S^1$ we obviously find an embedded non-separating 2-sphere S such that $\widehat{M}_S=M_0$; simply take S to be a fibre of $S^2\times S^1$ not affected by the connected sum construction.

In the argument proving that the number of summands $S^2\times S^1$ in a prime decomposition of M is uniquely determined by M, the crucial lemma is that for any two non-separating 2-spheres $S_0,S_1\subset M$ there is a diffeomorphism of M sending S_0 to S_1 [8, Lemma 3.18]. In the presence of a contact structure, this statement needs to be weakened slightly; the following is sufficient for our purposes.

LEMMA 8. *Let (M,ξ) be a (connected) tight contact 3-manifold and $S_0,S_1\subset M$ two non-separating 2-spheres. Then $(\widehat{M}_{S_0},\hat{\xi})$ and $(\widehat{M}_{S_1},\hat{\xi})$ are contactomorphic.*

Proof. By the preceding lemma we may assume that S_0 and S_1 are in general position with respect to each other, so that $S_0\cap S_1$ consists of a finite number of embedded circles. We use induction on the number n of components of $S_0\cap S_1$.

If $n=0$, we find disjoint product neighbourhoods $S_i\times[-1,1]\subset M$, $i=0,1$. In case $M\setminus(S_0\cup S_1)$ is not connected, we may assume that the identifications of these neighbourhoods with a product have been chosen in such a way that $S_0\times\{1\}$ and $S_1\times\{1\}$ lie in the same component of $M\setminus(S_0\cup S_1)$. As described above, we then obtain a well-defined tight contact manifold $(\widetilde{M},\tilde{\xi})$ by capping off the boundary components $S_i\times\{\pm1\}$ of

$$M\setminus\big(S_0\times(-1,1)\cup S_1\times(-1,1)\big)$$

with 3-discs $D_0^{\pm},D_1^{\pm}$. Our assumptions imply that $D_0^-\sqcup D_0^+$ is isotopic to $D_1^-\sqcup D_1^+$ in $\widetilde{M}$. By performing index 1 surgery with respect to these embeddings of $S^0\times D^3$, we obtain $(\widehat{M}_{S_1},\hat{\xi})$ and $(\widehat{M}_{S_0},\hat{\xi})$, respectively, so the result follows from Lemma 4.

If $n>0$, then some component J of $S_0\cap S_1$ bounds a 2-disc $D\subset S_1$ with $\mathrm{Int}(D)\cap S_0=\varnothing$. Let E' and E'' be the 2-discs in S_0 bounded by J, and set $S_0'=D\cup E'$ and $S_0''=D\cup E''$.

CLAIM 1. *At least one of S_0' and S_0'' is non-separating.*

Proof. Since S_0 is non-separating, there is a loop γ in M (in general position with respect to all spheres in question) that intersects S_0 in a single point, say one contained in the interior of E'. If S_0'' is separating, then γ intersects it in an even number of points. Since γ does not intersect E'', these points all lie in D. So γ intersects S_0' in an odd number of points, which means that S_0' is non-separating. □

Thus, continuing with the proof of Lemma 8, we may assume without loss of generality that S_0' is non-separating. Move S_0' slightly so that it becomes a smoothly embedded sphere disjoint from S_0 and intersecting S_1 in fewer than n circles. Then two applications of the inductive assumption prove the inductive step. □

Proof of Theorem 5. As indicated above, it only remains to prove the uniqueness statement. Thus, let

$$(M_1, \xi_1) \# \cdots \# (M_k, \xi_k)$$

and

$$(M_1^*, \xi_1^*) \# \cdots \# (M_l^*, \xi_l^*)$$

be two prime decompositions of a given tight contact 3-manifold (M, ξ). Without loss of generality we assume[1]) $k \leq l$, and use induction on k. For $k = 1$ there is nothing to prove. Now assume $k > 1$ and the assumption to be proved for prime decompositions with fewer than k summands.

