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### SOLUBLE GROUPS OF TYPE VF

by Conchita MARTÍNEZ-PÉREZ and Brita E. A. NUCINKIS

Cohomological finiteness conditions for soluble groups are very well understood for torsion-free soluble groups, but there remains a gap for soluble groups with torsion. This can be summarised by the following conjecture:

CONJECTURE 51.1. *Every soluble group  $G$  of type VF admits a cocompact model for  $\underline{EG}$ .*

A group is said to be *of type VF* if it has a finite index subgroup admitting a finite  $K(G, 1)$ .

Finiteness conditions for soluble groups have been attracting wide attention ever since the celebrated result by Stammbach [9], that for a torsion-free soluble group the homological dimension  $\text{hd } G$  is equal to the Hirsch length  $\text{h } G$  of the group. For polycyclic groups the Hirsch length is equal to the cohomological dimension,  $\text{cd } G$ . (For countable groups, the homological dimension and the cohomological dimension differ by at most one.)

This result was extended by Gruenberg, see [1], who showed that a torsion-free nilpotent group is finitely generated if and only if  $\text{cd } G = \text{h } G$ . This led to the question, which cohomological finiteness condition describes torsion-free soluble groups with  $\text{cd } G = \text{h } G$ .

There were partial answers to this question by numerous authors including Bieri, Gildenhuys and Strebel and it was finally answered by Kropholler [3]. Kropholler's later result [4] meant that this result could be phrased as follows:

THEOREM 51.2 ([3,4]). *Let  $G$  be a soluble group. Then the following are equivalent:*

- (1)  $G$  is of type  $\text{FP}_\infty$ ,
- (2)  $G$  is virtually of type  $\text{FP}$ ,
- (3)  $\text{vcd } G = \text{h } G < \infty$ ,
- (4)  $G$  is virtually torsion-free and constructible.

The fact that  $G$  is constructible implies that  $G$  is finitely presented. Thus any torsion-free group satisfying the conditions of the Theorem is of type  $\text{F}$ , i.e. it has a finite  $\text{K}(G, 1)$  or equivalently a cocompact model for  $\text{EG}$ .

In case  $G$  has torsion it cannot admit a finite dimensional model for  $\text{EG}$ , which is equivalent to saying that  $\text{cd } G = \infty$ . We say a  $G$ -CW-complex  $X$  is a model for  $\underline{\text{EG}}$  if  $X^H$  is contractible for all finite  $H < G$  and empty otherwise. The cohomological counterpart is Bredon (co)homology. It was shown [8] that for countable groups the Bredon cohomological dimension  $\underline{\text{cd}} G$  and the Bredon homological dimension  $\underline{\text{hd}} G$  differ by at most one. By using a spectral sequence of Martínez-Pérez [7], Flores and Nucinkis [2] proved the analogue to Stambach's result, namely that for soluble groups,  $\underline{\text{hd}} G = \text{h } G$ . This led us to pose the following conjecture:

CONJECTURE 51.3. *Let  $G$  be a soluble group. Then the following are equivalent:*

- (1)  $G$  is of type  $\underline{\text{FP}}_\infty$ ,
- (2)  $\underline{\text{cd}} G = \text{h } G < \infty$ ,
- (3)  $G$  is of type  $\text{FP}_\infty$ .

$\underline{\text{FP}}_\infty$  denotes the Bredon analogue to  $\text{FP}_\infty$ . It is not hard to see that (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3), see [2]. There are, however, examples by Leary and Nucinkis [5] showing that generally groups of type  $\text{VF}$  do not necessarily admit a cocompact model for  $\underline{\text{EG}}$ , but all available evidence leads us to believe that Conjecture 51.1 still holds for soluble groups. Lück [6] showed that a group admits a model of finite type for  $\underline{\text{EG}}$  if and only if it is finitely presented of type  $\text{FP}_\infty$ , has finitely many conjugacy classes of finite subgroups and all centralisers of finite subgroups are of type  $\text{FP}_\infty$ . But even with this reduction, an answer to both conjectures remains frustratingly elusive.

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