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THE CHROMATIC RED-SHIFT IN ALGEBRAIC K -THEORY

by Christian AUSONI and John ROGNES

The algebraic K -theory of the sphere spectrum \mathbf{S} is of interest in geometric topology, by Waldhausen's stable parametrized h -cobordism theorem [7] (ca. 1979). We wish to understand $K(\mathbf{S})$ like we understand $K(\mathbf{Z})$, via Galois descent. As a building block, the algebraic K -theory of the Bousfield localization $L_{K(n)}\mathbf{S}$ of \mathbf{S} with respect to the n -th Morava K -theory $K(n)$ might be more accessible. The second author has developed a theory of Galois extensions for \mathbf{S} -algebras, and in this framework he has stated extensions of the Lichtenbaum–Quillen conjectures. Their precise formulation is distilled from the clues provided by our computations of the algebraic K -theory of topological K -theory and related spectra, and it is to be expected that they will keep maturing in a cask of skepticism for a few years.

Writing X^{hG} for the homotopy fixed-point spectrum of a finite group G acting on a spectrum X , we recall:

DEFINITION 4.1 ([6]). A map $A \rightarrow B$ of commutative \mathbf{S} -algebras is a $K(n)$ -local G -Galois extension if G acts on B through commutative A -algebra maps, and the canonical maps $A \rightarrow B^{hG}$ and $B \wedge_A B \rightarrow \prod_G B$ are $K(n)$ -equivalences.

Let E_n be Morava's E -theory [5] with coefficients given by $(E_n)_* = W(\mathbf{F}_{p^n})[[u_1, \dots, u_{n-1}]][[u^{\pm 1}]]$. Then $L_{K(n)}\mathbf{S} \rightarrow E_n$ is an example of a $K(n)$ -local pro-Galois extension.

Let V be a finite CW-spectrum of chromatic type $n+1$, and let $T = v_{n+1}^{-1}V$ be the mapping telescope of its essentially unique v_{n+1} -self-map. For $n=0$ take $V = V(0) = \mathbf{S}/p$ (the Moore spectrum), and for $n=1$, $p \geq 3$ take $V = V(1) = V(0)/v_1$.

CONJECTURE 4.2. *Let $A \rightarrow B$ be a $K(n)$ -local G -Galois extension. Then there is a homotopy equivalence*

$$T \wedge K(A) \rightarrow T \wedge (K(B))^{hG}.$$

For $n = 0$, $A \rightarrow B$ is a G -Galois extension of commutative \mathbf{Q} -algebras, and Conjecture 4.2 is the descent conjecture of Lichtenbaum–Quillen (1973). For $n = 1$, Conjecture 4.2 holds by [1], [2], [4] for the $K(1)$ -local \mathbf{F}_p^\times -Galois extension $L_p \rightarrow KU_p$, where KU_p is the p -complete periodic K -theory spectrum and L_p its Adams summand.

CONJECTURE 4.3. *Let B be a suitably finite $K(n)$ -local commutative \mathbf{S} -algebra (for example $L_{K(n)}\mathbf{S} \rightarrow B$ could be a G -Galois extension). Then the map $V \wedge K(B) \rightarrow T \wedge K(B)$ induces an isomorphism on homotopy groups in sufficiently high degrees.*

If $n = 0$ and $B = HF$ for a reasonable field F , then

$$V \wedge K(F) = K(F; \mathbf{Z}/p) \rightarrow T \wedge K(F) \simeq K^{\text{ét}}(F; \mathbf{Z}/p)$$

induces an isomorphism on homotopy groups in sufficiently high degrees by Thomason’s theorem (1985). For $n = 1$, $p \geq 5$ and $B = L_p$, KU_p or their connective versions ℓ_p and ku_p , it is known ([2], [4]) that $V(1)_*K(B)$ is a finitely generated free $\mathbf{F}_p[v_2]$ -module in high degrees, hence Conjecture 4.3 holds for these \mathbf{S} -algebras. This is evidence for the “red-shift conjecture”, which, in a less precise formulation than Conjecture 4.3, asserts that algebraic K -theory increases chromatic complexity by one.

The algebraic K -theory of a ring of integers \mathcal{O}_F (in a number field F) can be computed from the K -theory of its residue fields and fraction field, by a localization sequence. To compute $K(F; \mathbf{Z}/p)$, one uses Suslin’s theorem (1983) that $K(\bar{F}; \mathbf{Z}/p) \simeq V(0) \wedge ku$, and descent with respect to the absolute Galois group G_F .

To generalize this program we wish to make sense of the $K(n)$ -local \mathbf{S} -algebraic fraction field \mathcal{F} of $L_{K(n)}\mathbf{S}$ (or one of its pro-Galois extensions), construct a separably closed extension Ω_n , and evaluate its algebraic K -theory.

CONJECTURE 4.4. *If Ω_n is a separable closure of the fraction field of $L_{K(n)}\mathbf{S}$, then there is a homotopy equivalence*

$$L_{K(n+1)}K(\Omega_n) \simeq E_{n+1}.$$

For $n = 0$ this reduces to $L_{K(1)}K(\overline{\mathbf{Q}}) \simeq E_1 \simeq KU_p$, a weaker formulation of Suslin's theorem. For $n = 1$, we did some computations [3] aimed at understanding what the fraction field \mathcal{F} of KU_p might be. We define $K(\mathcal{F})$ to sit in a hypothetical localization sequence $K(KU/p) \rightarrow K(KU_p) \rightarrow K(\mathcal{F})$, as the cofiber of the transfer map for $KU_p \rightarrow KU/p$. The result is that $V(1)_*K(\mathcal{F})$ is, in high enough degrees, a free $\mathbf{F}_p[v_2]$ -module on $2(p^2+3)(p-1)$ generators. In particular \mathcal{F} cannot be the $H\mathbf{Q}_p$ -algebra $KU_p[1/p]$. We rather believe that \mathcal{F} is an \mathbf{S} -algebraic analogue of a two-dimensional local field. For example, there appears to be a perfect arithmetic duality pairing in the Galois cohomology of \mathcal{F} , analogous to Tate–Poitou duality (1963) for local number fields.

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