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systeme geteilt. Vermesser und Geographen erfahren, welche tiefgreifenden Umwälzungen sich in ihren Wissenschaften durch den Einsatz der Informatik ergeben.

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Azimuth transport and the problem of orientation within geodetic traverses and geodetic networks

E. W. Grafarend

Mina herrar,
landmätaren måste stämplas som plattlus,
han kan inte se i rymden därför at han inte
läser beskrivande geometri.

Gentlemen, the surveyor must be considered a "plattlus", because he cannot see into the space having not taken my course in threedimensional geometry.

(R. Woxen, f. d. rektor,
Kungl. Tekn. Högskolan
Stockholm, Sverige)

Due to the variation of the local vertical, in which any local positioning system (LPS) operates, standard formulae for computing traverses and networks, separating horizontal and vertical control, generate model errors of the order of 10–30 cm in local regions of less than 1000 m. Here it is demonstrated that line-type traverses are impossible for computation as long as only distance and horizontal/vertical direction have been measured. In contrast, triangular chain-type traverses allow the computation of the variation of the local vertical, e. g. expressed in terms of astronomical longitude/latidude, the variation of the azimuth/vertical direction and therefore the threedimensional rectangular coordinates in the horizontal reference system attached to the initial point. In cases where prior information about model longitude/latitude and the variation of vertical deflections is available, a specific computational scheme for threedimensional rectangular coordinates is derived. Finally rigorous threedimensional observational equations for horizontal/vertical directions and distances within local geodetic networks are set-up. They are expressed in terms of a horizontal reference system attached to the chosen initial point and include three orientation unknowns like in photogrammetric networks, namely the conventional orientation unknown in the local horizontal plane and the differences of astronomical longitude/latitude between the LPS placement and the network initial point. Throughout the order of magnitude of any effect which can be attributed to the variation of the local vertical is estimated.

1. Introduction

For the determination of local point positions the local positioning system (LPS) consisting of a direction measurement system, the theodeolite, and a distance measurement system based on electromagnetic wave propagation (EDM) has been proven to be most useful. Powerful software has been developed in order to allow a nearly online positioning. All computer packages to be known to me from technical reports and engineering journal publications neglect the influence of the variation of the local vertical for local applications. Typical is the separation of local horizontal and local vertical control. Horizontal direction measurements are modelled by the arctan-function of the ratio of horizontal y- and x-coordinate difference, in contrast to the vertical direction measurements which are related exclusively by the vertical z-coordinate difference between target point and LPS placement. As the distance the Euclidean twodimensional metric, the twodimensional Pythagoras formula is used.

We have shown in another publication (1987) that in geodetic networks of about 500 m extension the systematic model errors which are caused by neglecting the change of the local vertical due to the earth's variable gravity field will amount to 10–30 cm! Thus the set-up of LPS-observational equations has to take into account the variation of the local vertical to which any LPS refers by the "horizontation procedure", the task of the present contribution. Once the "right" observational equations have been found and being classified as nonlinear they can be linearized with respect to prior positioning information in "geometry and gravity space" and be handed over to the standard linear adjustment procedures.

Chapter one is therefore devoted to set-up the computational equations for geodetic point positioning in a traverse. With refe-

Aufgrund der Variation der lokalen Vertikalen, auf die sichdie Horizontierung eines lokalen Positionierungssystems (LPS) bezieht, weisen die Standardformeln, mit denen Polygonzüge und Netze im lokalen Bereich berechnet werden, Modellfehler in der Grössenordnung 10–30 cm auf, falls das Gebiet, in dem sich das Netz erstreckt, kleiner als 1000 m ist. Wir zeigen insbesondere, dass linienförmige Polygonzüge überhaupt nicht ausgewertet werden können, falls ausschliesslich die Messungen von horizontalen und vertikalen Richtungen sowie von Distanzen vorliegen. Im Gegensatz dazu beweisen wir, dass dreieckige, kettenartige Polygonzüge perfekt berechenbar sind, da sie gestatten, die Variation der lokalen Vertikalen, z.B. parametrisiert durch astronomische Länge/Breite, ausserdem die Variation des Azimutes / der vertikalen Messrichtung und damit der dreidimensionalen rechtwinkligen Koordinaten in einem horizontalen Bezugssystem zu bestimmen, welches dem Anfangspunkt des Netzes angeheftet wird. In solchen Fällen, in denen Vorinformation über die sphärischen Längen/Breiten der Messpunkte sowie über die Variation der Lotabweichungen vorliegt, werden wir eine Rechenprozedur vorstellen. Schliesslich stellen wir die strengen nichtlinearen Beobachtungsgleichungen für horizontale / vertikale Richtungen und Distanzen innerhalb lokaler geodätischer Netze auf. Sie beziehen sich auf ein einheitliches horizontales Bezugssystem, welches am ausgezeichneten Anfangspunkt des Netzes angeheftet ist, und schliessen drei Orientierungsunbekannte wie in photogrammetrischen Netzen, nämlich die konventionelle Orientierungsunbekannte in der lokalen Horizontalebene und die Differenz der astronomischen Längen/Breiten zwischen dem LPS-Standpunkt und dem Anfangspunkt des Netzes, ein. Durchgehend geben wir die Grössenordnung derjenigen Effekte an, die der Variation der lokalen Vertikalen zuzuordnen sind.

A cause de la variation des verticales locales, verticales qui servent de référence au calage d'un système de positionnement local, les formules standards de calcul des polygonales et des réseaux peu étendus peuvent conduire à des erreurs de 10 à 30 cm, même lorsque la région sur laquelle s'étendent les mesures est plus petite que 1000 mètres. Nous montrons en particulier que des polygonales simples ne peuvent pas être traitées au cas où les mesures ne comprennent que des directions horizontales et verticales, ainsi que des distances. Nous prouvons, au contraire, que des chaînes de triangles sont parfaitement calculables car elles permettent de déterminer la variation des verticales locales (verticales données par exemple par la longitude et la latitude astronomique), ainsi que celle de l'azimut, la direction de l'axe principal de l'appareil de mesure, et par là les coordonnées orthogonales tridimensionnelles dans un système horizontal de référence calé à l'origine du réseau. Dans de tels cas, lorsque des informations sur les longitudes et les latitudes des points de mesure, ainsi que sur la variation des déviations de la verticale sont connues a priori, nous présentons une procédure decalcul. Nous établissons enfin des équations d'observation exactes et non linéaires pour les directions horizontales ou verticales et les distances à l'intérieur d'un réseau local. Elles se rapportent à un système de référence horizontal dépendant du point origine du réseau et à trois inconnues d'orientation comme en photogrammétrie, c'est-à-dire l'inconnue d'orientation conventionnelle dans le plan horizontal local et les différences de longitude et de latitude astronomiques entre le lieu de station du système de positionnement local et l'origine du réseau. Nous donnons finalement l'ordre de grandeur des influences qui dépendent de la variation des verticales locales.

rence to the Appendix it is proven that line-type traverses are impossible for computation if no prior information but the actual LPS observations is available. In contrast we prove that triangular chain-type traverses allow beside the computation of positions the determination of the local vertical variation, namely the variation in azimuth and vertical or the variation of astronomical longitude and astronomical latitude. On the other side if prior information of the variation of the local vertical is available, say in terms of variation of model longitude and latitude and the vertical deflection components, any type of traverse is ready for coordinate computation. (The variation of the vertical deflection vector is proportional to the second derivative of the gravity disturbing potential. Thus a triangular LPS can be considered a gravity gradiometer.) In chapter two we finally set-up the "right" nonlinear observational equations for direction measurements of horizontal and vertical type and for threedimensional distance measurements, namely in a horizontal reference system attached to the network initial point. We emphazise that in local geodetic networks there appear – like in photogrammetric networks – three orientation unknowns which describe the variation of the LPS horiziontal reference triad from point to point. These orientation unknowns are the conventional theodolite orientation unknown in the horizontal plane being referred to South and the differences in astronomical longitude and astronomical latitude between the initial point and the LPS placement.

Throughout we have tried to color the various equations by numerical examples. To some extend the contribution is a follow-up of my others (1975, 1981, 1985 and 1987). Here we have concentrated on the azimuth variation and the problem of orientation of a geodetic traverse and a geodetic network. R. Conzett has dealt with the threedimensional orientation problem in the report by R. Conzett and E. Frei (1985); he focussed on the LPS in the publication by R. Conzett and R. Scherrer (1985) and on the condition equations in geodetic networks in the publication R. Conzett (1985).

2. Geodetic traverses

For local positioning traverses play a central role, namely in inertial positioning. In surveying they are used in a line-type structure as outlined in Figure 1. Conventionally the Cartesian coordinates of points in a traverse are computed according to

$$x_\gamma = x_\alpha + r_{\alpha\beta} \cos\alpha_{\alpha\beta} \cos\beta_{\alpha\beta} + r_{\beta\gamma} \cos(\alpha_{\alpha\beta}+\alpha_{\alpha\beta\gamma})\cos(\beta_{\alpha\beta}+\beta_{\alpha\beta\gamma}) \quad (1.1)$$

$$y_\gamma = y_\alpha + r_{\alpha\beta} \sin\alpha_{\alpha\beta} \cos\beta_{\alpha\beta} + r_{\beta\gamma} \sin(\alpha_{\alpha\beta}+\alpha_{\alpha\beta\gamma})\cos(\beta_{\alpha\beta}+\beta_{\alpha\beta\gamma}) \quad (1.2)$$

$$z_\gamma = z_\alpha + r_{\alpha\beta} \sin\beta_{\alpha\beta} + r_{\beta\gamma} \sin(\beta_{\alpha\beta}+\beta_{\alpha\beta\gamma}) \quad (1.3)$$

where $(x_\alpha, y_\alpha, z_\alpha)$ are the coordinates of the initial point P_α, $\alpha_{\alpha\beta}$ the initial azimuth and $\beta_{\alpha\beta}$ the initial vertical direction of the line $P_\alpha P_\beta$ – here measured from the horizontal plane and being complementary to the zenith distance –, $r_{\alpha\beta}$ the initial threedimensional or Euclidean distance of the line $P_\alpha P_\alpha$. In contrast $(\alpha_{\alpha\beta\gamma}, \beta_{\alpha\beta\gamma})$ are the horizontal, vertical, respectively, angles measured at the point P_β between the lines $P_\beta P_\alpha$, $P_\beta P_\gamma$; $r_{\alpha\gamma}$ denotes the distance of the line $P_\beta P_\gamma$. $(x_\gamma, y_\gamma, z_\gamma)$ are the coordinates of the point P_γ, here arbitrarily being chosen. Conventionally $r_{\alpha\beta}\cos\beta_{\alpha\beta}$, $r_{\beta\gamma}\cos(\beta_{\alpha\beta}+\beta_{\alpha\beta\gamma})$ are called horizontal distances of the lines of sight $P_\alpha P_\beta$, $P_\beta P_\gamma$, respectively, while $r_{\alpha\beta}\sin\beta_{\alpha\beta}$, $r_{\beta\gamma}\sin(\beta_{\alpha\beta}+\beta_{\alpha\beta\gamma})$ are referred to as heights.

