L'Enseignement Mathématique (2) 54 (2008), 100–102

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METASTABLE EMBEDDING, 2-EQUIVALENCE AND GENERIC RIGIDITY OF FLAG MANIFOLDS

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CONJECTURE 34.1. *Any 2-equivalent manifolds embed in the same metastable dimension. I.e., let M^n and N^n be two simply connected closed differentiable manifolds such that their 2-localizations are homotopy equivalent. If M^n embeds in $\mathbf{R}^{n+k}$, $k \geq (n+3)/2$, then N^n embeds in euclidean space of the same dimension, cf.* [7].

R. Rigdon [15] proved this result in the case that there exists a global map, $f\colon M \to N$ realizing this 2-equivalence, e.g., an odd covering. Glover, Mislin [8] and independently Bendersky [1] proved an analogous result for immersing manifolds in euclidean space. Glover, Mislin [9] proved an analogous result for the number of linearly independent tangent vector fields on a smooth manifold. Although the embedding result would just be a technical generalization of Rigdon's result it still seems interesting and would apply to such situations as the Hilton, Roitberg criminal H-manifolds [11], or manifolds made by Zabrodsky mixing [16].

CONJECTURE 34.2. *All complex flag manifolds are generically rigid. I.e., given a simply connected space X of finite type, let $\mathcal{G}(X)$ denote the (Mislin) genus of X, the set of all homotopy types $[Y]$, of simply connected, finite type spaces Y, such that the p-localization of X is homotopy equivalent to the p-localization of Y, for all primes p. We say that a simply connected, finite type X is generically rigid or generically trivial if $\mathcal{G}(X) = \{[X]\}$, the single homotopy type. A complex flag manifold is any space G/H, where $G = U(n)$ and $H = U(n_1) \times U(n_2) \times \cdots \times U(n_k)$, with $\Sigma_{i=1}^{k} n_i = n$.*

See [9] for cases when Conjecture 34.2 has been proved. These include complex Grassmann manifolds and complete flag manifolds $U(n)/T^n$, where

$T^n = \Pi_{i=1}^n U(1)$. Note that Papadima has proved this result in the context of G any compact Lie group and H its maximal torus [14].

A survey of the Mislin genus is given in [13]. Many simply connected spaces of finite type fail to be generically trivial. First examples are $|\mathcal{G}(\mathbf{H}P^n)| = 2^k$, where k is the number of primes p, such that $2 \leq p \leq 2n-1$.

This conjecture began with the author's question to Albrecht Dold in 1973 of why we didn't know more manifolds with the *fixed point property* (every self map has a fixed point). The obvious ones at that point were the real, complex projective spaces of even dimension and all quaternionic projective spaces (except $\mathbf{H}P^1$) as shown by the Lefschetz fixed-point formula. Dold suggested the Grassmann manifold of complex 2-planes (through the origin) in 5-dimensional complex space, $U(5)/(U(2) \times U(3))$. This was correct as seen by applying the Lefschetz fixed-point formula to the integral cohomology ring

$$H^\star\big(U(p+q)/(U(p) \times U(q)); \mathbf{Z}\big) = \mathbf{Z}[c, \overline{c}]/\{c\overline{c} = 1\},$$

showing there were only Adams maps, $c_i \mapsto \lambda^i c_i$ for $i = 1, 2, \ldots, p$, in this case $p = 2$, $q = 3$. Here c is the total Chern class of the canonical p-plane bundle over this Grassmann manifold and $\overline{c}$ the total Chern class of the canonical q-plane bundle. In [4] it is shown that this result is true in general for $p \gg q$. This result led to the independent proofs by Stephen Brewster (OSU PhD thesis 1978) [2] and Mike Hoffman [12] that the only cohomology ring endomorphisms of Grassmann manifolds $U(p+q)/(U(p) \times U(q))$ were given by Adams maps when $p \neq q$, and $\lambda \neq 0$, and $c_i \mapsto \overline{c_i}$, $i = 1, 2, \ldots, p$, when $p = q$.

The results in [5] give a conjecture for all the integral cohomology ring endomorphisms of the general complex flag manifold and as a consequence give the conjecture that all the rational cohomology ring automorphisms are given by Adams maps, and actions of the Weyl group N/H, where N is the normalizer of $H = \Pi_{i=1}^k U(n_i)$, $\Sigma_{i=1}^k n_i = n$, in $G = U(n)$. It is this conjecture, proved in special cases, that gives the results in [10] and would prove Conjecture 34.2. Another consequence of the cohomology ring endomorphism conjecture would be a complete classification of which complex flag manifolds have the fixed point property (cf. [6]). There are a number of other applications of the cohomology ring endomorphism and automorphism theorems, e.g., by S. Papadima [14] to isometry invariant geodesics, and P. Gilkey [3] to the classification of Hermitian Riemannian manifolds.

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