(i) Suppose some M_i (say M_k) is diffeomorphic to $S^2 \times S^1$. Then M contains a non-separating 2-sphere. By applying the argument from Claim 1 to this non-separating 2-sphere and the 2-spheres defining the splitting of M into the connected sum of the M_j^*, one finds a non-separating 2-sphere in at least one of these summands, say M_l^*, which therefore must be a copy of $S^2 \times S^1$. By a folklore theorem of Eliashberg, there is a unique tight contact structure on $S^2 \times S^1$; cf. [4] for an outline proof and [6] for a complete proof. Thus, (M_k, ξ_k) is contactomorphic to (M_l^*, ξ_l^*). Let S_0, S_1 be a fibre in M_k, M_l^*, respectively. From Theorem 1 it follows that

$$(\widehat{M}_{S_0}, \hat{\xi}) = (M_1, \xi_1) \# \cdots \# (M_{k-1}, \xi_{k-1})$$

[1]) Of course, from the topological prime decomposition theorem, one already knows that $k = l$, but this does not help to simplify the present proof.

and

$$(\widehat{M}_{S_1}, \hat{\xi}) = (M_1^*, \xi_1^*) \,\#\cdots\#\, (M_{l-1}^*, \xi_{l-1}^*),$$

and by the preceding lemma these two manifolds are contactomorphic. So the conclusion of the theorem follows from the inductive assumption.

(ii) It remains to deal with the case where all the M_i are irreducible. Arguing as before (with the roles of the two connected sum decompositions reversed), we see that each M_j^* must be irreducible. Choose a separating 2-sphere $S \subset M$ such that the closures U, V of the components of $M \setminus S$ satisfy

$$(\widehat{U}, \widehat{\xi|_U}) = (M_1, \xi_1) \,\#\cdots\#\, (M_{k-1}, \xi_{k-1})$$

and $(\widehat{V}, \widehat{\xi|_V}) = (M_k, \xi_k)$. Observe that the contact structure $\widehat{\xi|_U}$ is the same as the restriction of the contact structure $\hat{\xi}$ (defined on $\widehat{M}_S = \widehat{U} \sqcup \widehat{V}$) to $\widehat{U}$.

Similarly, there exist pairwise disjoint 2-spheres $T_1, \ldots, T_{l-1}$ in M such that — with $W_1, \ldots, W_l$ denoting the closures of the components of $M \setminus (T_1 \cup \ldots \cup T_{l-1})$, and ξ_j the restriction of ξ to W_j — we have $(\widehat{W}_j, \hat{\xi}_j) = (M_j^*, \xi_j^*)$, $j = 1, \ldots, l$.

Suppose that the system $T_1, \ldots, T_{l-1}$ of embedded spheres has been chosen in general position with respect to S and with $S \cap (T_1 \cup \ldots \cup T_{l-1})$ having the minimal number of components among all such systems.

CLAIM 2. *The minimality condition implies* $S \cap (T_1 \cup \ldots \cup T_{l-1}) = \emptyset$.

Here we have to enter a caveat. The notation suggests that W_1 has boundary T_1, the W_j with $j \in \{2, \ldots, l-1\}$ have boundary $T_{j-1} \sqcup T_j$, and W_l has boundary T_{l-1}. In fact, some of the reasoning in the proof given in [8] seems to rely on such an assumption. However, under the minimality condition we have just described, it is perfectly feasible that some of the W_j have several boundary components (i.e., the connected sum looks like a tree rather than a chain). In particular, the numbering of the W_j is not meant to suggest any kind of order in which they are glued together.

Assuming Claim 2, we have $S \subset W_j$ for some $j \in \{1, \ldots, l\}$. Since $\widehat{W}_j = M_j^*$ is irreducible, S bounds a 3-cell B in M_j^*. Thus, S cuts W_j into two pieces X and Y, where say $\widehat{Y} = S^3$. By the uniqueness of the tight contact structure on S^3 we have in fact $(\widehat{Y}, \widehat{\xi|_Y}) = (S^3, \xi_{st})$. Moreover, $(\widehat{X}, \widehat{\xi|_X}) = (M_j^*, \xi_j^*)$ by Theorem 1.

Of the 3-discs in M_j^* used for forming the connected sum with one or several of the other prime manifolds, at least one has to be contained in B, otherwise S would bound a disc in M. This means that of the closures U, V of the two components of $M \setminus S$, the one containing Y must contain at least one of $W_1, \dots, W_{j-1}, W_{j+1}, \dots, W_l$. Thus, in the case $Y \subset V$, the numbering (including that of W_j) can be chosen in such a way that $W_1, \dots, W_{j-1}, X \subset U$ and $Y, W_{j+1}, \dots, W_l \subset V$, with $j \leq l-1$. (The case with $X \subset V$ and $Y \subset U$ is analogous; here $j \geq 2$.) With Theorem 1, and in particular the fact that (S^3, ξ_{st}) is the neutral element for the connected sum operation, we conclude that