We have outlined in the Appendix that

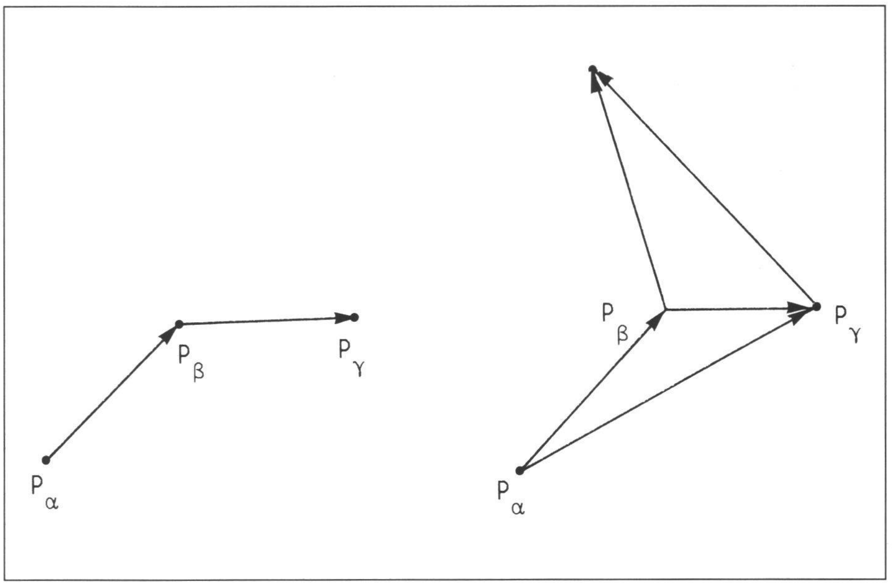

Figure 1: Line-type and triangular chain-type traverse (the line-type traverse does not allow a threedimensional computation in contrast to the triangular chain-type traverse)

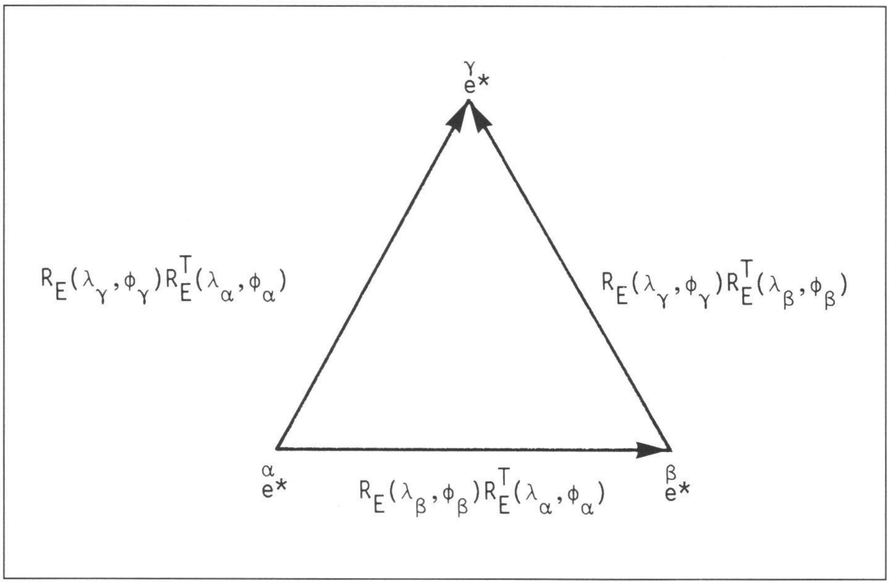

Figure 2: Commutative diagram of three orthonormal triads of horizontal type in a triangular traverse

once we take up the real local horizontal reference system as the true one being materialized in a theodolite or in a local positioning system (LPS) the formulae (1.1), (1.2) and (1.3) can no longer be applied. They are systematically being wrong due to the very nature of the local horizontal reference frame to change from point to point. Instead the above operational equations have to be replaced by (A20), (A21) and (A22): Beside the initial azimuth $\alpha_{\alpha\beta}$ there appear now also the variations ($\triangle\alpha$, $\triangle\beta$) of the local vertical!

2.1 Geodetic traverses without other information

In case just distances and horizontal/vertical directions are measured by a LPS in a line-type structure of a traverse we are unable to compute the coordinates

$$(\overset{\alpha}{x}_\gamma, \overset{\alpha}{y}_\gamma, \overset{\alpha}{z}_\gamma)$$

in the local reference frame $\overset{\alpha}{e}{}^*$

of horizontal type attached to the initial point, even if we assume the initial values for the azimuth $\alpha_{\alpha\beta}$ and the vertical direction $\beta_{\alpha\beta}$ to be known. The changes ($\triangle\alpha$, $\triangle\beta$) of the local vertical cannot be determined: With any new point in a line-type traverse we introduce two additional unknowns ($\triangle\alpha$, $\triangle\beta$)!

What can we do in order to overcome this situation?

The solution is simple: We have to change the geodetic traverse from line-type to triangular chain-type as we have illustrated in Figure 1. The result that now the variations of the local vertical ($\triangle\alpha$, $\triangle\beta$) are estimable will now been proven. We depart from the commutative diagram of Figure 2 where the three horizontal frames

$$\overset{\alpha}{e}{}^*,\ \overset{\beta}{e}{}^*,\ \overset{\gamma}{e}{}^*$$

at the points P_α, P_β and P_γ are connected. (All definitions are taken over from the Appendix.) From the commutative diagram we read

$$\overset{\alpha}{e}{}^* \rightarrow \overset{\beta}{e}{}^* = R_E(\lambda_\beta,\phi_\beta)R_E^T(\lambda_\alpha,\phi_\alpha)\overset{\alpha}{e}{}^* = \{I + A(\lambda_{\alpha\beta},\phi_{\alpha\beta})\}\overset{\alpha}{e}{}^* \quad (1.4)$$

$$\overset{\beta}{e}{}^* \rightarrow \overset{\gamma}{e}{}^* = R_E(\lambda_\gamma,\phi_\gamma)R_E^T(\lambda_\beta,\phi_\beta)\overset{\beta}{e}{}^* = \{I + A(\lambda_{\beta\gamma},\phi_{\beta\gamma})\}\ \overset{\beta}{e}{}^* \quad (1.5)$$

or

$$\overset{\gamma}{e}{}^{*} = R_E(\lambda_\gamma, \phi_\gamma) R_E^T(\lambda_\gamma, \phi_\alpha) \overset{\alpha}{e}{}^{*} = \{I + A(\lambda_{\alpha\gamma}, \phi_{\alpha\gamma})\} \overset{\alpha}{e}{}^{*} \tag{1.6}$$

where I is the unit matrix and A () an anti-symmetric matrix given by (A9) and once we have taylorized the Eulerian rotation matrices close to the identity. From the connection matrices we pick up the identities up to terms of second order o_2

$$\{I + A(\lambda_{\beta\gamma}, \phi_{\beta\gamma})\}\{I + A(\lambda_{\alpha\beta}, \phi_{\alpha\beta})\} = \{I + A(\lambda_{\alpha\gamma}, \phi_{\alpha\gamma})\} \tag{1.7}$$

$$I + A(\lambda_{\beta\gamma}, \phi_{\beta\gamma}) + A(\lambda_{\alpha\beta}, \phi_{\alpha\beta}) + o_2 = I + A(\lambda_{\alpha\gamma}, \phi_{\alpha\gamma}) \tag{1.8}$$

$$\lambda_{\alpha\beta} \sin\phi_\alpha + \lambda_{\beta\gamma} \sin\phi_\beta = \lambda_{\alpha\gamma} \sin\phi_\alpha \tag{1.9}$$

$$\phi_{\alpha\beta} + \phi_{\beta\gamma} = \phi_{\alpha\gamma} \tag{1.10}$$

or

$$\boxed{\lambda_{\alpha\beta} + \lambda_{\beta\gamma} \doteq \lambda_{\alpha\gamma} , \quad \phi_{\alpha\beta} + \phi_{\beta\gamma} = \phi_{\alpha\gamma}} \tag{1.11}$$

Thus the commutative diagram has led us to two constraints (or holonomity equations) which will help the estimability of $(\triangle\alpha, \triangle\beta)$. To this end we take advantage of the transformation (A10), (A11) $(\triangle\lambda, \triangle\Phi) \rightarrow (\triangle\alpha, \triangle\beta)$, namely its inverse

$$\lambda_{\alpha\beta} = a_{\alpha\beta} \, \Delta\alpha_{\alpha\beta} + b_{\alpha\beta} \, \Delta\beta_{\alpha\beta} \tag{1.12}$$

$$\phi_{\alpha\beta} = c_{\alpha\beta} \, \Delta\alpha_{\alpha\beta} + d_{\alpha\beta} \, \Delta\beta_{\alpha\beta} \tag{1.13}$$

or

$$\begin{bmatrix} \lambda_{\alpha\beta} \\ \phi_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \Delta\alpha_{\alpha\beta} \\ \Delta\beta_{\alpha\beta} \end{bmatrix} \tag{1.14}$$

$$= \frac{1}{\tan\beta_{\alpha\beta}\cos\phi_\alpha + \cos\alpha_{\alpha\beta}\sin\phi_\alpha} \begin{bmatrix} -\cos\alpha_{\alpha\beta} & +\sin\alpha_{\alpha\beta}\tan\beta_{\alpha\beta} \\ -\sin\alpha_{\alpha\beta}\cos\phi_\alpha & -\cos\alpha_{\alpha\beta}\tan\lambda_{\alpha\beta}\cos\phi_\alpha \\ & -\sin\phi_\alpha \end{bmatrix} \begin{bmatrix} \Delta\alpha_{\alpha\beta} \\ \Delta\beta_{\alpha\beta} \end{bmatrix}$$

or

$$a_{\alpha\beta} = -\frac{\cos\alpha_{\alpha\beta}}{\tan\beta_{\alpha\beta}\cos\phi_\alpha + \cos\alpha_{\alpha\beta}\sin\phi_\alpha}$$

$$b_{\alpha\beta} = \frac{\sin\alpha_{\alpha\beta}\tan\beta_{\alpha\beta}}{\tan\beta_{\alpha\beta}\cos\phi_\alpha + \cos\alpha_{\alpha\beta}\sin\phi_\alpha} \qquad (1.15)$$

$$c_{\alpha\beta} = -\frac{\sin\alpha_{\alpha\beta}\cos\phi_\alpha}{\tan\beta_{\alpha\beta}\cos\phi_\alpha + \cos\alpha_{\alpha\beta}\sin\phi_\alpha}$$

$$d\alpha_\beta = -\frac{\cos\alpha_{\alpha\beta}\tan\beta_{\alpha\beta}\cos\phi_\alpha + \sin\phi_\alpha}{\tan\beta_{\alpha\beta}\cos\phi_\alpha + \cos\alpha_{\alpha\beta}\sin\phi_\alpha}$$

Finally the two constraints (1.11) can now be transformed into variations of the azimuths and vertical directions:

$$\lambda_{\alpha\beta} + \lambda_{\beta\gamma} = \lambda_{\alpha\gamma} , \quad \phi_{\alpha\beta} + \phi_{\beta\gamma} = \phi_{\alpha\gamma}$$

$$a_{\alpha\beta}\Delta\alpha_{\alpha\beta} + b_{\alpha\beta}\Delta\beta_{\alpha\beta} + a_{\beta\gamma}\Delta\alpha_{\beta\gamma} + b_{\beta\gamma}\Delta\beta_{\beta\gamma} = a_{\alpha\gamma}\Delta\alpha_{\alpha\gamma} + b_{\alpha\gamma}\Delta\beta_{\alpha\gamma} \qquad (1.16)$$

$$c_{\alpha\beta}\Delta\alpha_{\alpha\beta} + d_{\alpha\beta}\Delta\beta_{\alpha\beta} + c_{\beta\gamma}\Delta\alpha_{\beta\gamma} + d_{\beta\gamma}\Delta\beta_{\beta\gamma} = c_{\alpha\gamma}\Delta\alpha_{\alpha\gamma} + d_{\alpha\gamma}\Delta\beta_{\alpha\gamma} \qquad (1.17)$$

