Zeitschrift: Helvetica Physica Acta

Band: 28 (1955)

Heft: V-VI

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DOI: https://doi.org/10.5169/seals-112622

Extension of McDougall-Stoner tables of the Fermi-Dirac functions

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(29. IV. 1955.)

Abstract. The McDougall-Stoner tables of the Fermi-Dirac functions $F_k(\eta)$ for $k = -1/2$, $1/2$, and $3/2$ have been extended to include $k = 5/2$, $7/2$, $9/2$, and $11/2$. Computations were carried out on digital computing equipment, using three terms of the Euler-Maclaurin numerical integration formula. Values are tabulated for the argument $\eta = -4\ (0{\cdot}1)\ 20$. Analysis of results, including use of numerous check points, suggests that the tables are accurate to five significant figures throughout. In order to provide a complete listing, the McDougall-Stoner values for $F_{-1/2}(\eta)$, $F_{1/2}(\eta)$, and $F_{3/2}(\eta)$ are also included in the tabulations.

Introduction.

The Fermi-Dirac functions $F_k(\eta)$ are defined by the following integrals:

$$F_k(\eta) \equiv \int_0^\infty \frac{x^k}{1+e^{x-\eta}}\,dx\,,\quad k > -1\,. \tag{1}$$

These functions occur in the expressions for electrical- or thermal-conductivity phenomena involving transport by electrons or holes whenever degeneracy is sufficient, so that the classical approximation to the Fermi-Dirac distribution function is not applicable. It is readily established that the use of Maxwell-Boltzmann statistics is equivalent to the evaluation of equation (1) for large negative values of η. This case is usually realized for practical purposes when $\eta < -4$, so that

$$F_k(\eta) \simeq k!\cdot e^{\eta},\quad \eta < -4\,. \tag{2}$$

When extreme degeneracy exists, as in the case of metals, the function approximates a power-law dependency. If, for example, $\eta > 20$, the Sommerfeld (1928) asymptotic expression gives

$$F_k(\eta) \approx \frac{\eta^{k+1}}{k+1}\,,\quad \eta > 20\,. \tag{3}$$

However, for the region of small positive values of η, which is of most interest in the analysis of degenerate semiconductors, no simple expression for $F_k(\eta)$ is wholly satisfactory*). Fortunately, however, a very complete evaluation of these functions for $k = 1/2$ and $3/2$ has been carried out by McDougall and Stoner (1938). Since derivatives are tabulated, it is also possible to obtain $F_{-1/2}(\eta)$, although with fewer significant figures, by means of the following general relationship:

$$\frac{\partial}{\partial\eta} F_k(\eta) = k F_{k-1}(\eta), \quad k > 0. \tag{4}$$

Additional evaluations of these Fermi-Dirac integrals appear in articles by Wright (1951a) for $k = 1, 2$; by Rhodes (1950) for $k = 1, 2, 3$, and 4; and by Johnson and Shipley (1953) for $k = 2, 3, 7/2$, and $9/2$. Use of the functions in specialized cases has been discussed in all of the above references as well as in many other publications, including articles by Hutner, Rittner, and Du Pré (1950), and by Wright (1951*b*). Wright develops expressions for a number of galvanomagnetic and thermomagnetic effects in semiconductors for mean-free-path dependencies on energy, which follow a power law. In cases where ionized impurity scattering is important, with mean free path dependent on the square of the energy, one encounters Fermi-Dirac functions with indices greater than 3/2. Accordingly, we have carried out an extension of the McDougall-Stoner tables of $F_k(\eta)$ for $k = 5/2, 7/2, 9/2$, and $11/2$, and $-4 \leqslant \eta \leqslant 20$, in steps of $\Delta\eta = 0{\cdot}1$. Details on the use of these functions will not be given here, since the references previously quoted may be consulted. Furthermore, there has recently appeared a publication by Madelung (1954) which treats extensively the conductivity phenomena in isotropic semiconductors, both for isothermal and adiabatic conditions, and for the commonly encountered scattering mechanisms.

Method of computation.

The basis of the computations is the numerical integration of the differential recursion formula given in equation (4). For simplicity, let us define

$$f_k(\eta) \equiv \beta_k F_k(\eta), \tag{5}$$

where

$$\frac{1}{\beta_k} = k(k-1)\ldots\frac{3}{2}. \tag{6}$$

*) See discussion by Blakemore (1952).

Then, if the range of interest is $a \leqslant \eta \leqslant b$, integration of equation (4) yields

$$f_k(\eta) = f_k(a) + \int_a^\eta f_{k-1}(x)\, dx \,. \tag{7}$$

The McDougall-Stoner tables, which supplied the integrands for the above relationship, list $f_{3/2}(\eta)$, $F_{1/2}(\eta)$, $w F'_{1/2}(\eta)$, $w^2 F''_{1/2}(\eta)$, and $w^3 F'''_{1/2}(\eta)$ for $-4{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$, the interval w being 0·1. The functions are given to six decimals for $-4{\cdot}0 \leqslant \eta \leqslant 4{\cdot}0$ and to five decimals for $4{\cdot}0 \leqslant \eta \leqslant 20{\cdot}0$.

Details of the Computing Process.

The numerical evaluation of equation (7) was carried out, using three terms of the Euler-Maclaurin formula for an interval of length w, namely,

$$\frac{1}{w}\int_\eta^{\eta+w} f(x)\, dx = \frac{1}{2}\left[f(\eta) + f(\eta+w)\right] - \frac{w}{12}\left[f'(\eta+w) - f'(\eta)\right] + \frac{w^3}{720}\left[f'''(\eta+w) - f'''(\eta)\right]. \tag{8}$$

Letting $\Delta_k(\eta_i) = f_k(\eta_{i+1}) - f_k(\eta_i)$, where $\eta_0 = -4{\cdot}0$, $\eta_{i+1} = \eta_i + w$, we have, by means of the preceding equations,

$$\Delta_k(\eta_i) = \frac{w}{2}\left[f_{k-1}(\eta_i) + f_{k-1}(\eta_{i+1})\right] - \frac{w^2}{12}\Delta_{k-2}(\eta_i) + \frac{w^3}{720}\Delta_{k-4}(\eta_i)\,. \tag{9}$$

The McDougall-Stoner tables were put onto cards by summing hand-punched differences. This gave a set of cards containing the values of $f_{3/2}(\eta_i)$, $\Delta_{1/2}(\eta_i)$, and $\Delta_{-3/2}(\eta_i)$ *), so that $\Delta_{5/2}(\eta_i)$ could readily be computed from equation (9). The interval w was again taken to be 0·1.

