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6

AUTOMATIC PERMUTATIONS

by Laurent BARTHOLDI

Finite state automata are wonderful little devices to produce interesting groups — for examples, see [1] or [2]. We fix a finite set X, called the *alphabet*. An *automaton* $\mathcal{A}$ is given by a finite set Q, a map $\tau\colon Q\times X \to X\times Q$, and an element $q \in Q$.

It defines an *automatic transformation*, still written $\mathcal{A}\colon X^* \to X^*$ on strings over X, by the following rule: given $w = x_1 \ldots x_n$, set $q_1 = q$, and compute $\tau(q_1, x_1) = (y_1, q_2)$, $\tau(q_2, x_2) = (y_2, q_3)$, etc. up to $\tau(q_n, x_n) = (y_n, q_{n+1})$. Then the image of w under $\mathcal{A}$ is $y_1 \ldots y_n$.

Part of the interest in these transformations is that they can be used to generate groups (and semigroups) with striking features, such as infinite finitely generated torsion groups, groups of intermediate word growth, of non-uniformly exponential word growth, etc.

It is easy to test whether an automatic transformation is the identity — assuming Q is minimal, this happens if $\tau(*, x) = (x, *)$ for all possible $*$'s. It is also easy to test whether it is invertible — this happens if the composition $x \to (q, x) \to \tau(q, x) = (y, *) \to y$ is a permutation of X for all $q \in Q$. It is finally easy to see that the product and inverse of automatic transformations are again automatic.

Therefore, in a group G generated by automatic invertible transformations, all elements of G are themselves represented by automatic transformations. It follows that the most basic decision problem — determining whether a word over a given generating set is trivial — is solvable for such groups.

What seems difficult to test, and is the object of this conjecture, is whether an automatic permutation $\mathcal{A}$ has finite order; i.e. whether the following decision problem is solvable: "determine whether the cyclic subgroup generated by a given group element is finite or infinite".

If the order of $\mathcal{A}$ is finite, then this can be checked by computing powers of $\mathcal{A}$ and seeing that one of them induces the identity transformation. But how can one determine that $\mathcal{A}$ has infinite order ? This would follow from

CONJECTURE 6.1. *If $\mathcal{A}$ has finite order, then its order divides* $(\#X!)^{\#Q}$.

A sharper bound should actually be $\exp(\mathrm{Sym}(X))^{\#Q}$. If $X = \mathbf{Z}/d\mathbf{Z}$ and $\sigma \in \mathrm{Sym}(X)$ has order e and $Q = \{0, \ldots, q\}$, the automaton defined by initial state q and

$$\tau(i,x) = \begin{cases} (x, 0) & \text{if } i = 0\,, \\ (x+1, i-1) & \text{if } i > 0 \text{ and } x = 0\,, \\ (x+1, 0) & \text{if } i > 0 \text{ and } x \neq 0 \end{cases}$$

has order e^q.

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