The two constraints (1.16), (1.17) contain three unknown variations $\Delta\alpha_{\alpha\beta}$, $\Delta\alpha_{\beta\gamma}$, $\Delta\alpha_{\alpha\gamma}$ of the azimuths since the variations of the vertical direction are determined directly from measurements, namely $\Delta\beta_{\alpha\beta} = \beta_{\beta\alpha} - \beta_{\alpha\beta}$, $\Delta\beta_{\beta\gamma} = \beta_{\gamma\beta} - \beta_{\beta\gamma}$, $\Delta\beta_{\alpha\gamma} = \beta_{\gamma\alpha} - \beta_{\alpha\gamma}$. Therefore we are missing one additional constraint (or holonomity condition). Since we are working in a threedimensional Euclidean space, there are three holonomity conditions, for instance

$$\oint dx^\alpha = 0 , \quad \oint dy^\alpha = 0 , \quad \oint dz^\alpha = 0 . \qquad (1.18)$$

In order to find the missing third constraint we recall the definition of horizontal angles (A12),

$$\left.\begin{aligned}
\alpha_{\alpha\beta\gamma} &= -\alpha_{\gamma\beta\alpha} = \alpha_{\beta\gamma} - \alpha_{\beta\alpha} = \alpha_{\beta\gamma} - \alpha_{\alpha\beta} - \Delta\alpha_{\alpha\beta} \\
\alpha_{\beta\gamma\alpha} &= -\alpha_{\alpha\gamma\beta} = \alpha_{\gamma\alpha} - \alpha_{\gamma\beta} = \alpha_{\gamma\alpha} - \alpha_{\beta\gamma} - \Delta\alpha_{\beta\gamma} \\
\alpha_{\gamma\alpha\beta} &= -\alpha_{\beta\alpha\gamma} = \alpha_{\alpha\beta} - \alpha_{\alpha\gamma} = \alpha_{\alpha\beta} - \alpha_{\gamma\alpha} - \Delta\alpha_{\gamma\alpha}
\end{aligned}\right] , \qquad (1.19)$$

which leads us to

$$\alpha_{\alpha\beta\gamma} + \alpha_{\beta\gamma\alpha} + \alpha_{\gamma\alpha\beta} = -(\Delta\alpha_{\alpha\beta} + \Delta\alpha_{\beta\gamma} + \Delta\alpha_{\gamma\alpha}) . \qquad (1.20)$$

The sum of the horizontal angles in a triangle equals the sum of azimuth variations. We refer to the sum of horizontal angles as the "horizontal excess": Once the azimuth variations vanish, so does the sum of horizontal angles. (Due to our peculiar definition of horizontal angles they do not sum up to 180°). Note that $\Delta\alpha_{\gamma\alpha} = \Delta\alpha_{\alpha\gamma}$ holds. The three constraints (1.16), (1.17) and (1.20) are finally written in the matrix form

$$\begin{bmatrix} -\alpha_{\alpha\beta\gamma} - \alpha_{\beta\gamma\alpha} - \alpha_{\gamma\alpha\beta} \\ b_{\alpha\gamma}\Delta\beta_{\alpha\gamma} - b_{\alpha\beta}\Delta\beta_{\alpha\beta} - b_{\beta\gamma}\Delta\beta_{\beta\gamma} \\ d_{\alpha\gamma}\Delta\beta_{\alpha\gamma} - d_{\alpha\beta}\Delta\beta_{\alpha\beta} - d_{\beta\gamma}\Delta\beta_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ a_{\alpha\beta} & a_{\beta\gamma} & a_{\alpha\gamma} \\ c_{\alpha\beta} & c_{\beta\gamma} & c_{\alpha\gamma} \end{bmatrix} \begin{bmatrix} \Delta\alpha_{\alpha\beta} \\ \Delta\alpha_{\beta\gamma} \\ \Delta\alpha_{\gamma\alpha} \end{bmatrix} \qquad (1.21)$$

From the inversion of (1.21) we directly derive the azimuths variations.

$$\begin{bmatrix} \Delta\alpha_{\alpha\beta} \\ \Delta\alpha_{\beta\gamma} \\ \Delta\alpha_{\gamma\alpha} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ a_{\alpha\beta} & a_{\beta\gamma} & a_{\alpha\gamma} \\ c_{\alpha\beta} & c_{\beta\gamma} & c_{\alpha\gamma} \end{bmatrix}^{-1} \begin{bmatrix} -\alpha_{\alpha\beta\gamma} - \alpha_{\beta\gamma\alpha} - \alpha_{\gamma\alpha\beta} \\ b_{\alpha\gamma}\Delta\beta_{\alpha\gamma} - b_{\alpha\beta}\Delta\beta_{\alpha\beta} - b_{\beta\gamma}\Delta\beta_{\beta\gamma} \\ d_{\alpha\gamma}\Delta\beta_{\alpha\gamma} - d_{\alpha\beta}\Delta\beta_{\alpha\beta} - d_{\beta\gamma}\Delta\beta_{\beta\gamma} \end{bmatrix} \qquad (1.22)$$

In summarizing we can make the following statement:
Within a triangular chain-type traverse we are able to compute station coordinates according to (A20), (A21), (A22) where the azimuth variations are computed from (1.22) and the variations of the vertical direction are directly determined from foreward and backward vertical direction measurements. Once the variations of the azimuths and vertical directions have been determined they are used to compute the variations of the local vertical, namely the variations in astronomical longitude and astronomical latitude according to (1.14).
A numerical example illustrates the coordinate computation in a triangular chain-type traverse.

Example 1

input

$$\alpha_{\alpha\beta} = 45°$$

$$b_{\alpha\beta} = 6.7°, \quad \Delta\beta_{\alpha\beta} = -0.25'' \sim -1.21 * 10^{-6} \text{ RAD}$$

$$\beta_{\beta\gamma} = 12.0°, \quad \Delta\beta_{\beta\gamma} = -0.25'' \sim -1.21 * 10^{-6} \text{ RAD}$$

$$\beta_{\gamma\alpha} = -13.6°, \quad \Delta\beta_{\gamma\alpha} = +0.25'' \sim +1.21 * 10^{-6} \text{ RAD}$$

$$\phi_{\alpha} = 48.783°$$

horizontal angles

$$\alpha_{\gamma\beta\alpha} = 270° + 0.75'', \quad \alpha_{\alpha\gamma\beta} = 149.036{,}243° + 0.5''$$

$$\alpha_{\beta\alpha\gamma} = -59.036{,}243° - 0.5''$$

inner triangular angles

$\pi - \alpha_{\gamma\beta\alpha} = -90° - 0.25''$, $\pi - \alpha_{\alpha\gamma\beta} = 30{,}963{,}757 - 0.5''$

$\pi - \alpha_{\beta\alpha\gamma} = 239{,}036{,}243 + 0.5''$

sum of inner triangular angles
180°–0.25″

horizontal excess

$$\alpha_{\alpha\beta\gamma} + \alpha_{\beta\gamma\alpha} + \alpha_{\gamma\alpha\beta} = -0.25'' \sim -1.21 * 10^{-6} \text{ RAD}$$

output

$a_{\alpha\beta} = -1.160{,}515{,}8$

$b_{\alpha\beta} = +0.136{,}329{,}2$

$c_{\alpha\beta} = -0.764{,}678{,}5$

$d_{\alpha\beta} = -1.324{,}384{,}6$

$a_{\beta\gamma} \doteq -1.804{,}565$

$b_{\beta\gamma} \doteq -0.383{,}572$

$c_{\beta\gamma} \doteq +1.666{,}954$

(here the following approximations have been used:

$$\cos\alpha_{\beta\gamma} \doteq \cos(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma}),\ \tan\beta_{\beta\gamma}\ \cos\phi_{\beta} \doteq \tan\beta_{\beta\gamma}\ \cos\phi_{\alpha})$$

$a_{\alpha\gamma} = -1.700{,}953$

$b_{\alpha\gamma} = +0.102{,}876$

$c_{\alpha\gamma} = +0.280{,}195$

$d_{\alpha\gamma} = -1.047{,}723$

(here, note $\cos\alpha_{\alpha\gamma} = \cos(\alpha_{\alpha\beta} + \alpha_{\beta\alpha\gamma})$ and the approximation $\tan\beta_{\alpha\gamma}\ \cos\phi_{\alpha} \doteq \tan\beta_{\gamma\alpha}\ \cos\phi_{\alpha}$)

:(1.22):

$$\begin{bmatrix} \Delta\alpha_{\alpha\beta} \\ \Delta\alpha_{\beta\gamma} \\ \Delta\alpha_{\gamma\alpha} \end{bmatrix} = 2.611 * \begin{bmatrix} 1.517 & 0.909 & 0.104 \\ 1.626 & 1.045 & 0.540 \\ -2.760 & -1.954 & -0.644 \end{bmatrix} \begin{bmatrix} 1.21 * 10^{-6} \text{ RAD} \\ -0.42 * 10^{-6} \text{ RAD} \\ 1.68 * 10^{-6} \text{ RAD} \end{bmatrix}$$

We have explicitly written the factor

$$\begin{vmatrix} 1 & 1 & 1 \\ a_{\alpha\beta} & a_{\beta\gamma} & a_{\alpha\gamma} \\ c_{\alpha\beta} & c_{\beta\gamma} & c_{\alpha\gamma} \end{vmatrix} = \frac{1}{2.611} = 0.383$$

which is the inverse of the determinant we have to invert within (1.22).

$$\Delta\alpha_{\alpha\beta} = 4.25 * 10^{-6} \text{ RAD} \sim 0.9''$$

$$\Delta\alpha_{\beta\gamma} = 6.37 * 10^{-6} \text{ RAD} \sim 1.3''$$

$$\Delta\alpha_{\gamma\alpha} = -9.41 * 10^{-6} \text{ RAD} \sim -1.9''$$

$$\Delta\alpha_{\alpha\beta} + \Delta\alpha_{\beta\gamma} + \Delta\alpha_{\gamma\alpha} = \alpha_{\gamma\beta\alpha} + \alpha_{\alpha\gamma\beta} + \alpha_{\beta\alpha\gamma} =$$
$$= -(\alpha_{\alpha\beta\gamma} + \alpha_{\beta\gamma\alpha} + \alpha_{\gamma\alpha\beta}) = 0.25'' \; .$$

At the end we depart, in addition to the horizontal and vertical direction, from measured distances

$$r_{\alpha\beta} = 42.720 \text{ m} \; , \quad r_{\beta\gamma} = 72.284 \text{ m} \; , \quad r_{\gamma\alpha} = 84.853 \text{ m}$$

in order to compute the coordinate differences

$$\begin{aligned} {}^{\alpha}x_{\alpha\beta} &= +\,30 \text{ m} \; , & {}^{\alpha}x_{\beta\gamma} &= +\,50 \text{ m} \; , & {}^{\alpha}x_{\gamma\alpha} &= -\,80 \text{ m} \\ {}^{\alpha}y_{\alpha\beta} &= +\,30 \text{ m} \; , & {}^{\alpha}y_{\beta\gamma} &= -\,50 \text{ m} \; , & {}^{\alpha}y_{\gamma\alpha} &= +\,20 \text{ m} \\ {}^{\alpha}z_{\alpha\beta} &= +\,5 \text{ m} \; , & {}^{\alpha}z_{\beta\gamma} &= +\,15 \text{ m} \; , & {}^{\alpha}z_{\gamma\alpha} &= -\,20 \text{ m} \; . \end{aligned}$$