The initial values $f_k(-4{\cdot}0)$ were obtained to eight decimals from four terms of the series

$$f_k(\eta) = \beta_k\, k! \sum_{r=1}^{\infty} (-1)^{r-1} \frac{e^{r\eta}}{r^{k+1}}\,, \tag{10}$$

which holds for $\eta \leqq 0$. Using the value so obtained for $f_{5/2}(-4{\cdot}0)$,

*) Although as originally formulated, $F_k(\eta)$ is defined by equation (1) for only $k > -1$, it has been shown by more general considerations that the function exists for all noninteger values of k [see Appendix, McDougall and Stoner (1938)]. Phil. Trans. Roy. Soc. **A 237**, 350, 1938. Likewise, it follows that equation (4) is also valid for all negative noninteger values of k.

the table of $f_{5/2}(\eta_i)$ was completed by summing the differences $\Delta_{5/2}(\eta_i)$.

The entire process was repeated, using appropriate functions, giving successively $f_{7/2}(\eta_i)$, $f_{9/2}(\eta_i)$, and $f_{11/2}(\eta_i)$.

Checks on the Computations.

Independent calculations of $F_k(\eta)$ were made, using the formula

$$F_k(0) = (1 - 2^{-k})\, k!\, \zeta(k+1) \equiv F_k^*(0) \tag{11}$$

for $\eta = 0$, and the asymptotic series

$$F_k(\eta) = \frac{\eta^{k+1}}{k+1}\left[1 + \sum_{r=1}^{5} a_{2r}^{(k)} \eta^{-2r}\right] + R_k(\eta) \equiv F_k^*(\eta) + R_k(\eta) \tag{12}$$

for $\eta = 10$, 12, 14, 16, 18, and 20.

The coefficients $a_{2r}^{(k)}$ in equation (12) are given by the formula

$$a_{2r}^{(k)} = 2\, C_{2r} (k+1)\, k \ldots (k - 2r + 2)\,,$$

where

$$C_{2r} = (1 - 2^{1-2r})\, \zeta\,(2r).$$

Ten-place tables of the Riemann zeta function required here are readily available, for example Dwight (1941). The error term in equation (12) satisfies the inequality

$$|R_k(\eta)| < L_k(\eta) = 12\, |a_{12}^{(k)}|\, \eta^{k+1}.$$

Values of $a_{2r}^{(k)}$, $L_k(\eta)$, $F_k^*(\eta)$, and $d_k(\eta) \equiv F_k^*(\eta) - F_k(\eta)$ are given in Tables 1, 2, 3, and 4, respectively. Because of the great variation in $F_k(\eta)$, the entries in these tables as well as the values of $F_k(\eta)$ in the main tabulation are given in a floating-decimal notation. The single digit, n, at the right of each entry indicates the power of ten by which that entry should be multiplied.

Table 1.

Coefficients $a_{2r}^{(k)}$.

$2r$	$k = 3/2$	n	$k = 5/2$	n	$k = 7/2$	n	$k = 9/2$	n	$k = 11/2$	n
2	6·1685028	0	1·4393173	1	2·5907712	1	4·0712118	1	5·8806393	1
4	–1·7756866	0	1·2429806	1	1·1186825	2	4·1018359	2	1·0664773	3
6	–6·9296561	0	9·7015186	0	–2·9104556	1	3·2015011	2	4·1619515	3
8	–1·1032502	2	8·5808350	1	–1·1032502	2	2·4271505	2	–1·0517652	3
10	–3·9552306	3	2·1297395	3	–1·7425142	3	2·1297395	3	–3·9552306	3
12	–2·5232721	5	1·0389944	5	–6·2339663	4	5·2748946	4	–6·2339663	4

Table 2. $L_k(\eta)$.

η	$k = 3/2$	n	$k = 5/2$	n	$k = 7/2$	n	$k = 9/2$	n	$k = 11/2$	n
10	1	–3	4	–3	2	–2	2	–1	2	0
12	2	–4	8	–4	6	–3	6	–2	9	–1
14	4	–5	2	–4	2	–3	2	–2	4	–1
16	1	–5	7	–5	7	–4	9	–3	2	–1
18	3	–6	3	–5	3	–4	4	–3	9	–2
20	2	–6	1	–5	1	–4	2	–4	5	–2

Table 3. $F^*_k(\eta)$.

η	$k = 3/2$	n	$k = 5/2$	n	$k = 7/2$	n	$k = 9/2$	n	$k = 11/2$	n
0	1·15280	0	3·08259	0	1·11837	1	5·12905	1	2·84904	2
10	1·34270	2	1·03468	3	8·92629	3	8·32807	4	8·26504	5
12	2·08062	2	1·88225	3	1·89204	4	2·04150	5	2·32528	6
14	3·02564	2	3·14983	3	3·62572	4	4·45804	5	5·75745	6
16	4·19458	2	4·94522	3	6·42490	4	8·88666	5	1·28666	7
18	5·60305	2	7·38434	3	1·06992	5	1·64645	6	2·64579	7
20	7·26568	2	1·05906	4	1·69419	5	2·87348	6	5·08033	7

Table 4. $d_k(\eta) \equiv F^*_k(\eta) - F_k(\eta)$.

