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## Quantum Theory in Real Hilbert Space

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*Abstract.* Relativistic Quantum Theory is brought to a form, where all operators, including time reversal, are linear: Hilbert space is real. Instead of the imaginary number  $i = \sqrt{-1}$ , an operator  $J$  ( $J^2 = -1$ ) is introduced, which commutes with all observables and with the orthogonal operators representing ortho-chronous Lorentz transformations, and anti-commutes with the orthogonal representation of pseudo-chronous Lorentz transformations. It is shown, that  $J$  is necessary in order to have an uncertainty principle (§ 2). Furthermore it follows that momentum-energy and angular momentum-centre of energy are pseudo-chronous quantities. Therefore, the Hamiltonian operator does not change sign under time reversal (§ 5). Lorentz transformations are considered as passive (= coordinate frame-) transformations (§ 7).

In the annexes the following topics are discussed: A possible generalisation of quantum theory involving non linear operators (A-1); The dictionary between conventional theory in complex Hilbert space and the proposed formalism in real Hilbert space (A-2) and (A-3); The dictionary between a quantum theory in quaternion Hilbert space and our real theory (A-4). Also an error, frequently found in literature, concerning the representation of the Lorentz group is pointed out (A-5).

### Introduction and Conclusion

This article presents the essential of the lectures on *Relativistic Quantum Theory (QT) of Fields*, given at the universities of Geneva and Lausanne during the past 20 years. The problem was to show students, *why the imaginary unit enters quantum theory*. We start therefore from a theory built entirely upon *real numbers* and are lead to introduce an operator  $J$  (with  $J^2 = -1$ ), in order to have an *uncertainty principle* (UP) between the *mean square errors*  $\langle \Delta F^2 \rangle$  and  $\langle \Delta G^2 \rangle$  of two observables  $F$  and  $G$ . Observables are *symmetric tensors* (or *symmetric linear operators*) in real Hilbert space (RHS)  $F_{ab} = F_{ba}$ , or

$$F^T = F, G^T = G, \dots \quad (0.1)**)$$

\*) Supported by the Swiss National Research Fund.

\*\*\*)  $A^T$  is the *transposed operator*:  $A_{ab}^T = A_{ba}$ , which plays the analogous role as the *hermitian conjugate*  $\hat{A}^\dagger$  ( $\hat{A}_{pq}^\dagger = \hat{A}_{qp}$ \*) in *Complex Hilbert Space* (CHS), see Annex (A-2).

The criterium for the impossibility of measuring  $F$  and  $G$  simultaneously, is a non vanishing commutator

$$FG - GF = [F, G] = - [F, G]^T \neq 0. \quad (0.2)$$

The expectation value  $\langle [F, G] \rangle$  vanishes, because  $[F, G]$  is an *antisymmetric tensor*. Therefore, only the *positive definit observable*

$$P = - [F, G]^2 = P^T \quad (0.3)$$

can occur in

$$\langle \Delta F^2 \rangle \langle \Delta G^2 \rangle \geq \lambda^2 \langle P \rangle. \quad (0.4)$$

$\lambda$  is a real number. *Unless otherwise mentioned* (Annexes (A-2), (A-3) and (A-4)) *all numbers occurring in this paper are real*.

We show, that this uncertainty principle leads to a contradiction, unless  $\lambda^2 = 0$ , in which case (0.4) is a triviality. We show, in § 2, that the only other possibility consists in introducing an *antisymmetric operator*  $J_{(FG)}$  which has an inverse  $J_{(FG)}^{-1}$  (and may therefore, without loss of generality, be normalised to  $-1$ ) and which commutes with  $F$  and  $G$ .

$$J_{(FG)}^T = - J_{(FG)}, \quad J_{(FG)}^2 = - J_{(FG)}^T J_{(FG)} = -1 \quad (0.5)$$

$$[J_{(FG)}, F] = [J_{(FG)}, G] = 0. \quad (0.6)$$

We may now form the symmetrical operator

$$C_{(FG)} = J_{(FG)} [F, G] = C_{(FG)}^T \quad (0.7)$$

and expect a uncertainty principle of the form

$$\langle \Delta F^2 \rangle \langle \Delta G^2 \rangle \geq \lambda^2 \langle C_{(FG)} \rangle^2. \quad (0.8)$$

Let  $H$  be a third observable.  $C_{(FG)}$  being an observable,  $J_{(CH)}$  has to commute with  $C_{(FG)}$  and  $H$ . Thus, the simplification to assume but one universal

$$J_{(FG)} = J \quad (0.9)$$

*commuting with all observables*, seems natural.

FINHELSTEIN, JAUCH and SPEISER<sup>3)</sup> have shown that only three possibilities: RHS, CHS (Complex Hilbert Space) and QHS (Quaternion Hilbert Space) are possible in Quantum theory (QT). Thus three anti-commuting  $J$ 's ( $J_1, J_2, J_3$ ) may exist. We have analysed QHS in terms of RHS in the annex (A-4).

We begin (§ 1) by an analysis of probability, which leads us necessarily to RHS. The linearity of the operators is a further assumption, which

may eventually be omitted, because classical statistical mechanics does not necessarily require the Jakobeian cyclic identity, but may give

$$\overrightarrow{FGH} \{F, \{G, H\}\} \neq 0 \tag{0.10)*}$$

for the generalised Poisson brackets (see annex (A-1) and 2)).

Therefore, the corresponding identity for linear operators

$$\overrightarrow{FGH} J [F, J [G, H]] = 0 \tag{0.10J}$$

may not hold. In § 2 we discuss the uncertainty principle (UP). In § 3 we introduce the representation of the linear group {L} (which leaves the metric tensor  $g^{\alpha\beta} = g^{(\alpha\beta)**}$  invariant) by orthogonal operators O in RHS:

$$O^T = O^{-1}, \quad L \rightarrow e^{J \lambda} O. \tag{0.11}$$

In § 4 we show, that the metric  $g^{\alpha\beta}$  of the differential manifold  $x = \{x^\alpha\}$ ,  $\alpha \beta \dots = 1 \ 2 \ \dots \ n$ , has necessarily the *thermodynamic signature* (STUECKELBERG and WANDERS<sup>4)5)</sup>) if the existence of *fundamental state*  $\Psi^{(0)}$ , the *vacuum*, is postulated:

$$\text{signat } (g^{\alpha\beta}) = \pm (1 \ 1 \ \dots \ 1 \ - \ 1). \tag{0.12}$$

This gives a preference to one coordinate  $x^n = t$ , the time. Thus, {L} is the *full Lorentz group in n-dimensional space* (including *time reversal*  $L = T$ ).

Furthermore it is shown, that  $J = \overset{\smile}{J}$  is a *pseudochronous operator*

$$\overset{\smile}{J} = O^{-1} \overset{\smile}{J} O = \text{sig } (L'^n_n) \overset{\smile}{J} \tag{0.13)***)}$$

if  $\overset{\smile}{x}'^\alpha = L'^\alpha_\alpha (x^\alpha + L^\alpha) : \overset{\smile}{x} = Lx; \det (L'^\alpha_\alpha) \neq 0 \tag{0.14)****)}$

satisfies  $g'^{\alpha'\beta} = L'^\alpha_\alpha L'^\beta_\beta g^{\alpha\beta}. \tag{0.15)}$

*Multilocal ortho-chronous observables* transform according to

$$\left. \begin{aligned} \overset{\smile}{F}'^{\alpha'\beta\dots} (\overset{\smile}{x}' y \dots) &= L'^\alpha_\alpha L'^\beta_\beta \dots F^{\alpha\beta\dots} (L^{-1} \overset{\smile}{x} L^{-1} \overset{\smile}{y} \dots) \\ &= O^{-1} \overset{\smile}{F}'^{\alpha'\beta\dots} (\overset{\smile}{x}' y \dots) O \end{aligned} \right\} \tag{0.16}$$

while *pseudo-chronous quantities* transform according to

$$\overset{\smile}{F}'^{\alpha\dots} (\overset{\smile}{x} \dots) = \text{sig } (L'^n_n) L'^\alpha_\alpha \dots \overset{\smile}{F}^\alpha (L^{-1} \overset{\smile}{x} \dots) = O^{-1} \overset{\smile}{F}'^{\alpha\dots} (\overset{\smile}{x} \dots) O \tag{0.17F)}$$

\*)  $\overrightarrow{ABC}$  stands for the cyclic sum.

\*\*\*)  $F^{(\alpha\beta\gamma\dots)}$  is a totally symmetric tensor, while  $F^{[\alpha\beta\gamma\dots]}$  is a totally antisymmetric tensor in  $\alpha$ -space.

\*\*\*\*)  $\text{sig } (\lambda)$  is the sign function  $\text{sig } (\lambda) = \pm 1$  for  $\lambda \gtrless 0$ .

\*\*\*\*\*) Frame transformations are written with the primes to the left:

$\overset{\smile}{x}'^\alpha \leftarrow x^\alpha; \overset{\smile}{\Psi}'_a \leftarrow \Psi_a.$

We shall later\*) also use *pseudo-chorous quantities*, which transform according to ( $ik \dots = 1 \ 2 \dots n - 1$ )

$$\left. \begin{aligned} \widehat{F}'^{\alpha \dots} ('x \dots) &= \text{sig}(\det(L'^i)) L'^{\alpha}_{\alpha} \dots \widehat{F}^{\alpha \dots}(L^{-1} 'x \dots) \\ &= O^{-1} \widehat{F}^{\alpha \dots} ('x \dots) O \end{aligned} \right\} (0.17\widehat{F})$$

and finally *pseudo-quantities*  $\overset{\circ}{F} = \overset{\circ}{F}$ :

$$\left. \begin{aligned} \overset{\circ}{F}'^{\alpha \dots} ('x \dots) &= \text{sig}(\det(L'^{\alpha}_{\alpha})) L'^{\alpha}_{\alpha} \dots \overset{\circ}{F}^{\alpha \dots}(L^{-1} 'x \dots) \\ &= O^{-1} \overset{\circ}{F}^{\alpha \dots} ('x \dots) O. \end{aligned} \right\} (0.17\overset{\circ}{F})$$

Let us remark, that we consider (§§ 3, 4 and 5)  $L$  always as *passive transformations*. As a matter of fact, we show (§ 7), that this *passive point of view* is perfectly reasonable in QT, because a statistical analysis of observations at two epochs  $t'$  and  $t''$ , is independent of whether  $t''$  is later or earlier than  $t'$  in the *thermodynamic time scale* ( $\overset{\circ}{S}(t'') > \overset{\circ}{S}(t')$ ;  $\overset{\circ}{S}(t) = \text{entropy} > 0$ , at epoch  $t$ , cf. 4) and 6)7)).