2.2 Geodetic traverse with prior information

In the previous paragraph we have demonstrated that line-type traverses cannot be computed from LPS-type measurements since we have no information about the variation of the local vertical available. Another situation is met once we get external information about the variation of azimuth and vertical direction, e.g. from a computation based on vertical deflection ($\delta\lambda, \delta\Phi$) according the threedimensional Laplace condition as we shall outline now. In order to simplify the notation we write only in this paragraph real quantities by capital letters, their model approximations by small letters, e.g.

$$\left[\begin{aligned} & A_{\alpha\beta} = \alpha_{\alpha\beta} + \delta\alpha_{\alpha\beta} \\ & B_{\alpha\beta} = \beta_{\alpha\beta} + \delta\beta_{\alpha\beta} \\ & \Lambda_{\alpha} = \lambda_{\alpha} + \delta\lambda_{\alpha} \\ & \Phi_{\alpha} = \phi_{\alpha} + \delta\phi_{\alpha} \\ & X_{\alpha} = x_{\alpha} + \Delta x_{\alpha} \; , \quad Y_{\alpha} = y_{\alpha} + \Delta x_{\alpha} \; , \quad Z_{\alpha} = z_{\alpha} + \Delta z_{\alpha} \; . \end{aligned}\right. \tag{1.23}$$

The astronomical azimuth is additively decomped into the model azimuth $\alpha_{\alpha\beta}$, also referred to as the geodetic azimuth, and the azimuth disturbance $\delta\alpha_{\alpha\beta}$. Or, astronomical longitude is additively decomposed into the model longitude, also called geodetic longitude λ, and the longitude disturbance, the vertical deflection component. Note that geodetic longitude and geodetic latitude are as spherical coordinates of the model gravity vector

$$\underset{\sim}{\gamma} = \text{grad } w$$

related to its rectangular coordinates by

$$\left[\begin{aligned} \lambda &= \arctan \gamma_y/\gamma_x \\ \phi &= \arctan \gamma_z/\sqrt{\gamma_x^2+\gamma_y^2} \end{aligned}\right. \qquad (1.24)$$

where w denotes the model gravity potential, also called "normal potential". In its most simple, the spherical form it is represented by

$$w = \frac{gm}{r} \qquad (1.25)$$

$$\gamma_i = \frac{\partial w}{\partial x_i} = -\frac{gm}{r^3} x_i \,. \qquad (1.26)$$

gm represents the product of the gravitational constant and the earth's mass. Within this model choice we derive

$$\left[\begin{aligned} \lambda &= \arctan y/x \\ \phi &= -\arctan z/\sqrt{x^2+y^2} \,. \end{aligned}\right. \qquad (1.27)$$

$$r = \sqrt{x^2+y^2+z^2}$$

defines the radial distance of the earth's centre from the topocentre of an LPS. (x, y, z) = (x•, y•, z•) are the topocentric coordinates in the equatorial frame of reference e•. Once (x, y, z) – prior coordinate information is available we are able to compute spherical longitude and latitude of the local vertical! For instance, we can read (λ, Φ) from a spherical map where we have located the topocentre. In a similar way we can compute the geodetic azimuth and the approximate vertical direction

$$\alpha_{\alpha\beta} = \arctan y^*_{\alpha\beta}/x^*_{\alpha\beta} = \arctan \frac{-\sin\lambda_\alpha x^\bullet_{\alpha\beta} + \cos\lambda_\alpha y^\bullet_{\alpha\beta}}{\cos\lambda_\alpha \sin\phi_\alpha x^\bullet_{\alpha\beta} + \sin\lambda_\alpha \sin\phi_\alpha y^\bullet_{\alpha\beta} - \cos\phi_\alpha z^\bullet_{\alpha\beta}}$$

$$\beta_{\alpha\beta} = \arctan z^*_{\alpha\beta}/\sqrt{x^{*2}_{\alpha\beta} + y^{*2}_{\alpha\beta}} =$$

$$= \arctan \frac{\cos\lambda_\alpha \cos\phi_\alpha x^\bullet_{\alpha\beta} + \sin\lambda_\alpha \cos\phi_\alpha y^\bullet_{\alpha\beta} + \sin\phi_\alpha z^\bullet_{\alpha\beta}}{\sqrt{(\cos\lambda_\alpha \sin\phi_\alpha x^\bullet_{\alpha\beta} + \sin\lambda_\alpha \sin\phi_\alpha y^\bullet_{\alpha\beta} - \cos\phi_\alpha z^\bullet_{\alpha\beta})^2 + (-\sin\lambda_\alpha x^\bullet_{\alpha\beta} + \cos\lambda_\alpha y^\bullet_{\alpha\beta})^2}} \qquad (1.28)$$

where $(x^\bullet_{\alpha\beta}, y^\bullet_{\alpha\beta}, z^\bullet_{\alpha\beta}) = (x^\bullet_\beta - x^\bullet_\alpha \,,\; y^\bullet_\beta - y^\bullet_\alpha \,, z^\bullet_\beta - z^\bullet_\alpha)$

are the relative equatorial coordinates in a spherical approximation of the model relative position vector

$$\underset{\sim}{x}_{\alpha\beta}.$$

For instance, in spherical coordinates

$$(x^\bullet_{\alpha\beta}, y^\bullet_{\alpha\beta}, z^\bullet_{\alpha\beta})$$

allow the standard representation

$$\left[\begin{aligned} x^\bullet_{\alpha\beta} &= r_\beta \cos\lambda_\beta \cos\phi_\beta - r_\alpha \cos\lambda_\alpha \cos\phi_\alpha \\ y^\bullet_{\alpha\beta} &= r_\beta \sin\lambda_\beta \cos\phi_\beta - r_\alpha \sin\lambda_\alpha \cos\phi_\alpha \\ z^\bullet_{\alpha\beta} &= r_\beta \sin\phi_\beta - r_\alpha \sin\phi_\alpha \,, \end{aligned}\right. \qquad (1.29)$$

being mainly used in "geometric geodesy" (Vermessungskunde).

Finally we are going to relate to disturbances of the azimuth $\delta\alpha_{\alpha\beta}$ and the vertical direction $\delta\beta_{\alpha\beta}$ to the disturbances of astronomical longitude $\delta\lambda_\alpha$ and astronomical latitude $\delta\Phi_\alpha$ via the threedimensional Laplace condition (for a derivation let us refer to E. Grafarend and B. Richter [1977])

$$\boxed{\begin{aligned} \delta\alpha_{\alpha\beta} &= -(\cos\alpha_{\alpha\beta} \tan\beta_{\alpha\beta} \cos\phi_\alpha + \sin\phi_\alpha)\delta\lambda_\alpha - \\ &\quad - \sin\alpha_{\alpha\beta} \tan\beta_{\alpha\beta}\, \delta\phi_\alpha \qquad (1.30) \\ \delta\beta_{\alpha\beta} &= \sin\alpha_{\alpha\beta} \cos\phi_\alpha\, \delta\lambda_\alpha - \cos\alpha_{\alpha\beta}\, \delta\phi_\alpha \qquad (1.31) \end{aligned}}$$

or in matrix form

$$\begin{bmatrix} \delta\alpha_{\alpha\beta} \\ \delta\beta_{\alpha\beta} \end{bmatrix} = (-1) * \begin{bmatrix} \cos\alpha_{\alpha\beta}\tan\beta_{\alpha\beta}\cos\phi_\alpha + \sin\phi_\alpha & \sin\alpha_{\alpha\beta}\tan\beta_{\alpha\beta} \\ -\sin\alpha_{\alpha\beta}\cos\phi_\alpha & \cos\alpha_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \delta\lambda_\alpha \\ \delta\phi_\alpha \end{bmatrix} \quad (1.32)$$

which looks formally similar to (A10), (A11), but of course with different contents. In conclusio, when the vertical deflection components $\delta\lambda\cos d\Phi$, $\delta\Phi$ are available from a map or can be computed by a program we are then able via (1.30), (1.31) or (1.32), respectively, to calculate the disturbances of the azimuth $\delta\alpha$ and the vertical direction $\delta\beta$.

We are left with the problem to compute the variations

$$\begin{bmatrix} \Delta A_{\alpha\beta} = A_{\beta\alpha} - A_{\alpha\beta} = \alpha_{\beta\alpha} - \alpha_{\alpha\beta} + \delta\alpha_{\beta\alpha} - \delta\alpha_{\alpha\beta} \\ \Delta B_{\alpha\beta} = B_{\beta\alpha} - B_{\alpha\beta} = \beta_{\beta\alpha} - \beta_{\alpha\beta} + \delta\beta_{\beta\alpha} - \delta\beta_{\alpha\beta} \end{bmatrix} \quad (1.33)$$

of the real azimuth and the real vertical direction. Namely in the spherical approximation (1.29) we obtain the important result that

$$\begin{bmatrix} \Delta A_{\alpha\beta} = \Delta\alpha_{\alpha\beta} + \delta\alpha_{\beta\alpha} - \delta\alpha_{\alpha\beta}\,, \quad \alpha_{\alpha\beta} \neq \alpha_{\beta\alpha} \\ \Delta B_{\alpha\beta} = \Delta\beta_{\alpha\beta} + \delta\beta_{\beta\alpha} - \delta\beta_{\alpha\beta}\,, \quad \beta_{\alpha\beta} \neq \beta_{\beta\alpha} \end{bmatrix} \quad (1.34)$$

the variation of the real azimuth and the real vertical direction is the sum of the variation of the spherical azimuth and the spherical vertical direction plus the difference of the disturbances of azimuth and vertical direction or via (1.32) can be computed from the difference or changes of vertical deflections $\delta\lambda_\beta - \delta\lambda_\alpha$, $\delta\Phi_\beta - \delta\Phi_\alpha$! Within the above framework of approximation and representation we rewrite the fundamental geodetic traverse equations (A16) and (A17) by

$$\boxed{\begin{aligned} A_i &= A_o + \sum_{j=1}^{i} A_{j-1,j,j+1} + \sum_{j=1}^{i} \Delta(\Delta\alpha_j + \delta\alpha_j) \\ B_i &= B_o + \sum_{j=1}^{i} B_{j-1,j,j+1} + \sum_{j=1}^{i} \Delta(\Delta\beta_j + \delta\beta_j) \end{aligned}} \quad (1.35)$$

where A_o is the initial azimuth, B_o the initial vertical direction, $A_{j-1,j,j+1}$ the horizontal angle (Brechungswinkel) and $B_{j-1,j,j+1}$ the vertical angle from the LPS-topocentre P_j to the target point P_{j-1} and P_{j+1} within a line-type traverse. $\triangle\delta\alpha_j = \delta\alpha_j - \delta\alpha_{j-1}$, $\triangle\delta\beta_j = \delta\beta_j - \delta\beta_{j-1}$ denotes the incremental variation of azimuth/vertical direction which can be replaced via the three-dimensional Laplace condition (1.32) by incremental variation of longitude $\delta\lambda_j - \delta\lambda_{j-1}$ and latitude $\delta\Phi_j - \delta\Phi_{j-1}$. Similarly (A20)–(A22) could be rewritten.

A basic question would be how accurate the information of vertical deflections would be needed. Especially the bias problem within prior deflections of the vertical is open for investigation, a future task.

An example, also including "trigonometric height determination" might be useful at the end.