η	$k = 3/2$	n	$k = 5/2$	n	$k = 7/2$	n	$k = 9/2$	n	$k = 11/2$	n
0	4	–6	–4	–6	2	–5	1	–4	8	–4
10	2	–4	4	–3	–8	–4	8	–3	–3	–1
12	–9	–5	4	–3	3	–2	1	–1	5	0
14	2	–4	–1	–3	–3	–2	–2	–1	–3	0
16	3	–4	2	–3	0	–3	1	–2	–4	1
18	–4	–5	7	–3	–3	–1	4	0	–3	1
20	9	–5	4	–2	2	–1	2	0	–6	0

Comparison of Tables 2, 3, and 4 indicates that the values of $F_k(\eta)$ for $k \geqslant 3/2$ are probably correct to five significant digits throughout and that for $\eta > 0$ there should only be an occasional error of l in the sixth digit. The cumulative nature of the computation makes it likely that any error in $F_k(\eta)$ other than a random rounding error would be carried on to larger values of η, so that the listed values of $d_k(\eta)$ provide a check on the entire table.

Tables of the Functions.

The values of $F_k(\eta)$ obtained here are listed to 5 or 6 significant digits in Table 5. In order to provide a complete listing, the McDougall-Stoner values for $F_{-1/2}(\eta)$, $F_{1/2}(\eta)$, and $F_{3/2}(\eta)$ are included in Table 5.

Table 5.

Tabulation of Fermi-Dirac Functions.

(a) Reprinted from McDougall-Stoner publication. See J. McDougall and E. C. Stoner, Phil. Trans. Royal Soc. (London) **A 237** 350 (1938).

(b) The single digit at the right of each entry indicates the power of ten by which that entry should be multiplied.

