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THE SMITH FILTRATION ON THE UNIVERSAL PRINCIPAL BUNDLE OF A COXETER GROUP

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DEFINITION 31.1. If G is any discrete group, let EG denote the *standard construction of a universal principal G-bundle.* That is, EG is the geometric realization of the simplicial set E_*G whose set of k-simplices is G^{k+1}, with faces given by deletion of coordinates and degeneracies given by repetition of coordinates and with G acting diagonally on the coordinates.

In his 1981 thesis [2], J. Smith constructed a natural filtration on the universal principal bundle $E\Sigma_n$ of the symmetric group Σ_n as follows.

For each $m \geq 1$, let $F^{(m)}E_*\Sigma_n$ denote the simplicial subset of $E_*\Sigma_n$ consisting of those simplices $(g_0, g_1, \dots, g_k)$, where for each pair $1 \leq i < j \leq n$, the number of times the pair gets reversed by the sequence of permutations $g_0 g_1^{-1}$, $g_1 g_2^{-1}, \dots, g_{k-1} g_k^{-1}$ is at most $m-1$ times. Let $F^{(m)}E\Sigma_n$ denote the geometric realization of this simplicial set.

Smith conjectured the following result, which was later proved by C. Berger [1].

THEOREM 31.2. *$F^{(m)}E\Sigma_n$ has the homotopy type of the configuration space of n-tuples of distinct points in $\mathbf{R}^{mn}$.*

This result can be reformulated in the context of Coxeter groups as follows. Let G be a finite Coxeter group generated by orthogonal reflections on $\mathbf{R}^n$. Then we can define the *Smith filtration* on E_*G by counting the number of times a generic point in $\mathbf{R}^n$ gets flipped around any one of the reflecting hyperplanes H_i by the sequence $g_0 g_1^{-1}$, $g_1 g_2^{-1}, \dots, g_{k-1} g_k^{-1}$ corresponding to a k-simplex $(g_0, g_1, \dots, g_k)$.

A natural generalization of the above result seems to be

CONJECTURE 31.3. *$F^{(m)}EG$ has the homotopy type of the complement of $\cup_i H_i \otimes \mathbf{R}^m$ in $\mathbf{R}^{mn}$.*

Moreover a considerable portion of Berger's proof carries through in this context. Berger's proof consists of constructing a certain poset and then decomposing the configuration space as a colimit of contractible subspaces indexed by this poset. He then shows that on the one hand the colimit has the same homotopy type as the homotopy colimit, and thus the same homotopy type as the nerve of the poset. On the other hand, he shows by a Quillen Theorem A argument that the nerve of the poset has the same homotopy type as $F^{(m)}E\Sigma_n$. Berger's poset has a natural interpretation in the Coxeter context. However there are certain technical difficulties in carrying out the complete proof.

It may also be the case that there are further generalizations possible for more general classes of reflection groups.

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