Example 2

(trigonometric height determination):

Once we introduce the additive decomposition of the change of the local vertical into a model (or normal) part and disturbing part, the height difference

$$\overset{\alpha}{Z}_{\beta\gamma}$$

between a point P_β and a point P_γ in the reference system $E^*(P_\alpha)$ of horizontal type attached to the point P_α can be represented from (A22) by

$$\overset{\alpha}{Z}_{\beta\gamma} = \overset{\alpha}{Z}_{\gamma} - \overset{\alpha}{Z}_{\beta} = R_{\beta\gamma} \sin B_{\beta\gamma}$$

$$+ \Phi_{\alpha\beta} R_{\beta\gamma} \cos(A_{\alpha\beta} + A_{\alpha\beta\gamma}) \cos B_{\beta\gamma} \quad (1.36)$$

$$- \Lambda_{\alpha\beta} \cos\Phi_{\alpha} R_{\beta\gamma} \sin(A_{\alpha\beta} + A_{\alpha\beta\gamma}) \cos B_{\beta\gamma}$$

where

$$\left.\begin{aligned} \Phi_{\alpha\beta} &= \phi_{\beta} - \phi_{\alpha} + \delta\phi_{\beta} - \delta\phi_{\alpha} = \\ &= \arcsin \frac{Z^{\bullet}_{\beta}}{R_{\beta}} - \arcsin \frac{Z^{\bullet}_{\alpha}}{R_{\alpha}} + \delta\phi_{\beta} - \delta\phi_{\alpha} \\ \Lambda_{\alpha\beta} &= \lambda_{\beta} - \lambda_{\alpha} + \delta\lambda_{\beta} - \delta\lambda_{\alpha} = \\ &= \arctan \frac{Z^{\bullet}_{\beta}}{\sqrt{X^{\bullet 2}_{\beta} + Y^{\bullet 2}_{\beta}}} - \arctan \frac{Z^{\bullet}_{\alpha}}{\sqrt{X^{\bullet 2}_{\alpha} + Y^{\bullet 2}_{\alpha}}} + \delta\lambda_{\beta} - \delta\lambda_{\alpha} \end{aligned}\right] \quad (1.37)$$

is the difference in latitude and longitude, respectively.

$(X^{\bullet}_{\alpha}, Y^{\bullet}_{\alpha}, Z^{\bullet}_{\alpha})$ and

$(X^{\bullet}_{\beta}, Y^{\bullet}_{\beta}, Z^{\bullet}_{\beta})$

denote the prior rectangular coordinates of the point P_{α} and P_{β} in the equatorial frame of reference $E^{\bullet}$. Once the arc sin-function within (1.37) is taylorized we meet a formula which corresponds to a known one in "geometric geodesy" (Vermessungskunde). Instead (1.36), (1.37) compromize the exact equations up to first order. Note that the difference in latitude and longitude (1.37) includes the differences in the vertical deflections between the point P_{α} and P_{β}. Let us summarize the observations and the prior information quantities being needed for the computation of local height differences

$\overset{\alpha}{Z}_{\beta\gamma}$:

(i) the distance $R_{\beta\gamma}$ and the vertical direction $B_{\beta\gamma}$ as well as the horizontal angle $A_{\gamma\beta\alpha}$ from the point P_{β} to the target point P_{γ} and the initial point P_{α} have to be measured;

(ii) the azimuth $A_{\alpha\beta}$ of the initial direction has to be known (or to be determined by a South seeking gyroscope or by star observations);

(iii) prior rectangular coordinates of the initial point and the LPS location in the equatorial frame of reference have to be known in order to be able to compute the model longitude and latitude differences;

(iv) the vertical deflection components have to be known with respect to a spherically symmetric gravity field between the initial point and the LPS location.

Similar to the above representation of (A22) for the vertical control in (1.36), (1.37) the formulae (A20) and (A21) can be transformed for the horizontal control.

3. Geodetic networks

In part one we have shown that line-type structured polygon traverses are impossible as long as we have no other information about the variation of the gravity field available. Instead we have emphasized that triangular chain-type polygon traverses can be computed without any other information. Once we move over to local geodetic networks we have to remember this fundamental result from geodetic traverses. Indeed triangular chain-type traverses have a network structure of band type. It is therefore not difficult to set-up observational equations in the most general form, say for direction and distance measurements of the line $P_{\beta}P_{\gamma}$, but whose reference coordinate system is attached to the initial point P_{α}.

The spherical coordinates of the relative position vector

$\underset{\sim}{x}_{\beta\gamma}$,

the horizontal direction $t_{\beta\gamma} = \alpha_{\beta\gamma} - \sigma_\beta - \sigma$ denotes the horizontal orientation unknown – the vertical direction $s_{\beta\gamma} = \beta_{\beta\gamma}$ (complement of zenith distance) and the distance $r_{\beta\gamma}$ in vacuo are related to their rectangular counterparts

$$(\overset{\beta}{x}_{\beta\gamma}, \overset{\beta}{y}_{\beta\gamma}, \overset{\beta}{z}_{\beta\gamma})$$

in the horizontal frame of reference

$$\overset{\beta}{e}{}^*$$

$$\left[\begin{array}{l} t_{\beta\gamma} = \alpha_{\beta\gamma} - \sigma_\beta = \text{arc tan}\ \overset{\beta}{y}_{\beta\gamma}/\overset{\beta}{x}_{\beta\gamma} - \sigma_\beta \\ s_{\beta\gamma} = \beta_{\beta\gamma} = \text{arc tan}\ \overset{\beta}{z}_{\beta\gamma}/\sqrt{\overset{\beta}{x}{}^2_{\beta\gamma} + \overset{\beta}{y}{}^2_{\beta\gamma}} \\ r_{\beta\gamma} = \sqrt{\overset{\beta}{x}{}^2_{\beta\gamma} + \overset{\beta}{y}{}^2_{\beta\gamma} + \overset{\beta}{z}{}^2_{\beta\gamma}} \end{array}\right. \tag{2.1}$$

In order to be able to compute the network in the coordinate system of the initial point P_α, the horizontal frame of reference

$$\overset{\alpha}{e}{}^*,$$

we have to apply to the transformation

$$\overset{\alpha}{e}{}^* \to \overset{\beta}{e}{}^*$$

according to (1.4) which leads us to

$$\begin{bmatrix} \overset{\beta}{x}_{\beta\gamma} \\ \overset{\beta}{y}_{\beta\gamma} \\ \overset{\beta}{z}_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & \lambda_{\alpha\beta}\sin\phi_\alpha & \phi_{\alpha\beta} \\ -\lambda_{\alpha\beta}\sin\phi_\alpha & 1 & -\lambda_{\alpha\beta}\cos\phi_\alpha \\ -\phi_{\alpha\beta} & & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{x}_{\beta\gamma} \\ \overset{\alpha}{y}_{\beta\gamma} \\ \overset{\alpha}{z}_{\beta\gamma} \end{bmatrix}, \tag{2.2}$$

once we assume that the parameters which describe the change of the local vertical, namely (λ, Φ), are close to the identity. (2.1), (2.2) combined give us the most general form of the LPS observational equations expressed in the chosen reference frame

$$\overset{\alpha}{e}{}^*$$

at the initial point P_α:

$$\boxed{\begin{array}{l} t_{\beta\gamma} = \text{arc tan}\ \dfrac{-\lambda_{\alpha\beta}\sin\phi_\alpha \overset{\alpha}{x}_{\beta\gamma} + \overset{\alpha}{y}_{\beta\gamma} - \lambda_{\alpha\beta}\cos\phi_\alpha \overset{\alpha}{z}_{\beta\gamma}}{\overset{\alpha}{x}_{\beta\gamma} + \lambda_{\alpha\beta}\sin\phi_\alpha \overset{\alpha}{y}_{\beta\gamma} + \phi_{\alpha\beta}\overset{\alpha}{z}_{\beta\gamma}} - \sigma_\beta \\ s_{\beta\gamma} = \text{arc tan}\ \dfrac{-\phi_{\alpha\beta}\overset{\alpha}{x}_{\beta\gamma} + \lambda_{\alpha\beta}\cos\phi_\alpha \overset{\alpha}{y}_{\beta\gamma} + \overset{\alpha}{z}_{\beta\gamma}}{\sqrt{\overset{\alpha}{x}{}^2_{\beta\gamma} + \overset{\alpha}{y}{}^2_{\beta\gamma} - 2\lambda_{\alpha\beta}\cos\phi_\alpha \overset{\alpha}{y}_{\beta\gamma}\overset{\alpha}{z}_{\beta\gamma} + 2\phi_{\alpha\beta}\overset{\alpha}{x}_{\beta\gamma}\overset{\alpha}{z}_{\beta\gamma}}} \\ r_{\beta\gamma} = \sqrt{\overset{\alpha}{x}{}^2_{\beta\gamma} + \overset{\alpha}{y}{}^2_{\beta\gamma} + \overset{\alpha}{z}{}^2_{\beta\gamma}} \end{array}} \tag{2.3}$$

$$
\begin{aligned}
t_{\beta\gamma} &= \arc\tan \frac{\overset{\alpha}{y}_{\beta\gamma}}{\overset{\alpha}{x}_{\beta\gamma}} - \sigma_\beta - \\
&\quad - \left(\sin\phi_\alpha + \frac{\overset{\alpha}{x}_{\beta\gamma}\overset{\alpha}{z}_{\beta\gamma}}{\overset{\alpha 2}{x}_{\beta\gamma}+\overset{\alpha 2}{y}_{\beta\gamma}} \cos\phi_\alpha\right)\lambda_{\alpha\beta} - \frac{\overset{\alpha}{y}_{\beta\gamma}\overset{\alpha}{z}_{\beta\gamma}}{\overset{\alpha 2}{x}_{\beta\gamma}+\overset{\alpha 2}{y}_{\beta\gamma}}\,\phi_{\alpha\beta} \\
s_{\beta\gamma} &= \arc\tan \frac{\overset{\alpha}{z}_{\beta\gamma}}{\overset{\alpha 2}{x}_{\beta\gamma}+\overset{\alpha 2}{y}_{\beta\gamma}} + \\
&\quad + \frac{\overset{\alpha}{y}_{\beta\gamma}}{\overset{\alpha 2}{x}_{\beta\gamma}+\overset{\alpha 2}{y}_{\beta\gamma}} \cos\phi_\alpha\,\lambda_{\alpha\beta} - \frac{\overset{\alpha}{x}_{\beta\gamma}}{\overset{\alpha 2}{x}_{\beta\gamma}+\overset{\alpha 2}{y}_{\beta\gamma}}\,\phi_{\alpha\beta} \\
r_{\beta\gamma} &= \sqrt{\overset{\alpha 2}{x}_{\beta\gamma} + \overset{\alpha 2}{y}_{\beta\gamma} + \overset{\alpha 2}{z}_{\beta\gamma}}
\end{aligned}
\qquad (2.4)
$$

Within the range where local networks apply we can consider the changes of the local vertical $\lambda_{\alpha\beta}$, $\Phi_{\alpha\beta}$ as small. Thus we are permitted to taylorize (2.3):

Obviously, the distance in vacuo is independent of the choice of the reference system, here of its orientation and origin. In contrast, the direction observations depend on it. The observational equations for horizontal and vertical directions contain three orientation unknowns, namely the conventional orientation unknown σ_β in the local horizontal plane between "zero" of the LPS's horizontal circle and South of astronomic type, and the differences in astronomical longitude $\lambda_{\alpha\beta}$ and astronomical latitude $\Phi_{\alpha\beta}$ between the initial point P_α and the point P_β of LPS's placement. In addition, the coordinate differences

$$(\overset{\alpha}{x}_{\beta\gamma}, \overset{\alpha}{y}_{\beta\gamma}, \overset{\alpha}{z}_{\beta\gamma})$$

between target point P_γ and set-up point P_β of the LPS appear as unknowns. Further linearization with respect to approximate placement coordinates is standard and will not be presented here.