In § 5 we analyse the infinitesimal group  $L(\delta \lambda \cdot \delta \omega^{[\dots]})$

$$'x'^{\alpha} = x'^{\alpha} + \delta \lambda'^{\alpha} + \delta \omega'^{\alpha}_{\alpha} x^{\alpha} = x'^{\alpha} + \delta \lambda'^{\alpha} + \frac{1}{2} \delta \omega^{[\mu\nu]} \Sigma_{\mu\nu}{}^{\alpha}_{\alpha} x^{\alpha} \quad (0.18)$$

$$\Sigma_{\mu\nu}{}^{\alpha}_{\alpha} = \Sigma_{[\mu\nu]}{}^{\alpha}_{\alpha} = \delta'_{\mu}{}^{\alpha} g_{\nu\alpha} - \delta'_{\nu}{}^{\alpha} g_{\mu\alpha} \quad (0.19)$$

generating the continuous group  $\{L_{(\text{cont})}\}$ . The generators of the corresponding Lie group  $\{O\}$ , with  $n + (1/2)n(n-1)$ , parameters  $\lambda^{\mu}$  and  $\omega^{\mu\nu} = \omega^{[\mu\nu]}$  are  $-\overset{\circ}{J} \overset{\circ}{\Pi}_{\mu}$  and  $\overset{\circ}{J} \overset{\circ}{M}_{\mu\nu} = \overset{\circ}{J} \overset{\circ}{M}_{[\mu\nu]}$ . The pseudo-chorous observables are the *pseudo-chorous momentum-energy vector*  $\overset{\circ}{\Pi}_{\mu}$  and the *pseudo-chorous angular momentum-centre of energy tensor*  $\overset{\circ}{M}_{\mu\nu}$ . We arrive at the relation

$$(\partial_{\mu}^x + \partial_{\mu}^y + \dots) F^{\alpha \dots}(xy \dots) = -\overset{\circ}{J} [\overset{\circ}{\Pi}_{\mu}, F^{\alpha \dots}(xy \dots)] \quad (0.20)$$

$$\left. \begin{aligned} &(([\partial_{\mu}^x, \partial_{\nu}^y] + [\partial_{\mu}^y, \partial_{\nu}^x] + \dots) \delta_{\alpha'}^{\alpha} \delta_{\beta'}^{\beta} \dots + \Sigma_{\mu\nu}{}^{\alpha}_{\alpha'} \delta_{\beta'}^{\beta} \dots \\ &+ \delta_{\alpha'}^{\alpha} \Sigma_{\mu\nu}{}^{\beta}_{\beta'} \dots + \dots) F^{\alpha' \beta' \dots}(xy \dots) = -\overset{\circ}{J} [\overset{\circ}{M}_{\mu\nu}, F^{\alpha \beta \dots}(xy \dots)]. \end{aligned} \right\} (0.21)$$

From the structure relation of the generators of  $\{L_{(\text{cont})}\}$  ( $-\partial_{\mu}$  and  $N_{\mu\nu}{}^{\alpha}_{\alpha'} = [\partial_{\mu}^x, \partial_{\nu}^y] \delta_{\alpha'}^{\alpha} + \Sigma_{\mu\nu}{}^{\alpha}_{\alpha'}$ ), the commutation relations

$$\overset{\circ}{J} [\overset{\circ}{\Pi}_{\mu}, \overset{\circ}{\Pi}_{\nu}] = 0 \quad (0.22)$$

\*) In a following article on *real representations of the spinor group*  $\{\pm \delta_B^A, \pm \gamma^{\alpha A}_B, \pm \gamma^{[\alpha, \alpha_2] A}_B, \dots, \pm \gamma^{[\alpha, \dots, \alpha_n] A}_B\}$ .

$$\check{J}[\check{M}_{\mu\nu}, \check{M}_{\sigma\tau}] = -g_{\mu\sigma} \check{M}_{\nu\tau} - g_{\nu\tau} \check{M}_{\mu\sigma} + g_{\mu\tau} \check{M}_{\nu\sigma} + g_{\nu\sigma} \check{M}_{\mu\tau} \quad (0.23)$$

$$\check{J}[\check{\Pi}_\mu, \check{M}_{\sigma\tau}] = g_{\mu\sigma} \check{\Pi}_\tau - g_{\mu\tau} \check{\Pi}_\sigma \quad (0.24)$$

follow\*).

In § 6, we show, that  $\check{\Pi}_\mu$  and  $\check{M}_{\mu\nu}$  can be expressed in terms of an ortho-chronous observable, the *momentum energy tensor*

$$\Theta^{\alpha\beta}(x) = \Theta^{(\alpha\beta)}(x), \quad \partial_\alpha \Theta^{\alpha\beta}(x) = 0 \quad (0.25)$$

as integrals over a surface  $\tau(x) = 0$ , whose surface element  $d\check{\sigma}_\alpha(x)$  is a *time-like pseudo-chronous vector* ( $\text{signat}(g^{\alpha\beta}) = (1, 1 \dots 1 - 1)$ )

$$d\check{\sigma}_\alpha(x) d\check{\sigma}^\alpha(x) < 0; \quad d\check{\sigma}_n(x) > 0 \quad (0.26)$$

$$\check{\Pi}^\mu = \int_{\tau(x)=0} d\check{\sigma}_\alpha \Theta^{\alpha\mu}(x); \quad \check{M}^{\mu\nu} = \int_{\tau(x)=0} d\check{\sigma}_\alpha (x^\mu \Theta^{\alpha\nu} - x^\nu \Theta^{\alpha\mu})(x). \quad (0.27)$$

In the annexes, we consider *classical statistical mechanics* with  $F_{GH} \{F, \{G, H\}\} \neq 0$  (A-1), *hermitian CHS* (A-2), *unitary and antiunitary transformations*  $U$  and  $V$  in CHS (A-3), *quaternion Hilbert space (QHS)* in (A-4), and an *error frequently found in literature* due to a wrong definition of the representation  $O$  (in RHS) or  $U$  (in CHS) (A-5).

### § 1. Analysis of Probability

Let  $F$  and  $G$  be two observables, whose spectra, assumed discrete, are

$$F: \{F^{(i)}\} = \{F^{(1)} < F^{(2)} < \dots < F^{(i)} < \dots < F^{(\omega_F)}\} \quad (1.1F)$$

$$G: \{G^{(k)}\} = \{G^{(1)} < G^{(2)} < \dots < G^{(k)} < \dots < G^{(\omega_G)}\} \quad (1.1G)$$

and let  $W^{(i)}$ , resp.  $W^{(k)}$  be the probabilities that  $F$  takes the value  $F^{(i)}$  (resp.  $G$  the value  $G^{(k)}$ )

$$W^{(i)}, W^{(k)} > 0 \quad \sum_i W^{(i)} = \sum_k W^{(k)} = 1. \quad (1.2)$$

Then we may, without loss of generality, write  $W^{(i)}$  as a sum of squares

$$W^{(i)} = \sum_{\alpha=1}^{\alpha=\omega} \Psi_{i\alpha}^2; \quad W^{(k)} = \sum_{\beta=1}^{\beta=\omega_k} \Psi_{k\beta}^2, \quad (1.3)$$

which introduces an  $\omega_i$ - (resp.  $\omega_k$ -) fold degeneracy of the spectral term  $F^{(i)}$  (resp.  $G^{(k)}$ ). Now let us introduce two indices  $a$  and  $a'$ :

\*) Due to a wrong sign in the representation  $L \rightarrow e^{\check{J}\lambda O}$ , these commutations relation are frequently wrong in several books on QT of fields (see Annex (A-5)).

$$\left. \begin{aligned} a\ b \dots &= 1\ 2 \dots \sum \omega_i = 1\ 2 \dots \omega_R \\ 'a\ 'b \dots &= '1\ '2 \dots \sum ' \omega_k = '1\ '2 \dots ' \omega_R \end{aligned} \right\} \quad (1.4)$$

and write the degeneracies in the form

$$F^{(a)} = F^{(i\alpha)}, \quad F^{(i\alpha)} = F^{(i)} \quad G^{(a)} = G^{(k\beta)}, \quad G^{(k\beta)} = G^{(k)}. \quad (1.5)$$

Now we may chose the arbitrary large numbers  $\omega_R$  and  $'\omega_R$  equal, and represent  $\Psi_a = \Psi_{i\alpha}$ , and  $'\Psi_a = '\Psi_{k\beta}$  as components of *the same abstract vector  $\Psi$  (state vector)*, referred to *two different orthogonal coordinate frames* in an *Euclidian space* of  $\omega_R = '\omega_R$  dimensions. In general, this number  $\omega_R$  will be infinite. Therefore, we call this space the *Real Hilbert Space* (RHS). The two sets of components are related to each other by an orthogonal matrix  $O = \{O_{aa}\}$ .

Using the summation convention, we write:

$$' \Psi_a = O_{aa} \Psi_a; \quad ' \Psi = O \Psi; \quad O^T = O^{-1} \quad (1.6)$$

the expectation values are now

$$\langle F \rangle_{\Psi} = \sum_i W^{(i)} F^{(i)} = \Psi_a F_{ab} \Psi_b \equiv (\Psi, F \Psi) \quad (1.7)$$

$$F_{ab} = F^{(a)} \delta_{ab} \quad (1.8)$$

$$\langle G \rangle_{\Psi} = \sum_k W^{(k)} G^{(k)} = ' \Psi_a ' G_{a'b} ' \Psi_b = \Psi_a G_{ab} \Psi_b \equiv (\Psi, G \Psi) \quad (1.9)$$

$$' G_{a'b} = G^{(a)} \delta_{a'b}; \quad G_{ab} = O_{a'a}^T ' G_{a'b} O_{bb} \quad (1.10)$$

where

$$(\Phi, \Psi) = (\Psi, \Phi) = \Phi_a \Psi_a = ' \Psi_a ' \Phi_a \quad (1.11)$$

is the scalar product between *vectors in RHS*.  $F, G$  and all observables are *symmetrical tensors in RHS*:

$$F^T = F, \quad G^T = G, \quad H^T = H, \dots \quad (1.12)$$

In the  $a$ -frame,  $F$  is diagonal (1.8) and in the  $'a$ -frame  $'G$  is diagonal (1.10). The transposed operator of an operator  $A, A = \{A_{ab}\}$  is defined by

$$(\Phi, A \Psi) = (A^T \Phi, \Psi); \quad A_{ab}^T = A_{ba}. \quad (1.13)$$

Now,  $F$  and  $G$  are *two tensor ellipsoids in RHS or a-space*: The length of their principal axes are given by the spectra (1.1F) and (1.1G). The length of the axes are thus independent of the orientation of the  $a$ -space vector  $\Psi$ . However the *relative orientation of the two ellipsoids  $F$  and  $G$*  i. e. the *relative orientation of the  $a$ -frame and  $'a$ -frame* in abstract RHS is not necessarily independent of  $\Psi$ . Thus,  $O = \{O_{aa}\}$  may depend on  $\Psi$ . This introduces the possibility of assuming  $F$  and  $G$  to be more general operators than linear ones<sup>2)</sup> (see (0.10) (0.10J) and Annex (A-1)).

§ 2. The Uncertainty Principle

In order to express the uncertainty principle (UP), we introduce in (0.4) the error operators

$$\Delta F = F - 1 \langle F \rangle_\Psi; \quad \Delta G = G - 1 \langle G \rangle_\Psi \tag{2.1}$$

from which we form the *mean square errors*  $\langle \Delta F^2 \rangle_\Psi$  and  $\langle \Delta G^2 \rangle_\Psi$  in (0.4). There are two possibilities:

$$\langle \Delta F^2 \rangle_\Psi \langle \Delta G^2 \rangle_\Psi \geq \begin{cases} \lambda^2 \langle P \rangle_\Psi & (2.2P) \\ \lambda^2 \langle C \rangle_\Psi^2 & (2.2C) \end{cases}$$

where  $P$  is a positive observable of dimension  $[F]^2[G]^2$  and  $C$  is an observable of dimension  $[F][G]$ .  $\lambda$  is a number to be determined.

