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THE DEHN FUNCTION OF $\mathrm{SL}_n(\mathbf{Z})$

by Tim RILEY

For a word w on $a_1{}^{\pm 1}, \dots, a_m{}^{\pm 1}$ representing 1 in a finite presentation $\mathcal{P} = \langle a_1, \dots, a_m \mid \mathcal{R} \rangle$ of a group Γ, define $\mathrm{Area}(w)$ to be the minimal $A \in \mathbf{N}$ such that there is an equality $w = \prod_{i=1}^{A} u_i{}^{-1} r_i{}^{\varepsilon_i} u_i$ in the free group $F(a_1, \dots, a_m)$ for some $\varepsilon_i = \pm 1$, some words u_i, and some $r_i \in \mathcal{R}$. Equivalently, $\mathrm{Area}(w)$ is the minimal A such that there is a van Kampen diagram for w over $\mathcal{P}$ with at most A 2-cells. Defining $\mathrm{Area}(n)$ to be the maximum of $\mathrm{Area}(w)$ over all w that have length at most n and represent 1 in Γ, gives the *Dehn function* $\mathrm{Area}\colon \mathbf{N} \to \mathbf{N}$ of $\mathcal{P}$. Whilst $\mathrm{Area}\colon \mathbf{N} \to \mathbf{N}$ is defined for $\mathcal{P}$, a different finite presentation $\mathcal{P}'$ for Γ will yield a Dehn function $\mathrm{Area}'\colon \mathbf{N} \to \mathbf{N}$ that is qualitatively the same — for example, $(\exists C > 1,\ \forall n,\ (1/C)n^2 \leq \mathrm{Area}'(n) \leq Cn^2)$ if and only if the same is true for $\mathrm{Area}\colon \mathbf{N} \to \mathbf{N}$. (The C may differ.)

QUESTION 55.1. *Is the Dehn function of* $\mathrm{SL}_n(\mathbf{Z})$ *quadratic when* $n \geq 4$ *?*

Presenting this as a question, rather than a claim, conjecture, or the like, may be unduly conservative. In his 1993 survey article [5], Gersten describes the quadratic Dehn function as an *assertion* of W.P. Thurston.

I am not even aware of a proof that the Dehn function of $\mathrm{SL}_n(\mathbf{Z})$ is bounded above by a polynomial when $n \geq 4$. By contrast, the Dehn function of $\mathrm{SL}_2(\mathbf{Z})$ is known to grow linearly — $\mathrm{SL}_2(\mathbf{Z})$ is hyperbolic — and that of $\mathrm{SL}_3(\mathbf{Z})$ grows like $n \mapsto \exp(n)$: Epstein–Thurston [4] proved the lower bound and a result sketched by Gromov [6] §$5A_7$ gives the upper bound. (An elementary proof might be a step towards 55.1.)

Of course, 55.1 presupposes $\mathrm{SL}_n(\mathbf{Z})$ is finite presentable, but that has been long known. The $n^2 - n$ matrices e_{ij} with 1's on the diagonal, the off-diagonal ij-entry 1, and all others 0, generate $\mathrm{SL}_n(\mathbf{Z})$. Milnor [11], following J.R. Silvester and in turn Nielsen and Magnus, explains that the Steinberg relations $\{[e_{ij}, e_{kl}] = 1\}_{i \neq l, j \neq k}$ and $\{[e_{jk}, e_{kl}] = e_{jl}\}_{j \neq l}$ together with $\{(e_{ij} e_{ji}{}^{-1} e_{ij})^4 = 1\}_{i \neq j}$ are defining relations. A proof of 55.1 would be an exacting quantitative proof of finite presentability.

One can regard 55.1 as a higher dimensional version of the Lubotzky–Mozes–Raghunathan Theorem ([9], [10]) establishing the existence of efficient words representing elements g of $\mathrm{SL}_n(\mathbf{Z})$ for $n \geq 3$, that is, words of length like the log of the maximum of the absolute values of the matrix entries. As a word representing g amounts to a path in the Cayley graph from 1 to g, the L.–M.–R. Theorem can be thought of as saying that 0-spheres admit efficient fillings by 1-discs. A word w representing 1 in a finite presentation $\mathcal{P}$ corresponds to a loop ρ_w in the Cayley graph; a van Kampen diagram for w can be regarded as a combinatorial homotopy disc for ρ_w in the Cayley 2-complex of $\mathcal{P}$. So 55.1 is, roughly speaking, the claim that 1-spheres admit efficient fillings by 2-discs in $\mathrm{SL}_n(\mathbf{Z})$ for $n \geq 4$.

Gromov [6], §5D(5)(c), cf. §2B$_1$, takes this further and suggests that in $\mathrm{SL}_n(\mathbf{Z})$, Euclidean isoperimetric inequalities concerning filling k-spheres by $(k+1)$-discs persist up to $k = n-3$. (For $k = n-2$, the exponential lower bound of [4] applies.)

One attack on 55.1 is that whilst $\mathrm{SL}_n(\mathbf{Z})$ is not a *cocompact* lattice in the symmetric space $X := \mathrm{SL}_n(\mathbf{R})/\,\mathrm{SO}(n)$, and so the quadratic isoperimetric inequality enjoyed by X does not immediately pass to $\mathrm{SL}_n(\mathbf{Z})$, open horoballs can be removed from X to give a space X_0 on which $\mathrm{SL}_n(\mathbf{Z})$ acts cocompactly. Druţu [2] and Leuzinger–Pittet [8] have made progress in this direction, including a quadratic isoperimetric inequality for the boundary horosphere of each removed horoball.

Chatterji has asked whether for $n \geq 4$, $\mathrm{SL}_n(\mathbf{Z})$ enjoys her property L_δ for some $\delta \geq 0$, which would imply a sub-cubic Dehn function [3].

The author's efforts towards 55.1 have, to date, yielded [12] a version of L.–M.–R. giving explicit efficient words. This may aid the construction of van Kampen diagrams, but that remains to be seen. However it has led to progress elsewhere [7].

Finally, we mention that for $n > 3$, the Dehn functions of the cousins $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ of $\mathrm{SL}_n(\mathbf{Z})$ are also unknown [1].

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