η	$F_{-1/2}(\eta)$(a)	$F_{1/2}(\eta)$(a)	$F_{3/2}(\eta)$(a)	$F_{5/2}(\eta)$	$F_{7/2}(\eta)$	$F_{9/2}(\eta)$	$F_{11/2}(\eta)$
−4·0	0·3204 −1(b)	0·16128 −1	2·42685 −2	6·07731 −2	2·12877 −1	9·58334 −1	5·27190 0
−3·9	0·3536 −1	0·17812 −1	2·68125 −2	6·71529 −2	2·35245 −1	1·05908 0	5·82621 0
−3·8	0·3904 −1	0·19670 −1	2·96220 −2	7·42014 −2	2·59961 −1	1·17040 0	6·43879 0
−3·7	0·4306 −1	0·21721 −1	3·27240 −2	8·19882 −2	2·87271 −1	1·29342 0	7·11577 0
−3·6	0·4752 −1	0·23984 −1	3·61485 −2	9·05902 −2	3·17447 −1	1·42937 0	7·86390 0
−3·5	0·5240 −1	0·26480 −1	3·99300 −2	1·00092 −1	3·50788 −1	1·57960 0	8·69065 0
−3·4	0·5778 −1	0·29233 −1	4·41060 −2	1·10588 −1	3·87627 −1	1·74560 0	9·60431 0
−3·3	0·6372 −1	0·32269 −1	4·87140 −2	1·22181 −1	4·28327 −1	1·92903 0	1·06140 1
−3·2	0·7022 −1	0·35615 −1	5·38020 −2	1·34985 −1	4·73294 −1	2·13173 0	1·17297 1
−3·1	0·7740 −1	0·39303 −1	5·94165 −2	1·49126 −1	5·22971 −1	2·35570 0	1·29627 1
−3·0	0·8526 −1	0·43366 −1	6·56115 −2	1·64742 −1	5·77852 −1	2·60317 0	1·43253 1
−2·9	0·9390 −1	0·47842 −1	7·24470 −2	1·81985 −1	6·38479 −1	2·87662 0	1·58309 1
−2·8	1·0336 −1	0·52770 −1	7·99860 −2	2·01024 −1	7·05450 −1	3·17875 0	1·74948 1
−2·7	1·1374 −1	0·58194 −1	8·83020 −2	2·22043 −1	7·79426 −1	3·51257 0	1·93333 1
−2·6	1·2512 −1	0·64161 −1	9·74715 −2	2·45246 −1	8·61134 −1	3·88139 0	2·13650 1
−2·5	1·3758 −1	0·70724 −1	1·07580 −1	2·70857 −1	9·51377 −1	4·28886 0	2·36099 1
−2·4	1·5118 −1	0·77938 −1	1·18722 −1	2·99122 −1	1·05104 0	4·73903 0	2·60905 1
−2·3	1·6606 −1	0·85864 −1	1·30998 −1	3·30312 −1	1·16110 0	5·23635 0	2·88314 1
−2·2	1·8228 −1	0·94566 −1	1·44521 −1	3·64725 −1	1·28263 0	5·78574 0	3·18600 1
−2·1	1·9994 −1	1·04116 −1	1·59410 −1	4·02686 −1	1·41682 0	6·39261 0	3·52062 1
−2·0	2·1918 −1	1·14588 −1	1·75800 −1	4·44554 −1	1·56497 0	7·06296 0	3·89035 1
−1·9	2·4010 −1	1·26063 −1	1·93836 −1	4·90723 −1	1·72851 0	7·80339 0	4·29883 1
−1·8	2·6278 −1	1·38627 −1	2·13674 −1	5·41623 −1	1·90902 0	8·62116 0	4·75013 1
−1·7	2·8736 −1	1·52373 −1	2·35484 −1	5·97724 −1	2·10825 0	9·52431 0	5·24872 1
−1·6	3·1394 −1	1·67397 −1	2·59451 −1	6·59544 −1	2·32810 0	1·05217 1	5·79953 1
−1·5	3·4262 −1	1·83802 −1	2·85773 −1	7·27646 −1	2·57066 0	1·16230 1	6·40800 1
−1·4	3·7352 −1	2·01696 −1	3·14666 −1	8·02644 −1	2·83825 0	1·28390 1	7·08016 1
−1·3	4·0674 −1	2·21193 −1	3·46361 −1	8·85212 −1	3·13339 0	1·41815 1	7·82261 1
−1·2	4·4236 −1	2·42410 −1	3·81110 −1	9·76079 −1	3·45886 0	1·56636 1	8·64268 1