Let us demonstrate, that the first choice (0.4) or (2.2P) leads to a contradiction: We express (2.2P) or (0.3), (0.4) in the  $a$ -frame, where  $F$  is diagonal. Then, if  $[F, G] \neq 0$ ,  $G$  has nondiagonal elements in this frame. Suppose further that  $F$  has the value  $F^{(a)}$  i.e.  $\Psi_a = \Psi'_a = \pm \delta_{aa}$ . Then we have

$$[F, G]_{ab} = (F^{(a)} - F^{(b)}) G_{ab}; \quad G_{ab} \neq 0 \tag{2.3}$$

and (on account of  $G^T = G$ )

$$P_{ab} = - [F, G]_{ab}^2 = \sum_c (F^{(a)} - F^{(c)}) (F^{(b)} - F^{(c)}) G_{ac} G_{bc}. \tag{2.4}$$

Therefore the expectation value is

$$\langle P \rangle_{\Psi'} = \sum_c (F^{(a)} - F^{(c)})^2 (G_{a'c})^2 = (\text{finite}')^2 > 0. \tag{2.5}$$

Now  $\langle \Delta F^2 \rangle_{\Psi'} = 0$ . Let the spectre of  $G$  (1.1 G) be bounded. Then we have  $\langle \Delta G^2 \rangle_{\Psi'} \leq (G^{(1)} - G^{(\omega G)})^2 = (\text{finite})^2$  and (2.2P) (or (0.4)) reads

$$0 \cdot (\text{finite})^2 \geq \lambda^2 (\text{finite}')^2 \tag{2.6}$$

which has only the trivial solution  $\lambda = 0$ , corresponding to the trivial statement

$$\langle \Delta F^2 \rangle_\Psi \langle \Delta G^2 \rangle_\Psi \geq 0 \tag{2.7}$$

The only other possibility, (2.2C) is to introduce an observable  $C$ , linear in  $F$  and linear in  $G$ . This implies the existence of an antisymmetric tensor in  $a$ -space  $J_{(FG)} = - J_{(FG)}^T$  commuting with  $F$  and  $G$ :

$$C = J_{(FG)} [F, G] = C^T \tag{2.8}$$

In order to deduce the UP, we form, with an arbitrary number  $\xi$ ,

$$\begin{aligned} |(\Delta F + \xi J \Delta G) \Psi|^2 &= (\Psi, (\Delta F - \xi \Delta G J) (\Delta F + \xi J \Delta G) \Psi) \\ &= \langle \Delta F^2 \rangle_\Psi - \langle J^2 \Delta G^2 \rangle_\Psi \xi^2 + \langle J [F, G] \rangle_\Psi \xi \\ &= f(\xi) \geq f_{\min} (\xi') \geq 0. \end{aligned} \tag{2.9)*}$$

\*) We have written  $J$  for  $J_{(FG)}$ .



In order to make appear  $\langle \Delta G^2 \rangle_\Psi$  in (2.9), it is necessary that  $J_{(FG)}^2$  is a number  $\neq 0$ . Being antisymmetric, this number must be negative. As  $\xi$  is an arbitrary number, we lose no generality in normalising  $J_{(FG)}^2 = -1$ . Thus, we arrive at the conditions (0.5) and (0.6).

The minimum  $f_{\min}(\xi')$  of  $f(\xi)$  is easily found to be at

$$\xi' = -1/2 \langle J_{(FG)} [F, G] \rangle_\Psi \langle \Delta G^2 \rangle_\Psi^{-1} \quad (2.10)$$

which we insert in the last inequality (2.9). Multiplying with  $\langle \Delta G^2 \rangle_\Psi$ , we find the inequality (0.8) with  $\lambda^2 = 1/4$ . Assuming but one universal  $J$  (see text following (0.8)), we arrive at the UP:

$$\langle \Delta F^2 \rangle_\Psi \langle \Delta G^2 \rangle_\Psi \geq \frac{1}{4} \langle J[F, G] \rangle_\Psi^2. \quad (2.11)$$

### § 3. The linear inhomogenous group $\{L\}$ in $x$ -space

$\{L\}$  is defined by its general element  $L$  (0.14) and the condition (0.15), stipulating the invariance of the metric tensor  $g^{\alpha\beta}$ . A classical observable transforms according to (0.16) or (0.17). It will be useful to combine the indices'

$$\{\alpha \beta \dots x y \dots\} \equiv X \quad (3.1)$$

and define

$$\left. \begin{aligned} 'F^X &\equiv 'F'^{\alpha\beta\dots} ('x'y\dots) = L^X_X F^X \\ &= L'^\alpha_\alpha L'^\beta_\beta \dots \int d^n x \delta('x - Lx) \int d^n y \delta('y - Ly) \dots F^{\alpha\beta\dots} (xy\dots) \\ &= L'^\alpha_\alpha L'^\beta_\beta \dots F^{\alpha\beta\dots} (L^{-1}'x L^{-1}'y\dots). \end{aligned} \right\} (3.2)$$

Let us now consider how the expectation value

$$\langle F^{\alpha\dots} (x\dots) \rangle_\Psi = (\Psi, F^{\alpha\dots} (x\dots) \Psi) = \Psi_a F^X_{ab} \Psi_b \quad (3.3)$$

transforms under  $\{L\}$ . There are *two possibilities*: Either we leave  $\Psi_a$  *unchanged* and write

$$\langle 'F'^{\alpha\dots} ('x\dots) \rangle_\Psi = L'^\alpha_\alpha \dots (\Psi, F^{\alpha\dots} (L^{-1}'x\dots) \Psi) \quad (3.4)$$

which expresses the fact that  $'F^X$  is the *transformed operator* (3.2) (0.16) (0.17). Or, we may express the transformed expectation value in terms of the initial operator  $F^X$  with the index  $'X = (\alpha\dots 'x\dots)$  and in terms of a *transformed vector*  $'\Psi_a$

$$'\Psi_a = O_{aa} \Psi_a = \Psi_a O_a^T \quad (3.5)$$

in the form

$$\langle 'F'^{\alpha\dots} ('x \dots) \rangle_{\Psi} = \langle F'^{\alpha\dots} ('x \dots) \rangle_{\Psi} = ' \Psi'_a F'^X_{a'b} ' \Psi'_b. \tag{3.6}$$

Equating (3.4) and (3.6) we obtain an identity

$$\Psi_a L'^{\alpha}_{\alpha} \dots F'^{\alpha\dots}_{ab} (L^{-1} 'x \dots) \Psi_b \equiv \Psi_a O^T_{a'a} F'^{\alpha\dots}_{a'b} ('x \dots) O_{b'b} \Psi_b \tag{3.7}$$

between two quadratic forms in  $\Psi_a$ . Or, these forms are equal, for an arbitrary  $\Psi$ , if and only if\*

$$L'^{\alpha}_{\alpha} \dots F'^{\alpha\dots}_{ab} (L^{-1} 'x \dots) = O^T_{a'a} F'^{\alpha\dots}_{a'b} ('x \dots) O_{b'b}. \tag{3.8}$$

We may write this identity (multiplying by  $O \dots O^{-1}$ )

$$F'^X_{a'b} = L'^X_X O_{a'a} F^X_{ab} O^T_{b'b} \tag{3.9}$$

Thus  $F^X_{ab}$  considered as a 'vector' in  $X = \{\alpha \dots x \dots\}$ -space and as a symmetric tensor in  $a$ -space, is left invariant, if it is transformed with respect to its three indices  $X$  and  $a, b$ . This is in perfect analogy to metric tensor  $g^{\alpha\beta}$  in (0.15) and to the  $\alpha$ -vector mixed bispinor  $\gamma^{\alpha A}_B$  (cf.<sup>1</sup>) and an article, to appear in this journal, on Spinor Calculus in RHS).

Now it is easily seen, that the  $\{O\}$  group is a ray representation of the  $\{L\}$ -group: Write  $L_{(1)}$  and  $O^{(1)}$  in (3.9) and consider a second transformation  $L_{(2)}$  and  $O^{(2)}$  leading from the frames 'X 'a to the frames "X "a:

$$F''^X_{a''b} = L''^X_{(2)'X} O^{(2)'}_{a'a} O^{(2)''}_{b'b} F'^X_{a'b} \tag{3.10}$$

and substitute (3.9) in (3.10): From

$$F''^X_{a''b} = (L_{(2)} L_{(1)})''^X_X (O^{(2)} O^{(1)})''_{a'a} (O^{(2)} O^{(1)})''_{b'b} F^X_{ab} \tag{3.11}$$

it is seen that  $L \rightarrow O$  and  $L \rightarrow (-O)$  is a two valued representation. However, since  $J$  commutes with any observable  $F, G$ , we may write

$$L \rightarrow e^{\lambda J} O, L^{-1} \rightarrow O^{-1} e^{-\lambda J} \tag{3.12}$$

where the number  $\lambda$  is an arbitrary phase. We see, that  $O$  does not necessarily commute with  $J$ .

To illustrate the identity

$$\langle 'F'^X \rangle_{\Psi} = \langle F'^X \rangle_{\Psi}$$

we have drawn the Fig. 1 and 2:

\*) In CHS, where  $\langle \widehat{\Psi}, \widehat{A} \widehat{\Psi} \rangle$  is a complex number (cf. Annex (A-2)) the unitary transformation  $U^\dagger = U^{-1}$  replaces  $O^T = O^{-1}$ . Thus we have two real identities, and the condition

$$L'^{\alpha}_{\alpha} \widehat{A}^{\alpha}_{pq} (L^{-1} 'x) = U^{-1}_{p'p} \widehat{A}^{\alpha}_{p'q} ('x) U_{q'q} \tag{3.8 \widehat{A}}$$

is valable for all operators  $\widehat{A}$ , whether hermitian or not.

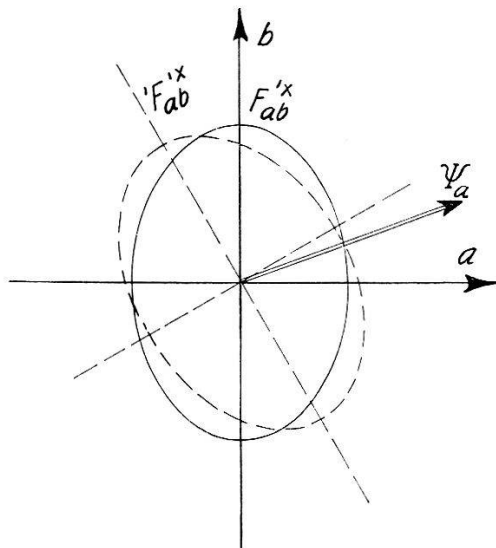


Fig. 1  
a-frame

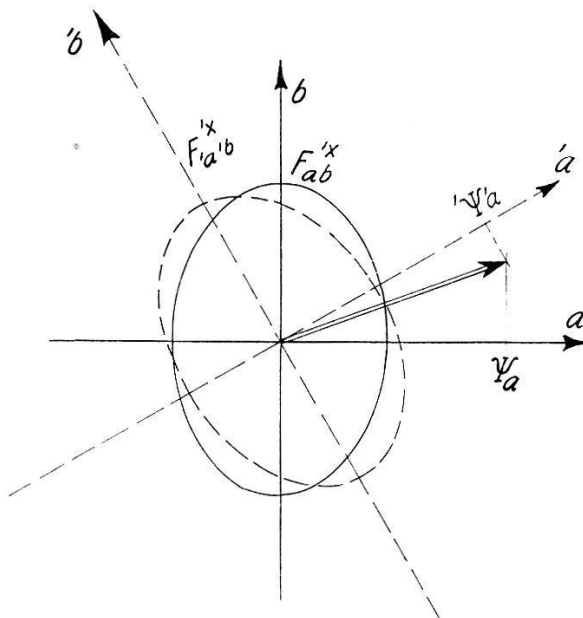


Fig. 2  
a-frame and 'a'-frame

Either (Fig. 1), we form  $\Psi_a 'F_{ab}^X \Psi_b$  in the *a-frame* from the ( $'X \leftarrow X$ )-transformed tensor

$$'F_{ab}^X = L^X_X F_{ab}^X$$

(which has, on account to the  $\{L\}$ -invariance, the same length of the principal axes as the untransformed tensor  $F_{ab}^X$ ) with respect to  $\Psi_a$ .