Appendix:

Transport of horizontal and vertical directions

Once we assume that the geodetic line of sight within a local positioning system (LPS, theodolite with EDM equipment) has been reduced to its Euclidean standard (refraction, instrumental errors, station-reference-point), then we can represent the relative position vector

$$\underset{\sim}{x}_{\alpha\beta} := \underset{\sim}{x}_\beta - \underset{\sim}{x}_\alpha$$

from the standpoint $\underset{\sim}{x}_\alpha$

to the target point $\underset{\sim}{x}_\beta$

with respect to the local horizontal triad { south, east, vertical } =

$$\{\underset{\sim}{e}_{1*}, \underset{\sim}{e}_{2*}, \underset{\sim}{e}_{3*}\}$$

by spherical coordinates distance $r_{\alpha\beta}$, south-azimuth $\alpha_{\alpha\beta}$ and vertical direction $\beta_{\alpha\beta}$ (measured from the horizontal plane, complement of zenith distance) such that

$$
\left[\begin{aligned}
\overset{\alpha}{x}_{\alpha\beta} &= r_{\alpha\beta} \cos\alpha_{\alpha\beta} \cos\beta_{\alpha\beta} \\
\overset{\alpha}{y}_{\alpha\beta} &= r_{\alpha\beta} \sin\alpha_{\alpha\beta} \cos\beta_{\alpha\beta} \\
\overset{\alpha}{z}_{\alpha\beta} &= r_{\alpha\beta} \sin\beta_{\alpha\beta}
\end{aligned}\right.
\qquad (A1)
$$

holds. The superscript α indicates that the coordinate differences $x_{\alpha\beta} := x_\beta - x_\alpha$, $y_{\alpha\beta} := y_\beta - y_\alpha$, $z_{\alpha\beta} := z_\beta - z_\alpha$ with respect to the horizontal triad

$$\overset{\alpha}{\underset{\sim}{e}}{}^{*}$$

refer to the reference frame at the point P_α. Correspondingly, the reverse relative position vector

$$\underset{\sim}{x}_{\beta\alpha} = -\underset{\sim}{x}_{\alpha\beta}$$

can be represented by

$$
\left[\begin{aligned}
\overset{\beta}{x}_{\beta\alpha} &= - r_{\beta\alpha} \cos\alpha_{\beta\alpha} \cos\beta_{\beta\alpha} \\
\overset{\beta}{y}_{\beta\alpha} &= - r_{\beta\alpha} \sin\alpha_{\beta\alpha} \cos\beta_{\beta\alpha} \\
\overset{\beta}{z}_{\beta\alpha} &= - r_{\beta\alpha} \sin\beta_{\beta\alpha}
\end{aligned}\right.
\qquad (A2)
$$

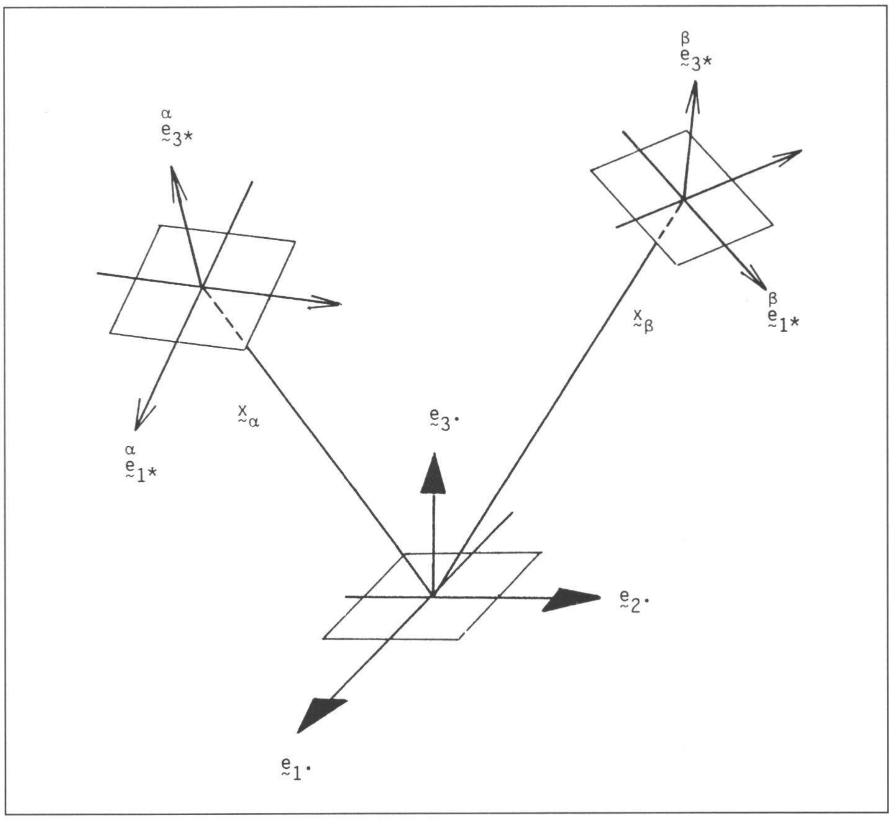

Figure A1: Moving versus fixed frame of reference (moving: e*, horizontal, fixed: e•, equatorial)

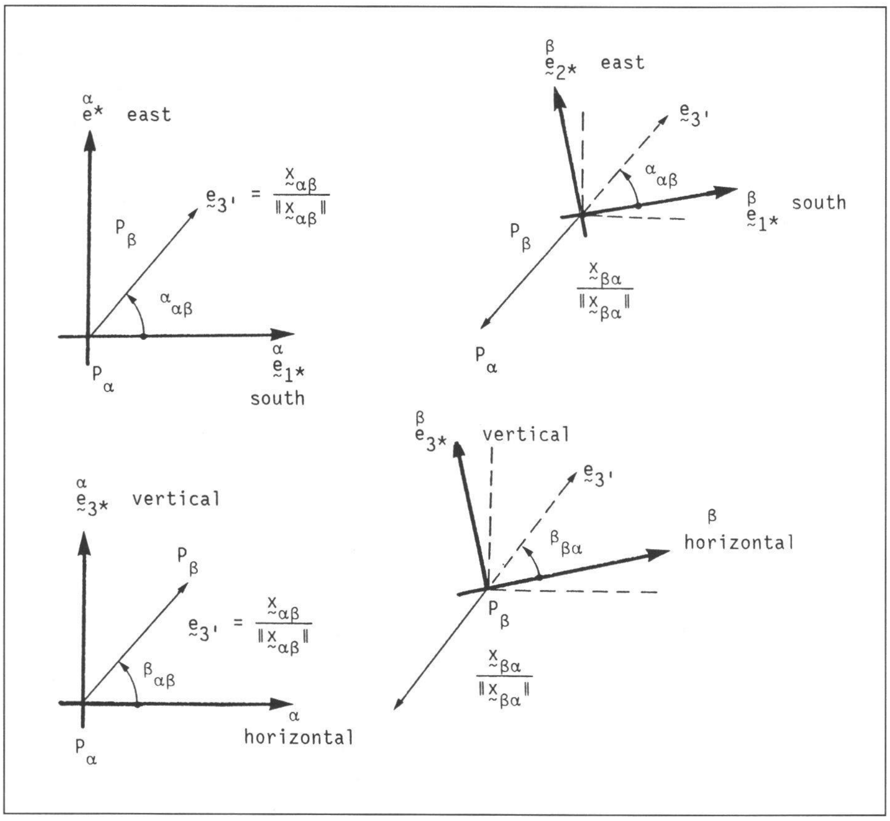

Figure A2: Horizontal and vertical projection of the Euclidean line of sight at points P_α versus P_β, horizontal $\alpha_{\alpha\beta}$, $\alpha_{\beta\alpha}$ and vertical $\beta_{\alpha\beta}$, $\beta_{\beta\alpha}$ directions

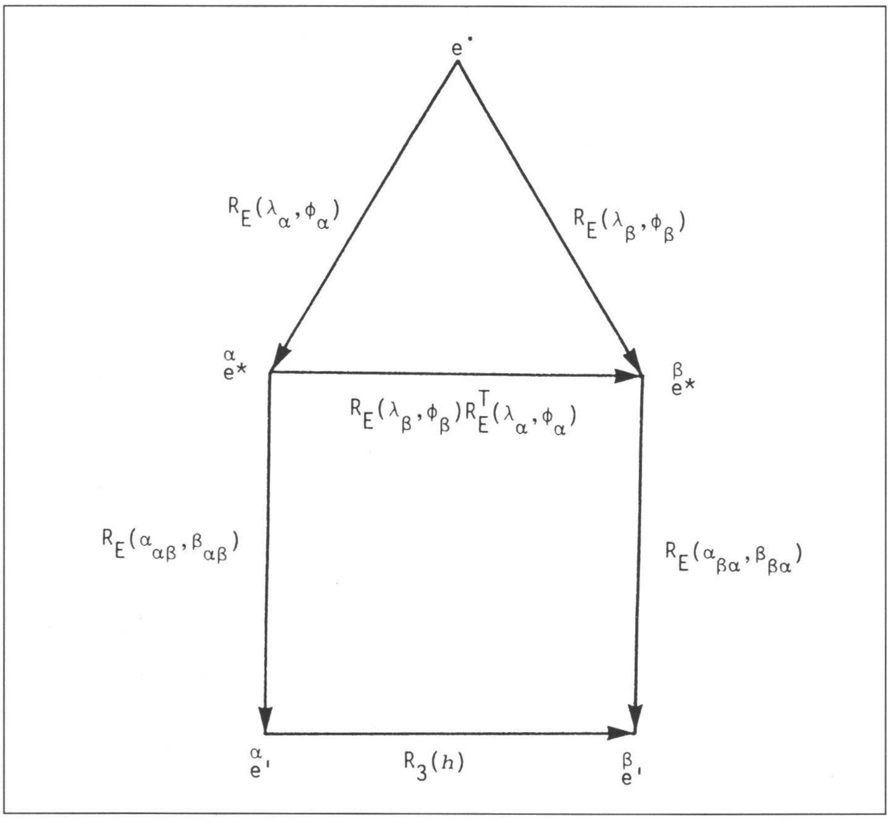

Figure A3: Commutative diagram of orthonormal triads (moving: e*, horizontal, fixed: e•, equatorial)

indicating that the coordinate differences $x_{\beta\alpha} := x_\alpha - x_\beta$, $y_{\beta\alpha} := y_\alpha - y_\beta$, $z_{\beta\alpha} := z_\alpha - z_\beta$ with respect to the horizontal traid

$$\overset{\beta}{e}{}^*$$

now refer to the reference frame at the point P_β. Figure A1 and A2 illustrate how we have defined the spherical coordinates $\alpha_{\alpha\beta}$, $\beta_{\alpha\beta}$, $\alpha_{\beta\alpha}$, $\beta_{\beta\alpha}$ of the direction sight

$$\underset{\sim}{x}_{\alpha\beta} \div \| x_{\alpha\beta} \|$$

where $r_{\alpha\beta} = \|\underset{\sim}{x}_{\alpha\beta}\|$

is the Euclidean standard distance. They have been chosen in such a way that they always parametrize

$$\underset{\sim}{e}_{3'} := \underset{\sim}{x}_{\alpha\beta} \div \|\underset{\sim}{x}_{\alpha\beta}\|$$

and that

$\alpha_{\alpha\beta} = \alpha_{\beta\alpha}$, $\beta_{\alpha\beta} = \beta_{\beta\alpha}$ holds if there is no change of the local vertical $\underset{\sim}{e}_{3*}$

and of the horizontal directions

$\underset{\sim}{e}_{1*}$, $\underset{\sim}{e}_{2*}$.