−1·1	4·8048 −1	2·65471 −1	4·19177 −1	1·07604 0	3·81771 0	1·72995 1	9·54842 1
−1·0	5·2114 −1	2·90501 −1	4·60848 −1	1·18597 0	4·21325 0	1·91050 1	1·05487 2
−0·9	5·6444 −1	3·17630 −1	5·06432 −1	1·30679 0	4·64916 0	2·10975 1	1·16534 2
−0·8	6·1040 −1	3·46989 −1	5·56250 −1	1·43954 0	5·12940 0	2·32959 1	1·28732 2
−0·7	6·5902 −1	3·78714 −1	6·10647 −1	1·58530 0	5·65835 0	2·57212 1	1·42201 2
−0·6	7·1034 −1	4·12937 −1	6·69989 −1	1·74527 0	6·24076 0	2·83964 1	1·57071 2
−0·5	7·6434 −1	4·49793 −1	7·34660 −1	1·92074 0	6·88184 0	3·13467 1	1·73488 2
−0·4	8·2096 −1	4·89414 −1	8·05065 −1	2·11308 0	7·58725 0	3·45997 1	1·91608 2
−0·3	8·8014 −1	5·31931 −1	8·81628 −1	2·32378 0	8·36314 0	3·81858 1	2·11608 2
−0·2	9·4182 −1	5·77470 −1	9·64796 −1	2·55444 0	9·21622 0	4·21381 1	2·33680 2
−0·1	1·0059 0	6·26152 −1	1·05503 0	2·80677 0	1·01538 1	4·64931 1	2·58034 2
0·0	1·0722 0	6·78094 −1	1·15280 0	3·08259 0	1·11837 1	5·12904 1	2·84903 2
0·1	1·1405 0	7·33403 −1	1·25862 0	3·38384 0	1·23146 1	5·65735 1	3·14542 2
0·2	1·2109 0	7·92181 −1	1·37300 0	3·71261 0	1·35556 1	6·23900 1	3·47232 2
0·3	1·2830 0	8·54521 −1	1·49646 0	4·07110 0	1·49168 1	6·87916 1	3·83279 2
0·4	1·3566 0	9·20505 −1	1·62954 0	4·46164 0	1·64091 1	7·58348 1	4·23020 2
0·5	1·4317 0	9·90209 −1	1·77279 0	4·88671 0	1·80440 1	8·35812 1	4·66826 2
0·6	1·5079 0	1·06369 0	1·92679 0	5·34893 0	1·98341 1	9·20977 1	5·15100 2
0·7	1·5851 0	1·14102 0	2·09209 0	5·85105 0	2·17929 1	1·01457 2	5·68288 2
0·8	1·6630 0	1·22222 0	2·26929 0	6·39597 0	2·39349 1	1·11739 2	6·26872 2
0·9	1·7415 0	1·30733 0	2·45895 0	6·98673 0	2·62754 1	1·23028 2	6·91385 2
1·0	1·8204 0	1·39638 0	2·66168 0	7·62653 0	2·88313 1	1·35419 2	7·62405 2
1·1	1·8995 0	1·48937 0	2·87806 0	8·31871 0	3·16201 1	1·49011 2	8·40566 2
1·2	1·9786 0	1·58632 0	3·10869 0	9·06675 0	3·46609 1	1·63915 2	9·26558 2
1·3	2·0575 0	1·68723 0	3·35416 0	9·87429 0	3·79738 1	1·80247 2	1·02113 3
1·4	2·1361 0	1·79207 0	3·61506 0	1·07451 1	4·15803 1	1·98135 2	1·12512 3
1·5	2·2144 0	1·90083 0	3·89198 0	1·16832 1	4·55032 1	2·17717 2	1·23939 3
1·6	2·2920 0	2·01350 0	4·18550 0	1·26925 1	4·97668 1	2·39139 2	1·36494 3
1·7	2·3691 0	2·13003 0	4·49622 0	1·37773 1	5·43968 1	2·62562 2	1·50281 3
1·8	2·4453 0	2·25039 0	4·82470 0	1·49421 1	5·94203 1	2·88155 2	1·65416 3
1·9	2·5208 0	2·37455 0	5·17152 0	1·61912 1	6·48661 1	3·16103 2	1·82022 3

Table 5 (continued).