Or (Fig. 2), we form  $'\Psi_a F_{a'b}^X \Psi_b$  in the '*a-frame*' from the tensor  $F_{a'b}^X$  (which has, in the '*a-frame*', the same components as  $F_{ab}^X$  has in the *a-frame*) with respect to  $'\Psi_a$ .

From the two figures follwes immediately:

$${}'\Psi_{,a} = O_{,aa} \Psi_a; F_{,a'b}^X = O_{,aa} O_{,bb} {}'F_{ab}^X \tag{3.13}$$

or 
$${}'F_{ab}^X = L^X_X F_{ab}^X = O_{,a'a}^{-1} F_{,a'b}^X O_{,bb}; \tag{3.14}$$

$${}'F^{\alpha\dots} ({}'x \dots) = L^{\alpha}_{\alpha} \dots F^{\alpha\dots} (L^{-1} {}'x \dots) = O^{-1} F^{\alpha\dots} ({}'x \dots) O$$

**§ 4. The Thermodynamic Signature of  $g^{\alpha\beta}$  and the Pseudo-Chronous Character of  $J = \overset{\smile}{J}$**

We need the hypotheses that a *particular state*  $\Psi^{(0)}$  of the cosmos exists, the vacuum, which is *homogeneous and isotropic*. Let us consider, for the simplicity, a *local scalar observable*  $F(x)$ , and write the UP for two events  $x$  and  $y$  in the form

$$\langle J [F(x), F(y)] \rangle_{\Psi^{(0)}} \equiv \overset{\smile}{f}(x y) = - \overset{\smile}{f}(y x). \tag{4.1}$$

Homogeneity requires:

$$\overset{\smile}{f}(x y) = \overset{\smile}{f}(x - y). \tag{4.2}$$

Isotropy would further require:

$$\overset{\smile}{f}(x y) = \overset{\smile}{f}((x - y)^2), \tag{4.3}$$

$$(x - y)^2 = (x - y)_{\alpha} (x - y)^{\alpha}. \tag{4.4}$$

However, (4.3) is in contradiction with the antisymmetry (4.1) of the commutator. There is only one way to turn this difficulty: We have to give to the differentiable manifold  $\{x^{\alpha}\}$  *one privileged axis*,  $x^n = t$ , the time. By this we understand that the metric has the *thermodynamic signature*:

$$\text{signat } (g^{\alpha\beta}) = \pm (1 1 \dots 1 - 1). \tag{4.5}$$

We have shown in an earlier paper<sup>4)</sup>, that this signature is necessary for a phenomenological relativistic thermodynamics. Thus we may define a function

$$\left. \begin{aligned} \overset{\smile}{f}(x y) &= \text{sig } (x^n - y^n) f((x - x)^2) \\ f((x - y)^2) &= 0 \quad \text{for } x - y = \text{spatial} \end{aligned} \right\} \tag{4.6}$$

which is homogeneous and ,quasi-isotropic'. Now it is easily seen, that  $\overset{\smile}{f}$  is a pseudo-chronous bilocal scalar

$${}'\overset{\smile}{f}({}'x {}'y) = \text{sig } (L^n_n) \overset{\smile}{f}(L^{-1} {}'x L^{-1} {}'y) \tag{4.7}$$

because, for  $L^n_n > 0$ , we have

$${}'\overset{\smile}{f}({}'x {}'y) = \overset{\smile}{f}(L^{-1} {}'x L^{-1} {}'y) = \overset{\smile}{f}(x y); \quad L^n_n > 0 \tag{4.8}$$

while, for  $L'^n_n < 0$ , the relation ( $x = L^{-1} 'x$ ) is

$$\left. \begin{aligned} \overset{\smile}{f}('x 'y) &= - \text{sig} (x^n - y^n) f((x - y)^2) \\ &= - \overset{\smile}{f}(x y); \quad L'^n_n < 0. \end{aligned} \right\} \quad (4.9)$$

Thus we may write, using the homogeneity and the pseudo-chronous isotropy of  $\overset{\smile}{f}$ :

$$\overset{\smile}{f}('x 'y) = \text{sig} (L'^n_n) \overset{\smile}{f}(x y). \quad (4.10)$$

Now, consider the transformed value of the observable  $J[F(x), F(y)]$ , which is, according to (3.5) and (3.6), the expectation value with respect to  $\Psi^{(0)}$  of

$$\left. \begin{aligned} O^{-1} (J [F('x), F('y)]) O \\ = O^{-1} J O ['F('x), 'F('y)] \end{aligned} \right\} \quad (4.11)$$

in  $\langle O^{-1} J O ['F('x), 'F('y)] \rangle_{\Psi^{(0)}} = \overset{\smile}{f}('x 'y) \quad (4.12)$

or  $'F('x) = F(L^{-1} 'x) = F(x)$ , according to (0.16). Thus, making use of the relation (4.10), we obtain

$$\langle (O^{-1} J O) [F(x), F(y)] \rangle_{\Psi^{(0)}} = \text{sig} (L'^n_n) \overset{\smile}{f}(x y). \quad (4.13)$$

Comparing this relation to (4.1), we find (0.13):

$$J \equiv \overset{\smile}{J}; \quad O^{-1} \overset{\smile}{J} O = \text{sig} (L'^n_n) \overset{\smile}{J} \equiv 'J. \quad (4.14)$$

(4.14) defines the transformed operator  $'J$ :

The transformation law  $\overset{\smile}{J} \rightarrow 'J$  is now analogous to the law (0.17) for an  $x$ -independent operator. Note however, that  $'J$  is but a *definition*, because we have established the identities expressed in the second equation (0.16) (and (0.17)) by comparing (3.4) and (3.6) in (3.7) only for observables, i. e. for symmetric  $a$ -space tensors  $F, G...$  and not for antisymmetric  $a$ -space tensors like  $\overset{\smile}{J}$ .

### § 5. Infinitesimal Lorentz transformations

After having introduced the pseudo-euclidian signature with one privileged axis  $x^n = t$  in § 4, our group  $\{L\}$  is the *full Lorentz-group in  $n$  dimensions*. Writing down the infinitesimal transformation  $L = L(\delta\lambda \cdot \delta\omega^{[\cdot\cdot]})$  ((0.18) (0.19)) we find

$$\left. \begin{aligned} L'^\alpha_\alpha F^\alpha(L^{-1} 'x) &= (\delta'^\alpha_\alpha + \frac{1}{2} \delta\omega^{\mu\nu} \Sigma_{\mu\nu}{}'^\alpha_\alpha) F^\alpha('x \cdot - \delta\lambda \cdot - \delta\omega \cdot \nu 'x^\nu) \\ &= F'^\alpha('x) + \delta\lambda^\mu (- ' \partial_\mu) F'^\alpha('x) + \frac{1}{2} \delta\omega^{\mu\nu} 'N_{\mu\nu}{}'^\alpha_\alpha F^\alpha('x) \end{aligned} \right\} \quad (5.1)$$

where  $-\partial_\mu$  and

$$N_{\mu\nu}{}^\alpha = [x_\mu, \partial_\nu] \delta^\alpha_\nu + \Sigma_{\mu\nu}{}^\alpha \tag{5.2}$$

are the generators of the  $n + (1/2)n(n-1)$ -parameter Lie-group  $\{L_{(cont)} = L(\lambda, \omega^{[\cdot\cdot]})\}$ , which is the continuous subgroup of  $\{L\}$ . The generators satisfy the Lie structure relations:

$$[-\partial_\mu, -\partial_\nu] = 0 \tag{5.3}$$

$$[N_{\mu\nu}, N_{\sigma\tau}] = -g_{\mu\sigma} N_{\nu\tau} - g_{\nu\tau} N_{\mu\sigma} + g_{\mu\tau} N_{\nu\sigma} + N_{\nu\sigma} g_{\mu\tau} \tag{5.4}$$

$$[-\partial_\mu, N_{\sigma\tau}] = g_{\mu\sigma}(-\partial_\tau) - g_{\mu\tau}(-\partial_\sigma). \tag{5.5}$$

Now the corresponding orthogonal operator  $O(\delta\lambda, \delta\omega^{[\cdot\cdot]})$  can be written as

$$O(\delta\lambda, \delta\omega^{[\cdot\cdot]}) = 1 + \delta\lambda^\mu (-\overset{\smile}{J}\overset{\smile}{\Pi}_\mu) + \frac{1}{2} \delta\omega^{\mu\nu} (\overset{\smile}{J}\overset{\smile}{M}_{\mu\nu}). \tag{5.6}$$

The symmetric operators  $\overset{\smile}{\Pi}_\mu$  and  $\overset{\smile}{M}_{[\mu\nu]}$  are pseudo-chronous observables and commute with  $\overset{\smile}{J}$ , because  $O_{(cont)} \rightarrow L_{(cont)}$ , contains neither time- nor space-reflections. In particular, the generators of the group  $\{O_{(cont)}\}$ :  $-\overset{\smile}{J}\overset{\smile}{\Pi}_\mu$  and  $\overset{\smile}{J}\overset{\smile}{M}_{\mu\nu}$  must satisfy the Lie structure relations (5.3)–(5.5) of  $L_{(cont)}$ .

Multiplying by  $\overset{\smile}{J}^{-1}$ , these are (0.22)–(0.24).

The identity (3.9), which relates  $L$  and  $O$  is, for the infinitesimal element

$$\left. \begin{aligned} F'^\alpha(x) &= F'^\alpha(x) + \delta\lambda^\mu (-\partial_\mu F'^\alpha(x) - \overset{\smile}{J}[\overset{\smile}{\Pi}_\mu, F'^\alpha(x)]) \\ &+ \frac{1}{2} \delta\omega^{\mu\nu} (\overset{\smile}{N}_{\mu\nu}{}^\alpha F'^\alpha(x) + \overset{\smile}{J}[\overset{\smile}{M}_{\mu\nu}, F'^\alpha(x)]) \end{aligned} \right\} \tag{5.10}$$

and leads thus to (0.20) and (0.21).