$$\begin{aligned}(M_1, \xi_1) \# \cdots \# (M_{k-1}, \xi_{k-1}) &= (\widehat{U}, \widehat{\xi|_U}) \\ &= (\widehat{W}_1, \hat{\xi}_1) \# \cdots \# (\widehat{W}_{j-1}, \hat{\xi}_{j-1}) \# (\widehat{X}, \widehat{\xi|X}) \\ &= (M_1^*, \xi_1^*) \# \cdots \# (M_j^*, \xi_j^*)\end{aligned}$$

and

$$\begin{aligned}(M_k, \xi_k) &= (\widehat{V}, \widehat{\xi|_V}) \\ &= (\widehat{Y}, \widehat{\xi|_Y}) \# (\widehat{W}_{j+1}, \hat{\xi}_{j+1}) \# \cdots \# (\widehat{W}_l, \hat{\xi}_l) \\ &= (M_{j+1}^*, \xi_{j+1}^*) \# \cdots \# (M_l^*, \xi_l^*) .\end{aligned}$$

Since M_k is prime, we must have $j = l-1$, hence $(M_k, \xi_k) = (M_l^*, \xi_l^*)$. Once again, the theorem follows from the inductive assumption. Modulo Claim 2 this concludes the proof of the unique decomposition theorem. □

Proof of Claim 2. Arguing by contradiction, we assume that $T_1, \dots, T_{l-1}$ is a system of 2-spheres as described, with $S \cap (T_1 \cup \ldots \cup T_{l-1})$ having the minimal number of components among all such systems, and that this minimal number is positive. Then we find a 2-disc $D \subset S$ with $\partial D \subset T_i$ for some $i \in \{1, \dots, l-1\}$, and $\mathrm{Int}(D) \cap (T_1 \cup \ldots \cup T_{l-1}) = \varnothing$. This disc is contained in W_j for some $j \in \{1, \dots, l\}$. For ease of notation we assume that $i = j = 1$, and that W_2 is the other component adjacent to T_1.

Let E', E'' be the 2-discs in T_1 bounded by ∂D. Since $\widehat{W}_1$ is irreducible, the sets $D \cup E'$ and $D \cup E''$ (which are homeomorphic copies of S^2) bound 3-cells B', B'' in $\widehat{W}_1$. One of these must contain the other, otherwise it would follow that $\widehat{W}_1$ can be obtained by capping off the 3-cell $B' \cup_D B''$, and thus would be a 3-sphere.

So suppose that $B'' \subset B'$. Then $D \cup E'$ can be deformed into a smooth 2-sphere T_1' that meets S in fewer components than T_1, see Figure 1. In the complement $M \setminus (T_1' \cup T_2 \cup \ldots \cup T_{l-1})$ we still find $W_3, \dots, W_l$, but W_1, W_2 have been changed to new components W_1', W_2'. Write ξ_1', ξ_2', respectively, for the restriction of ξ to these components. We are done if we can show that

$$(\widehat{W}'_i, \hat{\xi}'_i) = (\widehat{W}_i, \hat{\xi}_i), \quad i = 1, 2,$$

because then the new system of spheres $T'_1, T_2, \ldots, T_{l-1}$ contradicts the minimality assumption on $T_1, T_2, \ldots, T_{l-1}$.

