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POWER SERIES THAT GENERATE CLASS NUMBERS

by Warren SINNOTT

Let k be a CM field, i.e., k is a totally imaginary quadratic extension of a totally real number field k^+ . Let p be a prime, and let K be the basic \mathbf{Z}_p -extension of k : then $K \subset k(\mu_{p^\infty})$ (here μ_{p^∞} is the group of p -power roots of unity), and k has a unique extension k_n in K of degree p^n over k . Let h_n^* denote the relative class number of k_n/k_n^+ . Then Iwasawa [1] showed that there are integers $\mu \geq 0, \lambda \geq 0$ and ν such that

$$(1) \quad \text{ord}_p(h_n^*) = \mu p^n + \lambda n + \nu$$

for n greater than or equal to some integer n_0 . One way to show this (not Iwasawa's original method, which gives more general results) is to use Hecke's analytic class number formula and the theory of p -adic L -functions (see for example Sinnott [3]): these results imply that there is a power series $F(T) \in \mathbf{Z}_p[[T-1]]$ such that

$$(2) \quad h_n^* = h_{n_0}^* \prod_{\substack{\zeta^{p^n}=1 \\ \zeta^{p^{n_0}} \neq 1}} F(\zeta) \text{ for } n \geq n_0 .$$

The Weierstrass Preparation Theorem implies that we may write $F(T) = p^\mu Q(T)u(T)$, where $\mu \geq 0$, $Q(T)$ is a monic polynomial of degree λ congruent to $(T-1)^\lambda \pmod{p}$, and $u(T)$ is a unit in $\mathbf{Z}_p[[T-1]]$. From this one can see that (2) \implies (1).

But (2) contains much more information than (1), since it gives a formula for the whole relative class number. My questions (basically just questions about formal power series) are:

QUESTION 59.1. *What does (2) tell us about class numbers? I.e., what constraints are imposed on the sequence $\{h_n^*\}$ by the formula (2)?*

For example, (2) has the following curious consequence: let $(h_n^*)'$ denote the “prime-to- p ” part of h_n^* . Then (2) implies that

$$(3) \quad \lim_{n \rightarrow \infty} (h_n^*)' \text{ exists in } \mathbf{Z}_p^\times.$$

H. Kisilevsky [2] pointed out that one can show that the limit (3) exists for the prime-to- p part (in fact for the ℓ -primary part for any $\ell \neq p$) of the class numbers of *any* \mathbf{Z}_p -extension.

Conversely, we can ask:

QUESTION 59.2. *What does (2) tell us about $F(T)$?*

For example, if $a \in \mathbf{Z}_p^\times$ then $F(T^a)$ gives the same sequence h_n^* , so $F(T)$ is not completely determined by (2). How much information about $F(T)$ is contained in (2)? The Newton polygon of $F(T^a)$ (as a power series in $\mathbf{Z}_p[[T-1]]$) is the same as the Newton polygon of $F(T)$: does (2) determine the Newton polygon of $F(T)$?

Finally, it would be interesting to know whether a power series as in (2) exists for other \mathbf{Z}_p -extensions:

QUESTION 59.3. *Suppose that K/k is a \mathbf{Z}_p -extension, h_n the class number of k_n : is there a power series $F(T) \in \mathbf{Z}_p[[T-1]]$ such that (2) holds (with h_n in place of h_n^*)?*

These questions are interesting since the “Main Conjecture” of Iwasawa theory (proved in the 1980s by Wiles [4]) relates $F(T)$ — up to a unit in $\mathbf{Z}_p[[T-1]]$ — to a characteristic polynomial defined from the action of $\text{Gal}(K/k) (\simeq \mathbf{Z}_p)$ on the p -primary part of the ideal class group of K .

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