In general, of course, $\overset{\alpha}{x}_{\alpha\beta} \neq -\overset{\beta}{x}_{\beta\alpha}$

etc. (but $\overset{\alpha}{x}_{\alpha\beta} = -\overset{\alpha}{x}_{\beta\alpha}$, $\overset{\beta}{x}_{\alpha\beta} = -\overset{\beta}{x}_{\beta\alpha}$

etc.) has to be taken into account, an effect which is due to the moving frame $\underset{\sim}{e}^*(\underset{\sim}{x})$.

According to Figure A3 we introduce the commutative diagram of reference frames, where $\overset{\alpha}{e}{}^*$, $\overset{\beta}{e}{}^*$

are moving with respect to the fixed equatorial frame {Greenwich direction in the equatorial plane, orthogonal to Greenwich in the equatorial plane, direction of the terrestrial rotation vector} =

$\{\underset{\sim}{e}_{1^\bullet}, \underset{\sim}{e}_{2^\bullet}, \underset{\sim}{e}_{3^\bullet}\}$.

All reference frame are chosen orthonormal such that they are connected by Eulerian rotation matrices $R_E(\alpha, \beta, \gamma) :=$

$R_3(\gamma) R_2(\frac{\pi}{2} - \beta) R_3(\alpha)$. $R_3(\gamma)$

indicates a rotation around the 3-axis by an angle γ etc. The observation triad e' is defined by

$$\overset{\alpha}{\underset{\sim}{e}}_{3'} = \overset{\beta}{\underset{\sim}{e}}_{3'} = \underset{\sim}{x}_{\alpha\beta} \div r_{\alpha\beta}$$

and the connection of

$$\overset{\alpha}{\underset{\sim}{e}}_{1'} \neq \overset{\beta}{\underset{\sim}{e}}_{1'}, \quad \overset{\alpha}{\underset{\sim}{e}}_{2'} \neq \overset{\beta}{\underset{\sim}{e}}_{2'}$$

to the horizontal frame, such that

$$\overset{\beta}{e}{}' = R_3(h)\overset{\alpha}{e}{}'.$$

We have to mention that the spherical coordinates of the gravity vector, also called astronomical longitude λ and astronomical latitude Φ, describe the directional movement of the local horizontal triad e* with respect to the global earth-fixed equatorial triad e•as indicated in the commutative diagram.

From the commutative diagram we read

$$\overset{\alpha}{e}{}^* \rightarrow \overset{\beta}{e}{}^* = R_E(\lambda_\beta,\phi_\beta)R_E^T(\lambda_\alpha,\phi_\alpha)\overset{\alpha}{e}{}^* = R_2(\tfrac{\pi}{2} - \phi_\beta)R_3(\lambda_\beta-\lambda_\alpha)R_2^T(\tfrac{\pi}{2} - \phi_\alpha)\overset{\alpha}{e}{}^* \qquad (A3)$$

$$\overset{\alpha}{e}{}^* \rightarrow \overset{\beta}{e}{}^* = R_E^T(\alpha_{\beta\alpha},\beta_{\beta\alpha})R_3(\zeta)R_E(\alpha_{\alpha\beta},\beta_{\alpha\beta})\overset{\alpha}{e}{}^* \qquad (A4)$$

or

$$\boxed{R_E^T(\alpha_{\beta\alpha},\beta_{\beta\alpha})R_3(\zeta)R_E(\alpha_{\alpha\beta},\beta_{\alpha\beta}) = R_E(\lambda_\beta,\phi_\beta)R_E^T(\lambda_\alpha,\phi_\alpha)} \qquad (A5)$$

or

$$R_3(\zeta) = R_E(\alpha_{\beta\alpha},\beta_{\beta\alpha})R_E(\lambda_\beta,\phi_\beta)R_E^T(\lambda_\alpha,\phi_\alpha)R_E^T(\alpha_{\alpha\beta},\beta_{\alpha\beta}) \ . \qquad (A6)$$

Now let us assume

$$\lambda_\beta = \lambda_\alpha + \lambda_{\alpha\beta} = \lambda_\alpha + \Delta\lambda, \ \phi_\beta = \phi_\alpha + \phi_{\alpha\beta} = \phi_\alpha + \Delta\phi$$

where Φλ, ΔΦ shall be small such that

$$\cos\lambda_\beta = \cos(\lambda_\alpha+\Delta\lambda) = \cos\lambda_\alpha \cos\Delta\lambda - \sin\lambda_\alpha \sin\Delta\lambda \doteq \cos\lambda_\alpha - \Delta\lambda \sin\lambda_\alpha \qquad (A7)$$

$$\sin\lambda_\beta = \sin(\lambda_\alpha+\Delta\lambda) = \sin\lambda_\alpha \cos\Delta\lambda + \cos\lambda_\alpha \sin\Delta\lambda \doteq \sin\lambda_\alpha + \Delta\lambda \cos\lambda_\alpha \qquad (A8)$$

holds close to the identity. The connection matrix

$$R_E(\lambda_\beta,\phi_\beta)R_E^T(\lambda_\alpha,\phi_\alpha) = \begin{bmatrix} 1 & +\Delta\lambda\sin\phi_\alpha & +\Delta\phi \\ -\Delta\lambda\sin\phi_\alpha & 1 & -\Delta\lambda\cos\phi_\alpha \\ -\Delta\phi & +\Delta\lambda\cos\phi_\alpha & 1 \end{bmatrix} \qquad (A9)$$

consists of the sum of the unit matrix and an antisymmetric matrix. Inserted into (A6) it leads us to a similar formula once we decompose analogously $\alpha_{\beta\alpha} = \alpha_{\alpha\beta} + \Delta\alpha$, $\beta_{\beta\alpha} = \beta_{\alpha\beta} + \Delta\beta$ and apply (A7), (A8), e.g. for the elements (3.1) and (3.2) of $R_3(h)$ after a tedious computation,

$$\boxed{\begin{aligned} \Delta\alpha &= -(\cos\alpha_{\alpha\beta}\tan\beta_{\alpha\beta}\cos\phi_\alpha + \sin\phi_\alpha)\Delta\lambda - \\ &\quad - \sin\alpha_{\alpha\beta}\tan\beta_{\alpha\beta}\Delta\phi \qquad (A10) \\ \Delta\beta &= \sin\alpha_{\alpha\beta}\cos\phi_\alpha\Delta\lambda - \cos\alpha_{\alpha\beta}\Delta\phi \ . \qquad (A11) \end{aligned}}$$

For the partisans of physical geodesy we better mention that (A10), (A11) is similar to the threedimensional Laplace condition whose contents is, of course, different.
For geodetic traverses and geodetic networks we need only a slightly more general computation. Once we have identified the origin of the coordinate system, e.g. P_α as the initial point, the reference directions, e.g. the horizontal direction $\alpha_{\alpha\beta}$ of azimuth type and the vertical directions $\beta_{\alpha\beta}$; $\beta_{\alpha\gamma}$ of the relative position vectors

$$\underset{\sim}{x}_{\alpha\beta}, \ \underset{\sim}{x}_{\alpha\gamma},$$

and the scale, e.g. the length

$$\|\underset{\sim}{x}_{\alpha\beta}\| = r_{\alpha\beta}$$

of the relative position vector $\underset{\sim}{x}_{\alpha\beta}$

– in short, we have decided upon an S-basis – we have to look into the horizontal and vertical directions of the other lines of sight. According to Figure A4 we place the local horizontal triad

$$\overset{\beta}{e}_{*}$$

at the point P_β and measure "backwards" to the reference point P_α and "forewards" to the point P_γ. Note that we have defined the directions $\alpha_{\alpha\beta}$, $\beta_{\alpha\beta}$, $\alpha_{\beta\alpha}$, $\beta_{\beta\alpha}$ both at the point P_α and P_β by e_3, =

$$= \overset{\alpha}{e}_{3'} = \overset{\beta}{e}_{3'}.$$

This type of definition is for two reasons helpful: (i) It avoids any factor π in the computation (see standard textbooks for a more cumbersome notation once we refer to two dimensions or see P. Teunissen (1985 p. 52 [2.52]) for a threedimensional approach); (ii) the transport equations of horizontal and vertical directions are formally the same. Finally from Figure A4 we read

$$\alpha_{\gamma\beta\alpha} = \alpha_{\beta\alpha} - \alpha_{\beta\gamma}, \quad \beta_{\gamma\beta\alpha} = \beta_{\beta\alpha} - \beta_{\beta\gamma} \tag{A12}$$

for the horizontal and vertical angles, or

$$\alpha_{\beta\gamma} = \alpha_{\beta\alpha} + \alpha_{\alpha\beta\gamma}, \quad \beta_{\beta\gamma} = \beta_{\beta\alpha} + \beta_{\alpha\beta\gamma} \ , \tag{A13}$$

$$\boxed{\alpha_{\beta\gamma} = \alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha \ ,} \tag{A14}$$

$$\boxed{\beta_{\beta\gamma} = \beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta \ .} \tag{A15}$$

In other words, we have found that the azimuth (alternatively: vertical direction) of the side

$$\underset{\sim}{x}_{\beta\gamma}$$

is determined by

(i) the azimuth (alternatively: vertical direction) of the side

$$\underset{\sim}{x}_{\alpha\beta} \ ,$$

(ii) the horizontal angle (alternatively: vertical angle) between the sides

$$\underset{\sim}{x}_{\beta\alpha}, \ \underset{\sim}{x}_{\beta\gamma},$$

(iii) and the change of azimuth (alternatively: vertical direction) from reference point $\underset{\sim}{x}_{\alpha}$

to the point $\underset{\sim}{x}_{\beta}$.

In turn, the changes of azimuth $\Delta\alpha$ and of vertical direction $\Delta\beta$ are caused by the changes of the local vertical as expressed in terms of $\Delta\lambda$, $\Delta\Phi$ according to (A10), (A11) which are due to gravity gradients.
Once we enlarge the geodetic traverse, for instance, azimuth and vertical direction transport is mastered by the equations

$$\boxed{\alpha_i = \alpha_o + \sum_{j=1}^{i} \alpha_{j-1,j,j+1} + \sum_{j=1}^{i} \Delta\alpha_j} \tag{A16}$$

$$\boxed{\beta_i = \beta_o + \sum_{j=1}^{i} \beta_{j-1,j,j+1} + \sum_{j=1}^{i} \Delta\beta_j} \tag{A17}$$

where we have used the condensed notation of a double index into one: α_o indicates the azimuth of the side P_oP_1 (alternatively: the vertical direction), $P_{j-1}P_jP_{j+1}$ (alternatively: the vertical angle) and $\Delta\alpha_i$ the azimuth variation at the point P_j (alternatively: variation of the vertical angle). The index i runs 1,2,…,n-1 where n is the total number of points P_n within the geodetic traverse.
With the equations of transport (A15), (A16) we have described the first effect of the influence of the local gravity field on the horizontal reference frame. Usually the terms $\Delta\alpha_j$, $\Delta\beta_j$ or, equivalently, $\Delta\lambda_j$, $\Delta\Phi_j$, the changes of astronomical longitude/latitude, are neglected. A numerical example might illustrate the seize of the first effect.