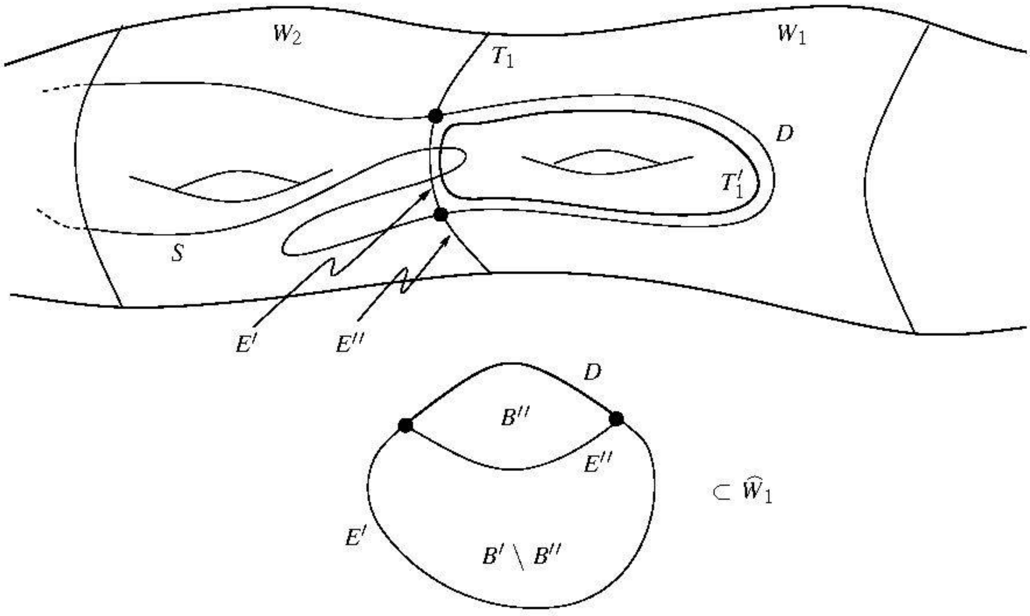

FIGURE 1
Modification of the prime decomposition

The 2-sphere T'_1 is isotopic to T_1 in $\widehat{W}_1$: simply move $D \subset T'_1$ across the ball B'' to E''. But beware that T'_1 need not be isotopic to T_1 in W_1 or M. However, B'' lies on the same side of T_1 as W_1, so T'_1 is isotopic to T_1 in

$$\widehat{W_1 \cup W_2} = \widehat{W'_1 \cup W'_2}.$$

Cutting this latter manifold open along T_1 and then capping off with discs gives the disjoint union of $(\widehat{W}_1, \hat{\xi}_1)$ and $(\widehat{W}_2, \hat{\xi}_2)$; cutting it open along T'_1 and capping off yields the disjoint union of $(\widehat{W}'_1, \hat{\xi}'_1)$ and $(\widehat{W}'_2, \hat{\xi}'_2)$. From Lemma 7 it follows that the results of either procedure are contactomorphic. □

There is no unique decomposition theorem for overtwisted contact 3-manifolds. For instance, start with a connected sum of two distinct prime tight contact 3-manifolds (M_0, ξ_0), (M_1, ξ_1). Choose arbitrary knots $K_i \subset M_i \setminus D^3 \subset M_0 \# M_1$. After a C^0-small isotopy we may assume that the K_i are transverse to ξ_i (or $\xi_0 \# \xi_1$) [5, Theorem 2.44]. Then, in suitable local coordinates (θ, r, φ) near $K_i \equiv S^1 \times \{0\} \subset S^1 \times \mathbf{R}^2$, the contact structure can be written as $\ker(d\theta + r^2\, d\varphi)$ [5, Example 2.33].

Now perform a full Lutz twist along either K_0 or K_1. This means that the contact structure $\ker(d\theta + r^2\, d\varphi)$ near the knot is changed to $\ker(h_1(r)\, d\theta + h_2(r)\, d\varphi)$ with $(h_1(r), h_2(r))$ as shown in Figure 2. It is required that $h_1(r) \equiv 1$ and $h_2(r) = r^2$ both near $r = 0$ and for large r.

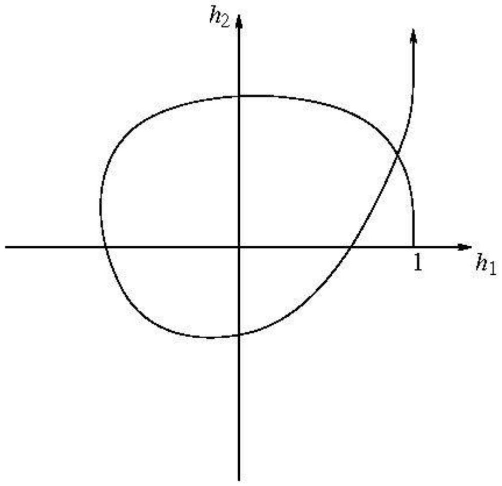

FIGURE 2
A full Lutz twist

It is not difficult to see that this local procedure does not change the homotopy class of the contact structure as a 2-plane field, but it makes the contact structure overtwisted [5, Lemma 3.17]. Thus, by performing such a full Lutz twist along either K_0 or K_1 in $(M_0 \# M_1, \xi_0 \# \xi_1)$, we obtain the contact manifolds $(M_0 \# M_1, \xi_0' \# \xi_1)$ and $(M_0 \# M_1, \xi_0 \# \xi_1')$, which are contactomorphic by Eliashberg's classification [2] of overtwisted contact structures (because, by construction, ξ_0', ξ_1' are overtwisted, and so are the contact structures $\xi_0' \# \xi_1$ and $\xi_0 \# \xi_1'$ on the connected sum). This obviously gives us two distinct connected sum decompositions.

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