The sign of  $\overset{\smile}{\Pi}_\mu$  is chosen in order to give, for signat  $(g^{\alpha\beta}) = (1\ 1\ \dots\ 1\ -1)$ , the relation  $(\partial_n = \partial_t, \overset{\smile}{\Pi}^n = -\overset{\smile}{\Pi}_n \equiv \overset{\smile}{H})$

$$\partial_t F(\vec{x}, t) = \overset{\smile}{J}[\overset{\smile}{H}, F(\vec{x}, t)], \tag{5.11}$$

where the Heisenberg operator  $F(\vec{x}, t)$  and the Schrodinger operator  $\bar{F}(\vec{x})$  are related by

$$\langle F(\vec{x}, t) \rangle_\Psi = \langle \bar{F}(\vec{x}) \rangle_{\bar{\Psi}(t)}; \bar{\Psi}(t) = e^{-\overset{\smile}{J}\overset{\smile}{H}t} \Psi \tag{5.12}$$

or

$$F(x) = F(\vec{x}, t) = e^{\overset{\smile}{J}\overset{\smile}{H}t} F(\vec{x}) e^{-\overset{\smile}{J}\overset{\smile}{H}t}. \tag{5.13}$$

We write  $\overset{\smile}{H}$  with the pseudo-chronous sign  $\smile$ , because it is the  $n - th$  component of  $\overset{\smile}{\Pi}^\alpha$ . For time reflection,  $(x^i = x^i, x^n = -x^n)$  we have therefore:

$$\overset{\smile}{H} = \overset{\smile}{H}, \overset{\smile}{\Pi} = -\overset{\smile}{\Pi}. \tag{5.14}$$

The *energy operator*  $\overset{\circ}{H} = \overset{\circ}{\Pi}^n$  does not change its sign, while the *momentum operator*  $\overset{\circ}{\Pi}$  changes sign, because velocities change sign.

In order to show that  $\overset{\circ}{M}^{ik}$  is the *angular momentum operator*, we consider the transformation for an infinitesimal displacement of the origine:

$$'x'^{\mu} = x'^{\mu} + \delta\lambda'^{\mu} \quad (5.15)$$

$$'M'^{\mu'v} = O^{-1} \overset{\circ}{M}^{\mu'v} O = \overset{\circ}{M}^{\mu'v} + \delta\lambda^{\sigma} J[\overset{\circ}{\Pi}_{\sigma}, \overset{\circ}{M}^{\mu'v}] \quad (5.16)$$

or, using (0.24)

$$'M'^{\mu'v} = \overset{\circ}{M}^{\mu'v} + (\delta\lambda'^{\mu} \overset{\circ}{\Pi}'^v - \delta\lambda'^v \overset{\circ}{\Pi}'^{\mu}). \quad (5.17)$$

This shows that the arm-length of the moment with respect to the primed frame ( $\alpha$ -frame) is larger by the amount  $\delta\lambda'^{\mu}$  than the arm-length with respect to the  $\alpha$ -frame.

### § 6. The momentum-energy density operator

$\overset{\circ}{\Pi}^{\mu}$  and  $\overset{\circ}{M}^{[\mu\nu]}$  can be expressed as integrals over an arbitrary time-like surface  $'\tau(x) = 0$ , whose covariant  $n$ -component  $d\overset{\circ}{\sigma}_n(x)$  of the surface-element  $d\overset{\circ}{\sigma}_{\alpha}(x)$  is positive in every  $\alpha$ -frame, if we choose the signature  $+(1\ 1 \dots 1 - 1)$ . This means that  $d\overset{\circ}{\sigma}_{\alpha}(x)$  is a *pseudo-chronous time-like vector* (0.26). Then it follows from Gauss' theorem, that the pseudo-chronous quantities  $\overset{\circ}{\Pi}^{\mu}$  and  $\overset{\circ}{M}^{\mu\nu}$  are independent of the surface  $'\tau(x) = 0$  chosen, if (0.25) holds.

To verify the transformation law, let us transform (6.2) according to

$$' \overset{\circ}{\Pi}^{\mu} = \int_{'\tau(x)=0} d\overset{\circ}{\sigma}_{\alpha}(x) O^{-1} \Theta^{\alpha\mu}(x) O = L^{\mu}_{\sigma} \int_{'\tau(x)=0} d\overset{\circ}{\sigma}_{\alpha}(x) L^{\alpha}_{\beta} \Theta^{\beta\sigma}(L^{-1}x) \quad (6.5)$$

and write  $y = L^{-1}x$ . Then, from the pseudo-chronous character of  $d\overset{\circ}{\sigma}_{\alpha}$  follows

$$d\overset{\circ}{\sigma}_{\alpha}(Ly) L^{\alpha}_{\beta} = \text{sig}(L^n_n) d\overset{\circ}{\sigma}_{\beta}(y) \quad (6.6)$$

where  $d\overset{\circ}{\sigma}_n(Ly)$  is orientated parallel to  $x^n = (Ly)^n$ , while  $d\overset{\circ}{\sigma}_n(y)$  is orientated parallel to  $y^n = (L^{-1}x)^n$ .

Thus we have finally, writing  $'\mu$  for  $\mu$  and  $\mu$  for  $\sigma$

$$' \overset{\circ}{\Pi}'^{\mu} = \text{sig}(L^n_n) L'^{\mu}_{\mu} \int_{'\tau(Ly)=\tau(y)=0} d\overset{\circ}{\sigma}_{\beta}(y) \Theta^{\beta\mu}(y). \quad (6.7)$$

The integral being independent of the particular surface  $\tau(x) = 0$  or  $'\tau(x) = 0$  chosen, we may write:

$$' \overset{\circ}{\Pi}'^{\mu} = \text{sig}(L^n_n) L'^{\mu}_{\mu} \overset{\circ}{\Pi}^{\mu}. \quad (6.8)$$

Analogously, we transform

$$\overset{\smile}{M}{}^{\mu\nu} = O^{-1} \overset{\smile}{M}{}^{\mu\nu} O = \int_{\tau(x)=0} d\overset{\smile}{\sigma}_\alpha(x) L^\alpha_\beta(x^\mu L^\nu_\tau \Theta^{\beta\tau}(L^{-1}x) - x^\nu L^\mu_\sigma \Theta^{\beta\sigma}(L^{-1}x)) \tag{6.9}$$

and substitute  $x = Ly$ :  $x^\mu = L^\mu_\sigma (y^\sigma + L^\sigma)$ . Using again (6.6), we obtain

$$\left. \begin{aligned} \overset{\smile}{M}'{}^{\mu\nu} &= \text{sig}(L'^n_n) L'^\mu_\mu L'^\nu_\nu \int_{\tau(y)=0} d\overset{\smile}{\sigma}_\beta(y) ((y^\mu + L^\mu) \Theta^{\beta\nu}(y) \\ &- (y^\nu + L^\nu) \Theta^{\beta\mu}(y)) = \text{sig}(L'^n_n) L'^\mu_\mu L'^\nu_\nu (\overset{\smile}{M}{}^{\mu\nu} + L^\mu \overset{\smile}{\Pi}{}^\nu - L^\nu \overset{\smile}{\Pi}{}^\mu) \end{aligned} \right\} \tag{6.10}$$

Thus we have verified the pseudo-chronous character of momentum-energy  $\overset{\smile}{\Pi}{}^\mu$  and of angular momentum-centre of energy  $\overset{\smile}{M}{}^{[\mu\nu]}$ , expressed as surface integrals of an ortho-chronous momentum-energy density  $\Theta^{(\alpha\beta)}(x)$  over a surface  $\tau(x) = 0$ , with a pseudo-chronous time-like surface element  $d\overset{\smile}{\sigma}_\alpha(x)$ .

**§ 7. Physical Meaning of the Passive Time Reversal**

To our *passive interpretation* of time reversal, it has been objected, that only the *active interpretation* has a physical sense, because an observation at an epoch  $t'$  changes the earlier state (in the thermodynamic sense) of the system  $\Psi$  into a later state  $\Psi'$  corresponding to the measure of an observable  $F = F^{(i)}$ . However, we may consider an observer which makes only correlation experiments:

This observer makes a great number, say  $N$ , of experiments, at two epochs  $t'_{(1)}$  and  $t''_{(1)}$ ,  $t'_{(2)}$  and  $t''_{(2)}$  etc. ..., separated always by  $\Delta t$ . Let us suppose first that  $t$  measures the thermodynamic time order, and that

$$t'' - t' = t''_{(1)} - t'_{(1)} = t''_{(2)} - t'_{(2)} = \dots = t''_{(N)} - t'_{(N)} = \Delta t > 0 \tag{7.1}$$

Every time, an observer observes  $F^{(i)}$  at the earlier epoch  $t'$ , he will observe  $G = \{G^{(k)}\}$  at the later epoch and thus be able to make a statistics

$$F^{(i)} \rightarrow G^{(1)}, G^{(2)}, \dots, G^{(k)}, \dots G^{(\omega_G)} \tag{7.2}$$

giving transition probabilities

$$W(\overleftarrow{k}, i') \geq 0; \sum_k W(\overleftarrow{k}, i') = 1. \tag{7.3}$$

However, he is free to evaluate his statistics the other way round: Every time he registers  $G^{(k'')}$  at the later epoch  $t''$ , he makes statistics of the corresponding measures of  $F = F^{(i)}$  at the earlier epoch  $t'$ . Thus, he obtains transition probabilities

$$W(\overrightarrow{k}'', i) \geq 0, \sum_i W(\overrightarrow{k}'', i) = 1. \tag{7.4}$$



The coefficients (7.3) and (7.4) are of course equal

$$W(\overrightarrow{k}, i) = W(\overleftarrow{k}, i) = W(k, i). \quad (7.4b)$$

The arrows  $\leftarrow$  (evolution in the thermodynamic sense) and  $\rightarrow$  (evolution in the opposite sense) are thus superfluous. This means, that quantum-mechanical 'predictions' can be made for the future as well as for the past. If the system is not degenerate, we may write  $F^{(a)}$  and  $G^{(a)}$  for  $F^{(i)}$  and  $G^{(k)}$  (see (1.5)) and our correlation coefficients are

$$W('a, a) = (O_{'a a})^2 = (O_{a 'a}^T)^2. \quad (7.5)$$

They correspond to the doubly normalised transition amplitudes:

$$W(k, i) \geq 0; \quad \sum_k W(k, i) = 1; \quad \sum_i W(k, i) = 1 \quad (7.6)$$

used by INAGAKI, PIRON and WANDERS<sup>9)</sup> to prove the Boltzmann  $H$  theorem for the most general case (STUECKELBERG<sup>8)</sup>), while the usual proof assumes, instead of (7.6), detailed balancing

$$W(k, i) = W(i, k) \quad (7.7)$$

which is known to be insufficient<sup>8)9)</sup>.