η	$F_{-1/2}(\eta)$(a)		$F_{1/2}(\eta)$(a)		$F_{3/2}(\eta)$(a)		$F_{5/2}(\eta)$		$F_{7/2}(\eta)$		$F_{9/2}(\eta)$		$F_{11/2}(\eta)$	
2·0	2·5954	0	2·50246	0	5·53725	0	1·75294	1	7·07645	1	3·46603	2	2·00234	3
2·1	2·6690	0	2·63407	0	5·92245	0	1·89615	1	7·71476	1	3·79864	2	2·20198	3
2·2	2·7417	0	2·76934	0	6·32766	0	2·04923	1	8·40491	1	4·16113	2	2·42074	3
2·3	2·8133	0	2·90822	0	6·75343	0	2·21270	1	9·15043	1	4·55591	2	2·66030	3
2·4	2·8839	0	3·05066	0	7·20030	0	2·38708	1	9·95507	1	4·98556	2	2·92253	3
2·5	2·9535	0	3·19660	0	7·66880	0	2·57290	1	1·08227	2	5·45281	2	3·20940	3
2·6	3·0219	0	3·34599	0	8·15946	0	2·77070	1	1·17575	2	5·96061	2	3·52308	3
2·7	3·0893	0	3·49878	0	8·67277	0	2·98106	1	1·27637	2	6·51206	2	3·86587	3
2·8	3·1557	0	3·65491	0	9·20925	0	3·20453	1	1·38458	2	7·11048	2	4·24027	3
2·9	3·2210	0	3·81433	0	9·76941	0	3·44172	1	1·50084	2	7·75939	2	4·64895	3
3·0	3·2852	0	3·97699	0	1·03537	1	3·69321	1	1·62566	2	8·46252	2	5·09479	3
3·1	3·3484	0	4·14283	0	1·09627	1	3·95961	1	1·75954	2	9·22384	2	5·58089	3
3·2	3·4106	0	4·31181	0	1·15967	1	4·24155	1	1·90302	2	1·00475	3	6·11056	3
3·3	3·4718	0	4·48388	0	1·22564	1	4·53966	1	2·05664	2	1·09381	3	6·68735	3
3·4	3·5320	0	4·65898	0	1·29420	1	4·85458	1	2·22099	2	1·19001	3	7·31506	3
3·5	3·5913	0	4·83707	0	1·36542	1	5·18698	1	2·39666	2	1·29387	3	7·99776	3
3·6	3·6497	0	5·01810	0	1·43933	1	5·53752	1	2·58429	2	1·40589	3	8·73981	3
3·7	3·7071	0	5·20202	0	1·51598	1	5·90687	1	2·78451	2	1·52664	3	9·54584	3
3·8	3·7637	0	5·38880	0	1·59540	1	6·29574	1	2·99800	2	1·65670	3	1·04208	4
3·9	3·8194	0	5·57838	0	1·67766	1	6·70481	1	3·22545	2	1·79667	3	1·13700	4
4·0	3·8743	0	5·77073	0	1·76277	1	7·13480	1	3·46758	2	1·94721	3	1·23991	4
4·1	3·9284	0	5·96580	0	1·85079	1	7·58644	1	3·72514	2	2·10898	3	1·35140	4
4·2	3·9818	0	6·16356	0	1·94176	1	8·06044	1	3·99889	2	2·28271	3	1·47212	4
4·3	4·0344	0	6·36396	0	2·03571	1	8·55756	1	4·28964	2	2·46914	3	1·60273	4
4·4	4·0862	0	6·56698	0	2·13269	1	9·07855	1	4·59820	2	2·66905	3	1·74397	4
4·5	4·1374	0	6·77257	0	2·23273	1	9·62416	1	4·92542	2	2·88326	3	1·89659	4
4·6	4·1878	0	6·98070	0	2·33588	1	1·01952	2	5·27219	2	3·11263	3	2·06141	4
4·7	4·2376	0	7·19134	0	2·44217	1	1·07924	2	5·63939	2	3·35806	3	2·23927	4
4·8	4·2868	0	7·40445	0	2·55163	1	1·14165	2	6·02797	2	3·62049	3	2·43110	4
4·9	4·3352	0	7·62001	0	2·66431	1	1·20684	2	6·43887	2	3·90091	3	2·63786	4
5·0	4·3832	0	7·83797	0	2·78024	1	1·27489	2	6·87309	2	4·20034	3	2·86055	4
5·1	4·4306	0	8·05832	0	2·89946	1	1·34588	2	7·33164	2	4·51986	3	3·10026	4
5·2	4·4774	0	8·28103	0	3·02201	1	1·41990	2	7·81556	2	4·86057	3	3·35813	4
5·3	4·5236	0	8·50606	0	3·14791	1	1·49701	2	8·32593	2	5·22365	3	3·63534	4
5·4	4·5694	0	8·73339	0	3·27720	1	1·57732	2	8·86384	2	5·61032	3	3·93316	4
5·5	4·6146	0	8·96299	0	3·40992	1	1·66090	2	9·43044	2	6·02183	3	4·25293	4
5·6	4·6594	0	9·19485	0	3·54610	1	1·74784	2	1·00269	3	6·45950	3	4·59604	4
5·7	4·7036	0	9·42893	0	3·68578	1	1·83823	2	1·06543	3	6·92471	3	4·96398	4
5·8	4·7474	0	9·66521	0	3·82898	1	1·93216	2	1·13140	3	7·41888	3	5·35829	4
5·9	4·7908	0	9·90367	0	3·97574	1	2·02971	2	1·20073	3	7·94348	3	5·78061	4
6·0	4·8338	0	1·01443	1	4·12610	1	2·13098	2	1·27353	3	8·50005	3	6·23266	4
6·1	4·8762	0	1·03870	1	4·28008	1	2·23605	2	1·34994	3	9·09020	3	6·71623	4
6·2	4·9182	0	1·06319	1	4·43772	1	2·34501	2	1·43010	3	9·71556	3	7·23323	4
6·3	4·9600	0	1·08789	1	4·59905	1	2·45797	2	1·51414	3	1·03779	4	7·78562	4
6·4	5·0012	0	1·11279	1	4·76410	1	2·57500	2	1·60220	3	1·10789	4	8·37550	4
6·5	5·0422	0	1·13790	1	4·93290	1	2·69620	2	1·69444	3	1·18205	4	9·00504	4
6·6	5·0828	0	1·16321	1	5·10548	1	2·82167	2	1·79099	3	1·26045	4	9·67653	4
6·7	5·1230	0	1·18873	1	5·28187	1	2·95151	2	1·89200	3	1·34330	4	1·03924	5
6·8	5·1628	0	1·21444	1	5·46211	1	3·08580	2	1·99764	3	1·43080	4	1·11550	5
6·9	5·2024	0	1·24035	1	5·64621	1	3·22464	2	2·10806	3	1·52316	4	1·19671	5
7·0	5·2416	0	1·26646	1	5·83422	1	3·36814	2	2·22342	3	1·62060	4	1·28314	5
7·1	5·2804	0	1·29277	1	6·02616	1	3·51639	2	2·34389	3	1·72335	4	1·37508	5
7·2	5·3190	0	1·31927	1	6·22206	1	3·66948	2	2·46963	3	1·83163	4	1·47281	5
7·3	5·3572	0	1·34596	1	6·42195	1	3·82752	2	2·60081	3	1·94569	4	1·57666	5
7·4	5·3952	0	1·37284	1	6·62586	1	3·99061	2	2·73761	3	2·06579	4	1·68695	5
7·5	5·4328	0	1·39991	1	6·83381	1	4·15885	2	2·88021	3	2·19217	4	1·80401	5
7·6	5·4702	0	1·42717	1	7·04584	1	4·33234	2	3·02879	3	2·32510	4	1·92821	5
7·7	5·5074	0	1·45461	1	7·26197	1	4·51118	2	3·18354	3	2·46485	4	2·05990	5
7·8	5·5442	0	1·48224	1	7·48223	1	4·69547	2	3·34464	3	2·61171	4	2·19947	5
7·9	5·5808	0	1·51005	1	7·70665	1	4·88532	2	3·51229	3	2·76597	4	2·34732	5