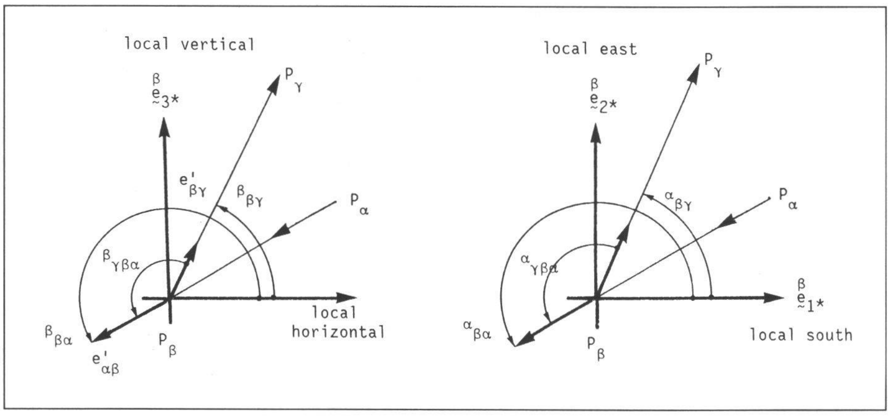

Figure A4: horizontal and vertical angles in a local horizontal triad

Example A1:

$\lambda_{\alpha\beta} = 1'' \sim 4.85 * 10^{-6}$ RAD

$\phi_{\alpha\beta} = -0.5'' \sim 2.42 * 10^{-6}$ RAD

$\phi_{\alpha} = 48.783°$ (Stuttgart)

$\alpha_{\alpha\beta} = 0°, \quad \beta_{\alpha\beta} = 0°$

$\Delta\alpha = -\sin\phi_{\alpha} \quad \lambda_{\alpha\beta} = -0.75'' = -3.65 * 10^{-6}$ RAD

$\Delta\beta = -\phi_{\alpha\beta} = +0.5'' = +2.42 * 10^{-6}$ RAD

Example A2:

$\lambda_{\alpha\beta} = 25'' \sim 12{,}12 * 10^{-5}$ RAD

$\phi_{\alpha\beta} = -15'' \sim -7{,}27 * 10^{-5}$ RAD

$\phi_{\alpha} = 48.783°$ (Stuttgart

$\alpha_{\alpha\beta} = 0°, \quad \beta_{\alpha\beta} = 0°$

$\Delta\alpha = -\sin\phi_{\alpha} \quad \lambda_{\alpha\beta} = -18.8'' = -9.12 * 10^{-5}$ RAD

$\Delta\beta = -\phi_{\alpha\beta} = +15'' = +7.27 * 10^{-5}$ RAD

The examples show that in spite of a South/horizontally orientated relative point placement there are most notable changes in the azimuth and the vertical direction.
The second effect we are going to describe now has to do with the coordinate changes when computed in the local reference frame of horizontal-vertical type.
As can be read from the commutative diagram of Figure A3 (coordinate differences transform like base vectors or triads) the coordinate differences transform from the frame ${}^{\beta}e^{*}$

into the frame ${}^{\alpha}e^{*}$

by

$$\begin{bmatrix} {}^{\alpha}x^{*}_{\beta\gamma} \\ {}^{\alpha}y^{*}_{\beta\gamma} \\ {}^{\alpha}z^{*}_{\beta\gamma} \end{bmatrix} = R_E(\lambda_\alpha, \phi_\alpha) R_E^T(\lambda_\beta, \phi_\beta) \begin{bmatrix} {}^{\beta}x^{*}_{\beta\gamma} \\ {}^{\beta}y^{*}_{\beta\gamma} \\ {}^{\beta}z^{*}_{\beta\gamma} \end{bmatrix} \qquad (A18)$$

or

$$\boxed{\begin{aligned} {}^{\alpha}x^{*}_{\beta\gamma} &\doteq {}^{\beta}x^{*}_{\beta\gamma} - \lambda_{\alpha\beta} \sin\phi_{\alpha} \, {}^{\beta}y^{*}_{\beta\gamma} - \phi_{\alpha\beta} \, z_{\beta\gamma} \\ {}^{\alpha}y^{*}_{\beta\gamma} &\doteq \lambda_{\alpha\beta} \sin\phi_{\alpha} \, {}^{\beta}x^{*}_{\beta\gamma} + {}^{\beta}y^{*}_{\beta\gamma} + \lambda_{\alpha\beta} \cos\phi_{\alpha} \, {}^{\beta}z_{\beta\gamma} \\ {}^{\alpha}z^{*}_{\beta\gamma} &\doteq \phi_{\alpha\beta} \, {}^{\beta}x^{*}_{\beta\gamma} - \lambda_{\alpha\beta} \cos\phi_{\alpha} \, {}^{\beta}y^{*}_{\beta\gamma} + {}^{\beta}z^{*}_{\beta\gamma} \end{aligned}} \quad . \qquad (A19)$$

(A19) as an approximation of (A18) close to the identity expresses that the coordinates of the relative position vector

$\underset{\sim}{x}_{\beta\gamma}$

have to be transformed from the frame

$\overset{\beta}{e}*$ to the reference frame $\overset{\alpha}{e}*$.

The transformation is necessary since the observed spherical coordinates $r_{\beta\gamma}$, $\alpha_{\beta\gamma}$, $\beta_{\beta\gamma}$ or, equivalently, the horizontal $\alpha_{\alpha\beta\gamma}$ and vertical $\beta_{\alpha\beta\gamma}$ angles refer to the local triad

$\overset{\beta}{e}*$,

but the holonomic coordinate computation is only admissible in the chosen (and then fixed) reference frame

$\overset{\alpha}{e}*$

at reference point P_α. Thus, in toto, summarizing the two basic effects we have discussed here lead to the coordinate computation (in the

$\overset{\alpha}{e}*$ -frame; the star "*" has been neglected)

$$\overset{\alpha}{x}_\gamma = \overset{\alpha}{x}_\alpha + r_{\alpha\beta} \cos\alpha_{\alpha\beta} \cos\beta_{\alpha\beta} + \tag{A20}$$
$$+ r_{\beta\gamma} \cos(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha)\cos(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$
$$- \lambda_{\alpha\beta} \sin\phi_\alpha\, r_{\beta\gamma} \sin(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha)\cos(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$
$$- \phi_{\alpha\beta}\, r_{\beta\gamma} \sin(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$

$$\overset{\alpha}{y}_\gamma = \overset{\alpha}{y}_\alpha + r_{\alpha\beta} \sin\alpha_{\alpha\beta} \cos\beta_{\alpha\beta} + \tag{A21}$$
$$+ \lambda_{\alpha\beta} \sin\phi_\alpha\, r_{\beta\gamma} \cos(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha)\cos(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$
$$+ r_{\beta\gamma} \sin(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha)\cos(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$
$$+ \lambda_{\alpha\beta} \cos\phi_\alpha\, r_{\beta\gamma} \sin(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$

$$\overset{\alpha}{z}_\gamma = \overset{\alpha}{z}_\alpha + r_{\alpha\beta} \sin\beta_{\alpha\beta} + \tag{A22}$$
$$+ \phi_{\alpha\beta}\, r_{\beta\gamma} \cos(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha)\cos(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$
$$- \lambda_{\alpha\beta} \cos\phi_\alpha\, r_{\beta\gamma} \sin(\alpha_{\alpha\beta} + \alpha_{\alpha\beta\gamma} + \Delta\alpha)\cos(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)$$
$$+ r_{\beta\gamma} \sin(\beta_{\alpha\beta} + \beta_{\alpha\beta\gamma} + \Delta\beta)\ .$$

We end up with an example for the second effect.

Example A3

$$\lambda_{\alpha\beta} = 1'' \sim 4.85 * 10^{-6}\ \mathrm{RAD}$$

$$\phi_{\alpha\beta} = -0.5'' \sim 2.42 * 10^{-6}\ \mathrm{RAD}$$

$$\phi_\alpha = 48.783^\circ\ \text{(Stuttgart)}$$

$$\overset{\alpha}{x}_{\beta\gamma} = 50\ \mathrm{m}\ ,\quad \overset{\alpha}{y}_{\beta\gamma} = -50\ \mathrm{m}\ ,\quad \overset{\alpha}{z}_{\beta\gamma} = +\ 15\ \mathrm{m}$$

$$\begin{bmatrix} \overset{\beta}{x}_{\beta\gamma} \\ \overset{\beta}{y}_{\beta\gamma} \\ \overset{\beta}{z}_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\lambda_{\alpha\beta}\sin\phi_\alpha & +\phi_{\alpha\beta} \\ -\lambda_{\alpha\beta}\sin\phi_\alpha & 1 & -\lambda_{\alpha\beta}\cos\phi_\alpha \\ -\phi_{\alpha\beta} & +\lambda_{\alpha\beta}\cos\phi_\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{x}_{\beta\gamma} \\ \overset{\alpha}{y}_{\beta\gamma} \\ \overset{\alpha}{z}_{\beta\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +3.65 * 10^{-6} & -2.42 * 10^{-6} \\ -3.65 * 10^{-6} & 1 & -3.19 * 10^{-6} \\ +2.42 * 10^{-6} & +3.19 * 10^{-6} & 1 \end{bmatrix} \begin{bmatrix} 50\ \mathrm{m} \\ -50\ \mathrm{m} \\ +15\ \mathrm{m} \end{bmatrix}$$

$$\overset{\beta}{x}_{\beta\gamma} = 50\ \mathrm{m} - 0.22\ \mathrm{mm}$$

$$\overset{\beta}{y}_{\beta\gamma} = -50\ \mathrm{m} - 0.23\ \mathrm{mm}$$

$$\overset{\beta}{z}_{\beta\gamma} = 15\ \mathrm{m} - 0.04\ \mathrm{mm}\ .$$

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Example A4

$$\lambda_{\alpha\beta} = 25'' \sim 12.12 * 10^{-5} \text{ RAD}$$

$$\phi_{\alpha\beta} = -15'' \sim 7.27 * 10^{-5} \text{ RAD}$$

$$\phi_{\alpha} = 48.783 \text{ (Stuttgart)}$$

$$\overset{\alpha}{x}_{\beta\gamma} = 800 \text{ m}, \quad \overset{\alpha}{y}_{\beta\gamma} = -800 \text{ m}, \quad \overset{\alpha}{z}_{\beta\gamma} = 150 \text{ m}$$

$$\begin{bmatrix} \overset{\beta}{x}_{\beta\gamma} \\ \overset{\beta}{y}_{\beta\gamma} \\ \overset{\beta}{z}_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\lambda_{\alpha\beta}\sin\phi_\alpha & +\phi_{\alpha\beta} \\ -\lambda_{\alpha\beta}\sin\phi_\alpha & 1 & -\lambda_{\alpha\beta}\cos\phi_\alpha \\ \phi_{\alpha\beta} & +\lambda_{\alpha\beta}\cos\phi_\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{x}_{\beta\gamma} \\ \overset{\alpha}{y}_{\beta\gamma} \\ \overset{\alpha}{z}_{\beta\gamma} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & +9.12 * 10^{-5} & -7.27 * 10^{-5} \\ -9.12 * 10^{-5} & 1 & -7.99 * 10^{-5} \\ +7.27 * 10^{-5} & +7.99 * 10^{-5} & 1 \end{bmatrix} \begin{bmatrix} 800 \text{ m} \\ -800 \text{ m} \\ 150 \text{ m} \end{bmatrix}$$

$$\overset{\beta}{x}_{\beta\gamma} = 800 \text{ m} - 83.9 \text{ mm}$$

$$\overset{\beta}{y}_{\beta\gamma} = -800 \text{ m} - 84.9 \text{ mm}$$

$$\overset{\beta}{z}_{\beta\gamma} = +150 \text{ m} - 5.8 \text{ mm} .$$

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