*Therefore, passive time reversal can be verified experimentally.*

### § 8. Acknowledgments

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#### Annex 1. Remark on Generalised Poisson Brackets

In order to establish the *Boltzmann H-theorem* in classical statistical mechanics, we have to start from a covariant theory of motion in *phase space*  $x = \{x^\alpha\}$  ( $\alpha \beta \dots = 1 2 \dots \omega$ ), which satisfies the *theorem of Liouville*. The *conservation of energy*  $H = H(x)$  leads to an equation of motion for  $x^\alpha = z^\alpha(t)$

$$\dot{z}^\alpha(t) = \partial_t z^\alpha(t) = (\partial_\beta H(z(t)) j^{\beta\alpha}(z(t))), \quad (A-1.1)$$

where

$$j^{\alpha\beta}(x) = j^{[\alpha\beta]}(x) \quad (A-2.2)$$

is an *antisymmetrical tensor in phase space*. The *scalar density* of the Gibbs ensemble is  $w(x, t) \geq 0$ . It satisfies the equation of continuity

$$\int d^\omega x w(x, t) = 1; \quad \partial_t w(z, t) + \partial_\alpha (\dot{z}^\alpha w(z, t)) = 0 \quad (A-2.2)$$

and transforms according to

$$w'(x, t) = |\det (A'^{\mu}_{\mu}(x))| w(x, t) \geq 0. \tag{A-2.3}$$

$$x'^{\alpha} = \psi'^{\alpha}(x); \quad A'^{\alpha}_{\alpha}(x) = \partial_{\alpha} \psi'^{\alpha}(x)$$

Let  $d\Omega(x) = g(x) |d\varphi^{[1^2 \dots \omega]}| = g d^{\omega}x$  be the invariant scalar volume element, where  $d\varphi^{[\alpha_1 \alpha_2 \dots \alpha_{\omega}]} = \det (dx^{\alpha})^{(\beta)}$  is the antisymmetric tensor of the parallelepiped, formed from  $\omega$  non coplanar vectors  $d x^{\alpha}$ . Then we may introduce the *scalar of the density*  $w(x, t)$

$$w(x, t) = \frac{dW(x, t)}{d\Omega(x)} = \frac{w}{g}(x, t) > 0 \tag{A-2.4}$$

where  $g(x)$  is the 'density of volume'. To form such a density we have only the antisymmetric contravariant tensor  $j^{[\mu\nu]}$  at our disposal, which is the *fundamental tensor in phase space*, analogous to the metric tensor  $g^{(\alpha\beta)}$  in Riemann space.

Therefore we put  $g = |\det (j^{\mu\nu})^{-1/2}| > 0$  (A-2.5)

because it has the right transformation property. It is  $\neq 0$  if, and only if,  $\omega = 2f$  is an *even number*. In terms of  $w(x, t)$ , the continuity equation (A-2.2) takes the form

$$\partial_t w + D_{\alpha}(z^{\alpha}w) = 0; \quad D_{\alpha} = \partial_{\alpha} + G_{\alpha}; \quad G_{\alpha} = \partial_{\alpha} \log g, \tag{A-2.6}$$

where  $D_{\alpha}$  is the *covariant divergence operator*. The *theorem of Liouville* states, that the *scalar of the density*  $w$  remains constant, if we follow an orbit  $x^{\alpha} = z^{\alpha}(t)$ :

$$\frac{d}{dt} w(z(t), t) \equiv \dot{w}(z, t) = (\partial_t w + z^{\alpha} \partial_{\alpha} w)(z, t) = 0. \tag{A-2.7}$$

This implies (see (A-2.6)):

$$D_{\alpha} z^{\alpha} = D_{\alpha} ((\partial_{\beta} H) j^{\beta\alpha}) = (\partial_{\alpha} \partial_{\beta} H) j^{[\alpha\beta]} + (\partial_{\beta} H) D_{\alpha} j^{[\alpha\beta]} = 0 \tag{A-2.8}$$

and is a covariant condition\*) for the fundamental tensor:

$$D_{\alpha} j^{[\alpha\beta]} = q^{\beta} = 0. \tag{A-2.9.j)*}$$

We may express it in terms of the density  $j^{\alpha\beta} = g j^{[\alpha\beta]}$

$$\partial_{\alpha} j^{[\alpha\beta]} = q^{\beta} = 0. \tag{A-2.9j)*}$$

(A-2.9) is formally analogous to the *second set of Maxwell's equations*, if no electric charges  $q^{\beta}$  are present.

\*)  $D_{\alpha} F^{[\alpha\beta\gamma \dots]} = G^{[\beta\gamma \dots]}$  or  $\partial_{\alpha} \mathfrak{F}^{[\alpha\beta \dots]} = \mathfrak{G}^{[\beta\gamma \dots]}$  (A-2.9\*)

and  $\partial_{[\alpha} F_{\beta\gamma \dots]} = G_{[\alpha\beta\gamma \dots]}$  are covariant relations. (A-2.13\*)

An observable  $F(x)$  varies with time, according to

$$\dot{F}(z(t)) = \dot{z}^\alpha \partial_\alpha F = (\partial_\beta H) j^{[\beta\alpha]} \partial_\alpha F \equiv \{H, F\} = -\{F, H\} \quad (\text{A-2.10})$$

where  $\{H, F\}$  defines a *generalised Poisson bracket*.

It immediately follows, from

$$\begin{aligned} \{F, \{G, H\}\} &= (\partial_\alpha F) j^{[\alpha\beta]} \partial_\beta ((\partial_\gamma G) j^{[\gamma\delta]} \partial_\delta H) \\ &= (\partial_\alpha F) j^{[\alpha\beta]} (\partial_\beta \partial_\gamma G) j^{[\gamma\delta]} \partial_\delta H + (\partial_\alpha F) j^{[\alpha\beta]} (\partial_\gamma G) j^{[\gamma\delta]} \partial_\beta \partial_\delta H \\ &\quad + (\partial_\alpha F) (\partial_\beta G) (\partial_\gamma H) j^{[\alpha\epsilon]} \partial_\rho j^{[\beta\gamma]} \end{aligned}$$

that the cyclic sum is

$$\underset{FGH}{\curvearrowright} \{F, \{G, H\}\} = (\partial_\alpha F) (\partial_\beta G) (\partial_\gamma H) \underset{\alpha\beta\gamma}{\curvearrowright} j^{[\alpha\epsilon]} \partial_\rho j^{[\beta\gamma]}. \quad (\text{A-2.11})$$

Thus, the Jacobi identity is not necessarily satisfied.

If we require this identity, we must have

$$\underset{\alpha\beta\gamma}{\curvearrowright} j^{\alpha\epsilon} \partial_\rho j^{\beta\gamma} \equiv q^{[\alpha\beta\gamma]} = 0, \quad (\text{A-2.12})$$

to which we may give the form of the *first set of Maxwell's equations* in the absence of magnetic charge  $q_{[\alpha\beta\gamma]}$

$$\underset{\alpha\beta\gamma}{\curvearrowright} \partial_\alpha j_{[\beta\gamma]} \equiv q_{[\alpha\beta\gamma]} = 0. \quad (\text{A-2.13}^*)$$

if we introduce the inverse tensor  $j_{\alpha\beta}$ :

$$j_{\alpha\rho} j^{\beta\rho} = \delta_\alpha^\beta; \quad j_{\alpha\beta} = \min(j^{\alpha\beta}) / \det(j^{\mu\nu}). \quad (\text{A-2.14})$$

From this definition follows

$$j_{\alpha\rho} j_{\beta\sigma} j^{\rho\sigma} = j_{\alpha\rho} \delta_\beta^\rho = j_{\alpha\beta}. \quad (\text{A-2.15})$$

We may thus raise and lower indices, with these antisymmetric fundamental tensors  $j^{[\alpha\beta]}$  and  $j_{[\alpha\beta]}$

$$F_{\alpha\beta\dots} = j_{\alpha\alpha'} j_{\beta\beta'} \dots F^{\alpha'\beta'} \dots \quad F^{\alpha\beta} = j^{\alpha\alpha'} j^{\beta\beta'} \dots F_{\alpha'\beta'} \dots \quad (\text{A-2.16})$$

In particular, it follows from

$$\partial_\rho \delta_{\gamma'}^\rho = (\partial_\rho j^{\beta\gamma}) j_{\beta\gamma'} + j^{\beta\gamma} \partial_\rho j_{\beta\gamma'} = 0$$

that

$$\partial_\rho j^{\beta\gamma} = j^{\beta\beta'} j^{\gamma\gamma'} \partial_\rho j_{\beta'\gamma'}. \quad (\text{A-2.17})$$

Introducing this expression into (A-2.12), we arrive at the first set of 'Maxwell's equations' (A-2.13), with the conditions that 'magnetic charges'  $q_{[\alpha\beta\gamma]}$  vanish.

This totally antisymmetric tensor of, magnetic charge'  $q^{[\alpha\beta\gamma]}$  in phase space with a fundamental tensor  $j^{[\alpha\beta]}$  plays an analogous role to the Riemann-Christoffel Tensor  $R_{([\alpha\beta][\gamma\delta])}$  in a space with a metric  $g_{(\alpha\beta)}$ : If  $q^{[\alpha\beta\gamma]}$  vanishes, a coordinate frame  $'x'^\alpha$  exists, where  $j^{[\alpha\beta]}$  has the form

$$\{ 'j'^{\alpha'\beta} \} = \left\{ \begin{array}{c|c|c} \begin{array}{c} 0-1 \\ 1 \quad 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 0-1 \\ 1 \quad 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \end{array} & \begin{array}{c} \dots \\ \dots \end{array} \end{array} \right\}; \det ('j'^{\alpha'\beta}) = 1. \tag{A-2.19}$$

This is analogous to the existence of an euclidian or pseudo-euclidian frame  $'g'_{\alpha'\beta} = \pm \delta_{\alpha'\beta}$  in the flat space, if  $R_{([\alpha\beta][\gamma\delta])} = 0$ . The proof of this theorem is given for instance by WHITTAKER<sup>10</sup> (see also PAULI<sup>12</sup>):

It states, that the Pfaff differential expression formed from  $a_\alpha$  in

$$\dot{j}_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \tag{A-2.20}$$

consisting of  $\omega$  terms

$$a_\alpha(x) dx^\alpha = \sum_{i=1}^{i=f} p_i(x) dq_i(x) + \begin{cases} 0; 2f \leq n \\ dq_{l+1}; 2f + 1 \leq n \end{cases} \tag{A-2.21}$$

can always be expressed in the form of the right-hand side.

Now put (as  $\omega = 2f$ )

$$p_i(x) = 'x^{2i-1}, q_i(x) = 'x^{2i}, i = 1 \ 2 \ \dots \ f. \tag{A-2.22}$$

Then we have

$$'a_{2i}('x) = 'x^{2i-1}; 'a_{2i-1}('x) = 0 \tag{A-2.22}$$

and the only non zero component of  $'j'^{\alpha'\beta}$

$$' \partial_{2i-1} 'a_{2i}('x) = 'j_{2i-1, 2i} = + 1, 'j^{2i, 2i-1} = \min ('j_{2i, 2i-1}) = - 1. \tag{A-2.23}$$

Introducing

$$'H('x) = H(x) = 'H(p, q) \tag{A-2.24}$$

we have

$$\left. \begin{aligned} \dot{z}^{2i-1} = \dot{p}_i = ' \partial_{2i} 'H('z) 'j^{2i, 2i-1} = - ' \partial_{2i} 'H('z) = - \frac{\partial}{\partial q_i} 'H(p, q) \\ \dot{z}^{2i} = \dot{q}_i = ' \partial_{2i-1} 'H('z) j^{2i-1, 2i} = ' \partial_{2i-1} 'H('z) = + \frac{\partial}{\partial p_i} 'H(p, q). \end{aligned} \right\} \tag{A-2.25}$$