Table 5 (continued).

η	$F_{-1/2}(\eta)$(a)		$F_{1/2}(\eta)$(a)		$F_{3/2}(\eta)$(a)		$F_{5/2}(\eta)$		$F_{7/2}(\eta)$		$F_{9/2}(\eta)$		$F_{11/2}(\eta)$	
8·0	5·6170	0	1·53805	1	7·93526	1	5·08084	2	3·68668	3	2·92792	4	2·50387	5
8·1	5·6532	0	1·56622	1	8·16808	1	5·28212	2	3·86801	3	3·09787	4	2·66954	5
8·2	5·6890	0	1·59458	1	8·40514	1	5·48928	2	4·05649	3	3·27615	4	2·84479	5
8·3	5·7240	0	1·62311	1	8·64646	1	5·70241	2	4·25233	3	3·46307	4	3·03008	5
8·4	5·7600	0	1·65183	1	8·89208	1	5·92164	2	4·45573	3	3·65897	4	3·22589	5
8·5	5·7950	0	1·68071	1	9·14202	1	6·14705	2	4·66692	3	3·86420	4	3·43273	5
8·6	5·8300	0	1·70978	1	9·39630	1	6·37877	2	4·88610	3	4·07911	4	3·65113	5
8·7	5·8646	0	1·73901	1	9·65496	1	6·61690	2	5·11351	3	4·30407	4	3·88162	5
8·8	5·8990	0	1·76842	1	9·91801	1	6·86156	2	5·34936	3	4·53945	4	4·12477	5
8·9	5·9334	0	1·79800	1	1·01855	2	7·11284	2	5·59389	3	4·78564	4	4·38116	5
9·0	5·9674	0	1·82776	1	1·04574	2	7·37087	2	5·84734	3	5·04304	4	4·65139	5
9·1	6·0012	0	1·85768	1	1·07338	2	7·63575	2	6·10993	3	5·31204	4	4·93610	5
9·2	6·0348	0	1·88777	1	1·10147	2	7·90760	2	6·38192	3	5·59307	4	5·23594	5
9·3	6·0682	0	1·91803	1	1·13002	2	8·18652	2	6·66355	3	5·88656	4	5·55157	5
9·4	6·1016	0	1·94845	1	1·15902	2	8·47264	2	6·95506	3	6·19294	4	5·88370	5
9·5	6·1346	0	1·97904	1	1·18847	2	8·76607	2	7·25672	3	6·51267	4	6·23304	5
9·6	6·1674	0	2·00980	1	1·21839	2	9·06692	2	7·56877	3	6·84620	4	6·60034	5
9·7	6·2002	0	2·04072	1	1·24877	2	9·37530	2	7·89149	3	7·19402	4	6·98638	5
9·8	6·2326	0	2·07180	1	1·27961	2	9·69134	2	8·22514	3	7·55660	4	7·39196	5
9·9	6·2650	0	2·10304	1	1·31092	2	1·00151	3	8·56998	3	7·93445	4	7·81789	5
10·0	6·2972	0	2·13445	1	1·34270	2	1·03468	3	8·92629	3	8·32807	4	8·26504	5
10·1	6·3290	0	2·16601	1	1·37495	2	1·06865	3	9·29435	3	8·73799	4	8·73428	5
10·2	6·3608	0	2·19774	1	1·40768	2	1·10344	3	9·67444	3	9·16474	4	9·22652	5
10·3	6·3926	0	2·22962	1	1·44089	2	1·13904	3	1·00668	4	9·60887	4	9·74272	5
10·4	6·4240	0	2·26166	1	1·47457	2	1·17548	3	1·04719	4	1·00709	5	1·02838	6
10·5	6·4554	0	2·29386	1	1·50874	2	1·21277	3	1·08898	4	1·05515	5	1·08509	6
10·6	6·4866	0	2·32622	1	1·54339	2	1·25092	3	1·13209	4	1·10512	5	1·14448	6
10·7	6·5176	0	2·35873	1	1·57853	2	1·28995	3	1·17655	4	1·15706	5	1·20669	6
10·8	6·5484	0	2·39139	1	1·61415	2	1·32985	3	1·22240	4	1·21103	5	1·27180	6
10·9	6·5792	0	2·42421	1	1·65027	2	1·37066	3	1·26965	4	1·26710	5	1·33994	6
11·0	6·6096	0	2·45718	1	1·68688	2	1·41237	3	1·31835	4	1·32532	5	1·41122	6
11·1	6·6402	0	2·49031	1	1·72398	2	1·45501	3	1·36853	4	1·38577	5	1·48576	6
11·2	6·6704	0	2·52359	1	1·76159	2	1·49858	3	1·42022	4	1·44851	5	1·56370	6
11·3	6·7006	0	2·55701	1	1·79969	2	1·54309	3	1·47344	4	1·51362	5	1·64514	6
11·4	6·7306	0	2·59059	1	1·83830	2	1·58856	3	1·52824	4	1·58115	5	1·73024	6
11·5	6·7604	0	2·62432	1	1·87741	2	1·63501	3	1·58465	4	1·65118	5	1·81912	6
11·6	6·7902	0	2·65820	1	1·91703	2	1·68244	3	1·64271	4	1·72379	5	1·91192	6
11·7	6·8196	0	2·69222	1	1·95716	2	1·73087	3	1·70243	4	1·79905	5	2·00878	6
11·8	6·8492	0	2·72639	1	1·99780	2	1·78030	3	1·76388	4	1·87704	5	2·10986	6
11·9	6·8784	0	2·76071	1	2·03895	2	1·83076	3	1·82707	4	1·95782	5	2·21531	6
12·0	6·9076	0	2·79518	1	2·08062	2	1·88225	3	1·89204	4	2·04150	5	2·32527	6
12·1	6·9368	0	2·82979	1	2·12281	2	1·93479	3	1·95884	4	2·12814	5	2·43993	6
12.2	6·9658	0	2·86455	1	2·16551	2	1·98840	3	2·02749	4	2·21782	5	2·55943	6
12·3	6·9946	0	2·89945	1	2·20874	2	2·04307	3	2·09804	4	2·31064	5	2·68394	6
12·4	7·0232	0	2·93449	1	2·25250	2	2·09884	3	2·17052	4	2·40667	5	2·81365	6
12·5	7·0518	0	2·96968	1	2·29678	2	2·15570	3	2·24497	4	2·50601	5	2·94874	6
12·6	7·0802	0	3·00501	1	2·34159	2	2·21368	3	2·32143	4	2·60875	5	3·08938	6
12·7	7·1086	0	3·04048	1	2·38693	2	2·27279	3	2·39994	4	2·71497	5	3·23576	6
12·8	7·1368	0	3·07610	1	2·43280	2	2·33303	3	2·48054	4	2·82478	5	3·38809	6
12·9	7·1650	0	3·11185	1	2·47921	2	2·39443	3	2·56327	4	2·93825	5	3·54656	6
13·0	7·1930	0	3·14775	1	2·52616	2	2·45700	3	2·64816	4	3·05550	5	3·71137	6
13·1	7·2210	0	3·18378	1	2·57365	2	2·52074	3	2·73527	4	3·17662	5	3·88273	6
13·2	7·2486	0	3·21996	1	2·62167	2	2·58568	3	2·82463	4	3·30171	5	4·06087	6
13·3	7·2764	0	3·25627	1	2·67024	2	2·65183	3	2·91628	4	3·43087	5	4·24600	6
13·4	7·3040	0	3·29272	1	2·71936	2	2·71920	3	3·01027	4	3·56421	5	4·43834	6
13·5	7·3314	0	3·32931	1	2·76903	2	2·78781	3	3·10664	4	3·70183	5	4·63814	6
13·6	7·3588	0	3·36603	1	2·81924	2	2·85766	3	3·20543	4	3·84385	5	4·84563	6
13·7	7·3860	0	3·40290	1	2·87001	2	2·92877	3	3·30669	4	3·99036	5	5·06105	6
13·8	7·4132	0	3·43989	1	2·92133	2	3·00116	3	3·41046	4	4·14149	5	5·28465	6
13·9	7·4402	0	3·47703	1	2·97321	2	3·07484	3	3·51679	4	4·29734	5	5·51670	6