Therefore we see, that the Jakobi identity is *by no means necessary to establish the H-theorem*: Only the 'second Maxwell set' (absence of "electric charges'  $q^\alpha$ ) has to be satisfied. The presence of 'magnetic charges'  $q_{[\alpha\beta\gamma]}$  in the "first Maxwell set' does *not invalidate* Liouville's theorem (from which the Boltzmann H-theorem follows). To this generalised statistical mechanics corresponds aQT, in which the observables are no more linear operators, because of:

$$\overset{\curvearrowright}{FGH} J [F, J [G, H]] \neq 0. \tag{A-2.26}$$

**Annex 2. Complex Hilbert Space (CHS)**

If we restrict ourselves to the ortho-chronous subgroup  $\{L_{(\text{ochr})}\}$  with  $L'_{(\text{ochr})n} > 0$ , all operators  $\{F, G, \dots, \overset{\sim}{J}, O_{(\text{ochr})}\}$  commute with  $\overset{\sim}{J}$ :

$$[\overset{\sim}{J}, O_{(\text{ochr})}] = [\overset{\sim}{J}, F] = \dots = 0; \quad \overset{\sim}{J}^T = -\overset{\sim}{J}; \quad \overset{\sim}{J}^2 = -1. \tag{A-2.1}$$

We may now establish a relation (dictionary) between our QT in RHS and the conventional QT in CHS. To do this, we consider the  $\omega_R$ -dimensional RHS as a product space between a 2-dimensional RHS and an  $\omega_C = 1/2 \omega_R$ -dimensional RHS. We write

$$\Phi = (\Phi_{(r)} \Phi_{(i)}); \quad \Psi = \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \tag{A-2.2}$$

and the arbitrary operator  $A$  ( $[\overset{\sim}{J}, A] = 0$ ) as the Kronecher product ( $\times$ )

$$A = 1 \times A_{(r)} + j \times A_{(i)}; \quad \overset{\sim}{J} = j \times 1 \tag{A-2.3}$$

$$A_{(r)} = \{A_{(r) pq}\} \quad \text{and} \quad A_{(i)} = \{A_{(i) pq}\}$$

are  $\omega_C = 1/2 \omega_R$ -dimensional matrices ( $pq \dots = 1 2 \dots \omega_C$ ) and

$$j = \begin{pmatrix} 0 & -\lambda \\ +\lambda & 0 \end{pmatrix}; \quad j^2 = -1; \quad \lambda^2 = 1. \tag{A-2.4}$$

Now let us consider the  $\omega_C$ -dimensional complex Matrix ( $i = +\sqrt{-1}$ )

$$\hat{A} = A_{(r)} + i A_{(i)}; \quad \hat{A}^\dagger = A_{(r)}^T - i A_{(i)}^T \tag{A-2.5)*}$$

and the two  $\omega_C$ -dimensional complex vectors

$$\hat{\Phi} = \Phi_{(r)} + i \Phi_{(i)}; \quad \hat{\Psi} = \Psi_{(r)} + i \Psi_{(i)} \tag{A-2.6}$$

where  $\Phi_{(r)}, \Phi_{(i)}, \Psi_{(r)}, \Psi_{(i)}, A_{(r)}$  and  $A_{(i)}$  are the real  $\omega_C$ -dimensional vectors and matrices. Now, we define the usual complex matrix element in CHS by

$$\left. \begin{aligned} \langle \hat{\Phi}, \hat{A} \hat{\Psi} \rangle &= ((\Phi_{(r)} - i \Phi_{(i)}), (A_{(r)} + i A_{(i)}) (\Psi_{(r)} + i \Psi_{(i)})) \\ &= ((\Phi_{(r)}, A_{(r)} \Psi_{(r)}) + (\Phi_{(i)}, A_{(r)} \Psi_{(i)}) - (\Phi_{(r)}, A_{(i)} \Psi_{(i)}) \\ &+ (\Phi_{(i)}, A_{(i)} \Psi_{(r)})) + i((\Phi_{(r)}, A_{(i)} \Psi_{(r)}) + (\Phi_{(i)}, A_{(i)} \Psi_{(i)}) + \\ &+ (\Phi_{(r)}, A_{(r)} \Psi_{(i)}) - (\Phi_{(i)}, A_{(r)} \Psi_{(r)})) = (\Phi_{(r)} \Phi_{(i)}) \\ &\left( \begin{matrix} A_{(r)} - A_{(i)} \\ A_{(i)} \quad A_{(r)} \end{matrix} \right) \left( \begin{matrix} \Psi_{(r)} \\ \Psi_{(i)} \end{matrix} \right) - i(\Phi_{(r)} \Phi_{(i)}) \left( \begin{matrix} -A_{(i)} - A_{(r)} \\ A_{(r)} - A_{(i)} \end{matrix} \right) \left( \begin{matrix} \Psi_{(r)} \\ \Psi_{(i)} \end{matrix} \right) \\ &= \langle \hat{A} \hat{\Psi}, \hat{\Phi} \rangle^* \end{aligned} \right\} \tag{A-2.6)*}$$

\*) An \* denotes the conjugate complex number. An † signifies the Hermitian conjugate operator:

$$\hat{A}_{pq}^\dagger = \hat{A}_{qp}^*; \quad \hat{A}^\dagger = (\hat{A}^*)^T = (\hat{A}^T)^* \tag{A-2.9}$$

This expression is equal to

$$\langle \widehat{\Psi}, \widehat{A} \widehat{\Phi} \rangle = (\Phi, A \Psi) - i(\Phi, \overset{\vee}{J} A \Psi) \tag{A-2.7}$$

if we choose  $\lambda = + 1$  in (A-2.3), i. e.

$$j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A-2.8}$$

(A-2.7) is the dictionary we proposed to establish.

It is worth noting, that the definition of  $j$  in (A-2.3) is univoque: (A-2.8).

### Annex 3. Unitary (U) and Anti-Unitary Operators (V) in CHS

We write our dictionary between CHS and RHS (A-2.7) in the form

$$\langle \widehat{\phi}, \widehat{A}^\alpha(x) \widehat{\Psi} \rangle = (\Phi, (1 - i \overset{\vee}{J}) A^\alpha(x) \Psi) \tag{A-3.1}$$

where the left hand side is the scalar product in complex Hilbert space (= CHS) and the right hand side is the scalar product in RHS (i. e. all symbols, except  $i$ , stand for real quantities (vectors, operators)). Let us consider the transformed quantity (matrix element of  $A^\alpha(x)$  between the states  $\Phi \leftarrow \Psi$ )

$$\left. \begin{aligned} \langle \widehat{\Phi}, \widehat{A}'^\alpha(x) \widehat{\Psi} \rangle &= L'^\alpha_\alpha (\Phi, (1 - i \text{sig}(L'^n_n) \overset{\vee}{J}) A^\alpha(L^{-1} 'x) \Psi) \\ \equiv \langle ' \widehat{\Phi}, \widehat{A}'^\alpha('x) ' \widehat{\Psi} \rangle &= (' \Phi, (1 - i \overset{\vee}{J}) A'^\alpha('x) ' \Psi) \end{aligned} \right\} \tag{A-3.2}$$

#### a) Orthochronous transformations

( $L'^n_{(\text{ochr})n} > 0$ ): We have

$$' \Psi_a = O_{aa} \Psi_a; \begin{pmatrix} ' \Psi_{(r)} \\ ' \Psi_{(i)} \end{pmatrix} = (1 \times O_{(r)} + j \times O_{(i)}) \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(i)} \end{pmatrix} \tag{A-3.4}$$

$$' \widehat{\Psi}_p = U_{pp} \widehat{\Psi}_p; ' \widehat{\Psi} = (O_{(r)} + i O_{(i)}) \widehat{\Psi} \equiv U \widehat{\Psi} \tag{A-3.5}$$

$$' \widehat{\Phi}'_p = \widehat{\Phi}'_p U'^*_{p'p} = \widehat{\Phi}'_p U^\dagger_{p'p}; ' \widehat{\Phi}^* = (O_{(r)} - i O_{(i)}) \widehat{\Phi}^* \equiv U^* \widehat{\Phi}^* \equiv \widehat{\Phi}^* U^\dagger \tag{A-3.5*}$$

From the orthogonality of  $O$

$$\left. \begin{aligned} O^T O &= (1 \times O^T_{(r)} - j \times O^T_{(i)}) (1 \times O_{(r)} + j \times O_{(i)}) = \\ 1 \times (O^T_{(r)} O_{(r)} + O^T_{(i)} O_{(i)}) &+ j \times (O^T_{(r)} O_{(i)} - O^T_{(i)} O_{(r)}) = 1 \times 1 = 1 \end{aligned} \right\} \tag{A-3.6}$$

follows

$$O^T_{(r)} O_{(r)} + O^T_{(i)} O_{(i)} = 1; \quad O^T_{(r)} O_{(i)} - O^T_{(i)} O_{(r)} = 0. \tag{A-3.8}$$

Thus, according to the definition of  $U$  and  $U^\dagger$  in (A-3.5), and (A-3.5\*),

$$U^\dagger U = 1 \tag{A-3.9}$$

$U = \{U_{pq}\}$  is an *unitary matrix* in  $\omega_C = 1/2 \omega_R$ - dimensional CHS ( $pq \dots = 1 2 \dots \omega_C$ ;  $ab \dots = 1 2 \dots \omega_R$ ). Because of  $\text{sig}(L'^n) = 1$  and of (A-3.1), we may write the second member of (A-3.2) in the form  $L'^\alpha_\alpha \langle \hat{\Phi}, \hat{A}^\alpha(L^{-1}'x) \hat{\Psi} \rangle$ . Thus we obtain finally (see (A-2.5)) the identity

$$\hat{A}'^\alpha('x) = L'^\alpha_\alpha U \hat{A}^\alpha(L^{-1}'x) U^\dagger \tag{A-3.10}$$

i. e. the tensor  $\hat{A}_{pq}(x) = \hat{A}_{pq}^X$  is invariant if it is transformed with respect to all its indices:

$$\hat{A}'^X_{p'q} = L'^X_{(\text{chr})X} U'_{pp} U'^*_{qq} \hat{A}^X_{pq} \tag{A-3.11}$$

and  $U$  is an unitary representation of  $L_{(\text{ochr})}$

$$L_{(\text{ochr})} \rightarrow e^{i\lambda} U. \tag{A-3.12}$$

b) *Pseudochronous Transformations* ( $L'^n_{(\text{pchr})n} < 0$ )

We try (A-3.2), posing for the transformed matrix element (A-3.2)

$$\left. \begin{aligned} \langle \hat{\Phi}, \hat{A}'^\alpha('x) \hat{\Psi} \rangle &= \langle \hat{\Phi}, \hat{A}^\alpha(L^{-1}'x) \hat{\Psi} \rangle = \\ L'^\alpha_\alpha \langle \hat{\Phi}, (1 + i \check{J}) A^\alpha(L^{-1}'x) \hat{\Psi} \rangle &= L'^\alpha_\alpha \langle \hat{\Phi}, \hat{A}^\alpha(L^{-1}'x) \hat{\Psi} \rangle^* \end{aligned} \right\} \tag{A-3.13}$$

It is, in virtue of the definition (A-3.1), a *linear function of the untransformed conjugate complex element*. We shall see however, that the identity in  $\hat{A}'^\alpha_{p'q}('x)$  leads now to a contradiction. In order to show that, write, introducing a *non-linear operator*  $V$

$$' \hat{\Psi} = (V \hat{\Psi}), \quad ' \hat{\Phi} = (V \hat{\Phi})$$

the second and fourth member of (A-3.13) in the Form:

$$\langle (V \hat{\Phi}), V(V^{-1} (\hat{A}'^\alpha('x) (V \hat{\Psi}))) \rangle = L'^\alpha_\alpha \langle \hat{A}^\alpha(L^{-1}'x) \hat{\Psi}, \hat{\Phi} \rangle. \tag{A-3.14}$$