Table 5 (continued).

η	$F_{-1/2}(\eta)$(a)		$F_{1/2}(\eta)$(a)		$F_{3/2}(\eta)$(a)		$F_{5/2}(\eta)$		$F_{7/2}(\eta)$		$F_{9/2}(\eta)$		$F_{11/2}(\eta)$	
14·0	7·4672	0	3·51430	1	3·02564	2	3·14983	3	3·62572	4	4·45804	5	5·75745	6
14·1	7·4940	0	3·55170	1	3·07864	2	3·22613	3	3·73729	4	4·62369	5	6·00717	6
14·2	7·5208	0	3·58924	1	3·13219	2	3·30376	3	3·85156	4	4·79443	5	6·26615	6
14·3	7·5474	0	3·62691	1	3·18631	2	3·38274	3	3·96857	4	4·97038	5	6·53465	6
14·4	7·5740	0	3·66471	1	3·24100	2	3·46308	3	4·08837	4	5·15165	5	6·81299	6
14·5	7·6006	0	3·70265	1	3·29626	2	3·54480	3	4·21100	4	5·33837	5	7·10144	6
14·6	7·6268	0	3·74072	1	3·35208	2	3·62790	3	4·33652	4	5·53068	5	7·40031	6
14·7	7·6532	0	3·77892	1	3·40848	2	3·71241	3	4·46497	4	5·72870	5	7·70992	6
14·8	7·6794	0	3·81725	1	3·46545	2	3·79833	3	4·59641	4	5·93257	5	8·03057	6
14·9	7·7054	0	3·85571	1	3·52300	2	3·88568	3	4·73087	4	6·14242	5	8·36261	6
15·0	7·7314	0	3·89430	1	3·58112	2	3·97448	3	4·86842	4	6·35840	5	8·70635	6
15·1	7·7574	0	3·93303	1	3·63983	2	4·06475	3	5·00910	4	6·58063	5	9·06215	6
15·2	7·7832	0	3·97188	1	3·69911	2	4·15648	3	5·15297	4	6·80926	5	9·43034	6
15·3	7·8090	0	4·01086	1	3·75898	2	4·24971	3	5·30007	4	7·04445	5	9·81129	6
15·4	7·8346	0	4·04997	1	3·81944	2	4·34443	3	5·45047	4	7·28632	5	1·02054	7
15·5	7·8602	0	4·08921	1	3·88048	2	4·44068	3	5·60420	4	7·53504	5	1·06129	7
15·6	7·8858	0	4·12857	1	3·94212	2	4·53846	3	5·76133	4	7·79075	5	1·10343	7
15·7	7·9112	0	4·16806	1	4·00434	2	4·63779	3	5·92191	4	8·05361	5	1·14700	7
15·8	7·9364	0	4·20768	1	4·06716	2	4·73869	3	6·08600	4	8·32377	5	1·19204	7
15·9	7·9618	0	4·24743	1	4·13057	2	4·84116	3	6·25364	4	8·60140	5	1·23858	7
16·0	7·9868	0	4·28730	1	4·19458	2	4·94522	3	6·42490	4	8·88666	5	1·28667	7
16·1	8·0120	0	4·32730	1	4·25919	2	5·05089	3	6·59982	4	9·17970	5	1·33634	7
16·2	8·0370	0	4·36742	1	4·32440	2	5·15818	3	6·77848	4	9·48070	5	1·38766	7
16·3	8·0618	0	4·40767	1	4·39021	2	5·26712	3	6·96092	4	9·78982	5	1·44065	7
16·4	8·0868	0	4·44804	1	4·45663	2	5·37770	3	7·14720	4	1·01072	6	1·49536	7
16·5	8·1116	0	4·48854	1	4·52366	2	5·48995	3	7·33737	4	1·04331	6	1·55184	7
16·6	8·1362	0	4·52916	1	4·59129	2	5·60389	3	7·53151	4	1·07677	6	1·61014	7
16·7	8·1608	0	4·56990	1	4·65953	2	5·71952	3	7·72967	4	1·11110	6	1·67030	7
16·8	8·1854	0	4·61076	1	4·72839	2	5·83687	3	7·93190	4	1·14634	6	1·73238	7
16·9	8·2098	0	4·65175	1	4·79785	2	5·95595	3	8·13827	4	1·18250	6	1·79642	7
17·0	8·2342	0	4·69286	1	4·86794	2	6·07677	3	8·34884	4	1·21959	6	1·86247	7
17·1	8·2586	0	4·73410	1	4·93864	2	6·19935	3	8·56366	4	1·25764	6	1·93059	7
17·2	8·2828	0	4·77545	1	5·00996	2	6·32370	3	8·78281	4	1·29667	6	2·00083	7
17·3	8·3070	0	4·81692	1	5·08190	2	6·44985	3	9·00634	4	1·33669	6	2·07324	7
17·4	8·3312	0	4·85852	1	5·15447	2	6·57780	3	9·23432	4	1·37773	6	2·14788	7
17·5	8·3552	0	4·90024	1	5·22766	2	6·70758	3	9·46681	4	1·41981	6	2·22481	7
17·6	8·3792	0	4·94207	1	5·30148	2	6·83919	3	9·70387	4	1·46294	6	2·30408	7
17·7	8·4030	0	4·98403	1	5·37592	2	6·97266	3	9·94558	4	1·50715	6	2·38575	7
17·8	8·4268	0	5·02610	1	5·45100	2	7·10799	3	1·01920	5	1·55246	6	2·46989	7
17·9	8·4506	0	5·06830	1	5·52671	2	7·24521	3	1·04432	5	1·59889	6	2·55654	7
18·0	8·4744	0	5·11061	1	5·60305	2	7·38433	3	1·06992	5	1·64645	6	2·64579	7
18·1	8·4980	0	5·15304	1	5·68003	2	7·52537	3	1·09601	5	1·69519	6	2·73768	7
18·2	8·5216	0	5·19559	1	5·75764	2	7·66834	3	1·12260	5	1·74510	6	2·83228	7
18·3	8·5450	0	5·23825	1	5·83589	2	7·81326	3	1·14969	5	1·79623	6	2·92966	7
18·4	8·5684	0	5·28104	1	5·91479	2	7·96014	3	1·17729	5	1·84858	6	3·02989	7
18·5	8·5918	0	5·32394	1	5·99433	2	8·10900	3	1·20541	5	1·90219	6	3·13303	7
18·6	8·6152	0	5·36696	1	6·07451	2	8·25986	3	1·23406	5	1·95708	6	3·23915	7
18·7	8·6384	0	5·41009	1	6·15533	2	8·41273	3	1·26323	5	2·01326	6	3·34833	7
18·8	8·6616	0	5·45334	1	6·23681	2	8·56764	3	1·29295	5	2·07078	6	3·46063	7
18·9	8·6846	0	5·49671	1	6·31894	2	8·72458	3	1·32321	5	2·12964	6	3·57614	7
19·0	8·7076	0	5·54019	1	6·40171	2	8·88359	3	1·35402	5	2·18987	6	3·69492	7
19·1	8·7306	0	5·58378	1	6·48514	2	9·04467	3	1·38540	5	2·25151	6	3·81705	7
19·2	8·7536	0	5·62749	1	6·56923	2	9·20785	3	1·41734	5	2·31457	6	3·94261	7
19·3	8·7764	0	5·67132	1	6·65397	2	9·37314	3	1·44985	5	2·37908	6	4·07168	7
19·4	8·7994	0	5·71526	1	6·73937	2	9·54055	3	1·48295	5	2·44506	6	4·20434	7
19·5	8·8220	0	5·75931	1	6·82543	2	9·71011	3	1·51664	5	2·51255	6	4·34066	7
19·6	8·8448	0	5·80348	1	6·91215	2	9·88183	3	1·55093	5	2·58157	6	4·48074	7
19·7	8·8674	0	5·84776	1	6·99953	2	1·00557	4	1·58582	5	2·65214	6	4·62466	7
19·8	8·8900	0	5·89215	1	7·08758	2	1·02318	4	1·62132	5	2·72430	6	4·77251	7
19·9	8·9124	0	5·93666	1	7·17630	2	1·04101	4	1·65744	5	2·79807	6	4·92437	7
20·0	8·9350	0	5·98128	1	7·26568	2	1·05906	4	1·69419	5	2·87348	6	5·08033	7

Acknowledgments.

The authors are indebted to Mr. JACK BELZER, director of the Battelle computing laboratory at Columbus, for valuable assistance in carrying out the numerical computations.

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