Thus, we must have a relation

$$(V \hat{\Psi})_p = U'_{pp} \hat{\Psi}^*_p; \quad (V^{-1} \hat{\Psi})_p = U^T_{p'p} \hat{\Psi}'^*_p \tag{A-3.15}$$

$$(V \hat{\Phi})^*_p = U'^*_p \hat{\Phi}_p = \hat{\Phi}_p U^\dagger_{p'p}; \quad (V^{-1} \hat{\Phi})^*_p = U^\dagger_{p'p} \hat{\Phi}'_p. \tag{A-3.15*}$$

where  $U$  is an unitary matrix, in order to have

$$\left. \begin{aligned} \langle V(\hat{\Phi}), V(\hat{\Psi}) \rangle &= \langle \hat{\Phi}, \hat{\Psi} \rangle^* = \langle \hat{\Psi}, \hat{\Phi} \rangle; \\ U^\dagger U &= U^T U^* = 1. \end{aligned} \right\} \tag{A-3.16}$$

On account of (A-3.16), the identity (A-3.14) (second = fourth member in (A-3.13)) has the form:

$$(V^{-1} (\hat{A}'^\alpha('x) (V \hat{\Psi})))^*_p \hat{\Phi}_p = (L'^\alpha_\alpha A^\alpha(L^{-1}'x) \hat{\Psi})^*_p \hat{\Phi}_p, \tag{A-3.17}$$

which must hold for all  $\widehat{\Phi}_q$ . Thus we have

$$(V^{-1} (\widehat{A}'^\alpha('x) (V \widehat{\Psi})))_p = (L'^\alpha_\alpha \widehat{A}^\alpha(L^{-1} 'x) \widehat{\Psi})_p \tag{A-3.18}$$

from this identity in  $\widehat{\Psi}_p$  (see (A-3.15)) follows

$$U^T_{p'p} \widehat{A}^*_{p'q}('x) U^*_{qq} = L'^\alpha_\alpha \widehat{A}^\alpha_{pq}(L^{-1} 'x) \tag{A-3.19}$$

or

$$A^*_{p'q}{}^X = L'^X_X U^*_{p'p} U_{qq} \widehat{A}^X_{pq}. \tag{A-3.20}$$

This identity can evidently not be satisfied in general, because no linear transformation  $(L'^\alpha_\alpha U^* \dots U^T)$  exists, which transforms *all tensors*  $\widehat{A}^X_{pq}$  in hermitian CHS into their conjugate complex. Thus we conclude, that pseudo-chronous transformations are not given by (A-3.13), but we must have

$$\langle \widehat{\Phi}, \widehat{A}'^\alpha('x) \widehat{\Psi} \rangle = \langle \widehat{\Phi}, \widehat{A}^*{}^\alpha('x) \widehat{\Psi} \rangle = L'^\alpha_\alpha \langle \widehat{\Phi}, \widehat{A}^\alpha(L^{-1} 'x) \widehat{\Psi} \rangle^*. \tag{A-3.13*}$$

In this case,  $\widehat{A}'^\alpha('x)$  (in the third member of (A-3.2) and in the first members of equations (A-3.14), (A-3.17)–(A-3.20)) has to be replaced by  $\widehat{A}^*{}^\alpha('x)$ .

(A-3.20) is now analogous to (A-3.11)

$$\widehat{A}^X_{p'q} = L'^X_{(pchr)X} U^*_{p'p} U_{qq} \widehat{A}^X_{pq}. \tag{A-3.20*}$$

This rather complicated formalism for time reversal in CHS shows clearly the advantage of RHS, where all transformations  $L \rightarrow e^{\lambda J} O$  are linear.

#### Annex 4. Quaternion Hilbert Space (QHS)

QHS has been introduced by FINKELSTEIN, JAUCH and SPEISER<sup>3)</sup>. They pose, with  $i_\alpha^2 = -1$ ;  $i_1 i_2 = -i_2 i_1 = i_3$  cycl.:

$$\widehat{\Phi} = \Phi_{(r)} + i_1 \Phi_{(1)} + i_2 \Phi_{(2)} + i_3 \Phi_{(3)} = \{\widehat{\Phi}_p\} \tag{A-4.1}$$

$$\widehat{\Psi} = \Psi_{(r)} + i_1 \Psi_{(1)} + i_2 \Psi_{(2)} + i_3 \Psi_{(3)} = \{\widehat{\Psi}_q\} \tag{A-4.2}$$

$$\widehat{A} = A_{(r)} + i_1 A_{(1)} + i_2 A_{(2)} + i_3 A_{(3)} = \{\widehat{A}_{pq}\}. \tag{A-4.3}$$

The QHS has  $\omega_Q = 1/4 \omega_R$ -dimensions:  $p q, \dots = 1 2 \dots \omega_Q \dots$ . The scalar product is defined by

$$\langle \widehat{\Phi}, \widehat{A} \widehat{\Psi} \rangle = \widehat{\Phi}_p^* \widehat{A}_{pq} \widehat{\Psi}_q \tag{A-4.4}$$

where  $\widehat{\Phi}_p^*$  is the conjugated quaternion of  $\widehat{\Phi}_p$  ( $i_\alpha \rightarrow -i_\alpha$ ). All numbers,  $\widehat{\Phi}_p, \widehat{A}_{pq}, \widehat{\Psi}_q$  and  $\langle \widehat{\Phi}, \widehat{A} \widehat{\Psi} \rangle$  are now quaternions ( $\Phi_{(r)} \dots A_{(r)} \dots$  are real vectors and tensors). A straight forward calculation gives

$$\langle \widehat{\Phi}, \widehat{A} \widehat{\Psi} \rangle_{(r)} = (\Phi, A \Psi) \tag{A-4.5}$$



with

$$\Phi = (\Phi_{(r)} \Phi_{(1)} \Phi_{(2)} \Phi_{(3)}); \Psi = \begin{pmatrix} \Psi_{(r)} \\ \Psi_{(1)} \\ \Psi_{(2)} \\ \Psi_{(3)} \end{pmatrix} \tag{A-4.6}$$

$$A = 1 \times A_{(r)} + j_1 \times A_{(1)} + j_2 \times A_{(2)} + j_3 \times A_{(3)}. \tag{A-4.7}$$

The  $j_\alpha$ 's are four by four matrices and satisfy the same commutation laws as the quaternions  $i_\alpha$ 's ( $j^2 = -1$ ;  $j_1 j_2 = -j_2 j_1 = j_3$  cycl). They are KRONECKER products of the 'pseudo-quaternions':

$$\left. \begin{aligned} 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad l = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ -j^2 &= k^2 = l^2 = 1; \quad jk = -kj = l, \quad kl = -lk = -j \\ lj &= -jl = k \end{aligned} \right\} \tag{A-4.8}$$

$$j_1 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} = 1 \times j; \quad j_2 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} = j \times k; \quad j_3 = \begin{pmatrix} 0 & -l \\ l & 0 \end{pmatrix} = j \times l. \tag{A-4.9}$$

However, in order to form the vector components of the quaternion, we need three further four by four matrices

$$\left. \begin{aligned} k_1 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = k \times 1; \quad k_2 = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = 1 \times k; \quad k_3 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} = k \times k \\ k_1 k_2 &= k_2 k_1 = k_3 \text{ cycl.}; \quad k_\alpha^2 = 1 \\ [k_\alpha, j_\alpha] &= 0; \quad (k_\alpha, j_\beta) = 0; \quad \alpha \neq \beta \end{aligned} \right\} \tag{A-4.10)*}$$

in order to write

$$\langle \widehat{\Phi}, \widehat{A} \widehat{\Psi} \rangle_\alpha = -(\Phi, (k_\alpha j_\alpha \times 1) A \Psi). \tag{A-4.11}$$

Introducing, analogous to (A-2.3)

$$J_\alpha = j_\alpha \times 1; \quad K_\alpha = k_\alpha \times 1, \tag{A-4.12}$$

we find

$$\left. \begin{aligned} \langle \widehat{\Phi}, \widehat{A} \widehat{\Psi} \rangle &= (\Phi, A \Psi) - i_1 (\Phi, K_1 J_1 \Psi) - i_2 (\Phi, K_2 J_2 \Psi) \\ &\quad - i_3 (\Phi, K_3 J_3 \Psi). \end{aligned} \right\} \tag{A-4.13}$$

Thus, QT in QHS is not equivalent to QT in RHS, because we need – in addition to the three anticommuting operators

$$J_\alpha^2 = -1; \quad J_1 J_2 = -J_2 J_1 = J_3 \text{ cycl.} \tag{A-4.14}$$

– three further operators

$$\left. \begin{aligned} K_\alpha^2 &= 1; \quad K_1 K_2 = K_2 K_1 = K_3 \text{ cycl.} \\ [K_\alpha, J_\alpha] &= 0; \quad (K_\alpha, J_\beta) = 0, \quad \alpha \neq \beta. \end{aligned} \right\} \tag{A-4.15)*}$$

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\*) We write  $[A, B] = AB - BA$  for the commutator and  $(A, B) = AB + BA$  for the anticommutator. This is in strict analogy to our notation for antisymmetric tensors  $j^{[\alpha\beta]} = -j^{[\beta\alpha]}$  and for symmetric tensors  $g^{(\alpha\beta)} = g^{(\beta\alpha)}$ .

**Annex 5. An Error in the Representation Theory  
Frequently Found in Literature**

Instead of our identity, following from (3.4) and (3.5)

$$(\Psi, 'F'^{\alpha}('x) \Psi) = ('\Psi, F^{\alpha}('x) '\Psi) \tag{A-5.1}$$

many authors start from the identity

$$('\bar{\Psi}, '\bar{F}'^{\alpha}('x) '\bar{\Psi}) = (\bar{\Psi}, \bar{F}^{\alpha}('x) \bar{\Psi}), \tag{A-5.2)*}$$

which is analogous to the relation (5.2) between the Schroedinger and the Heisenberg representation. We refer particularly to the otherwise excellent book by JAUCH and ROHRLICH<sup>11)</sup> (to be referred to as *J R*). They define

$$'\bar{\Psi} = \bar{O} \bar{\Psi}, \bar{O}^T = \bar{O}^{-1} \tag{A-5.3}$$

and, as we have done,

$$'\bar{F}'^{\alpha}('x) = L'^{\alpha}_{\alpha} \bar{F}^{\alpha}(L^{-1} 'x). \tag{A-5.4}$$

From the identity in  $'\bar{\Psi}'_a$  they find, of course

$$'F'^{\alpha}('x) = \bar{O} \bar{F}^{\alpha}('x) \bar{O}^{-1}. \tag{A-5.5}$$

Now, they pretend that  $\{e^{\int \lambda \bar{O}}\}$  is a ray representation of  $\{L\}$ . The identity which follows from (A-5.4) and (A-5.5) is, explicitly written:

$$L'^{\alpha}_{\alpha} \bar{F}'^{\alpha}_{a'b}(L^{-1} 'x) = \bar{O}_{aa} \bar{F}^{\alpha}_{ab}('x) \bar{O}^{-1}_{b'b} \tag{A-5.6}$$

(Eq. (1-43) and (1-42) p. 11 of *JR*). According to *JR*,  $\bar{O}$  (and not  $\bar{O}^{-1} = \bar{O}^T \equiv O$ , as we found in (3.8)) is a representation of  $L$ . They give no proof of their statement.

We shall give an argument, which may have lead them to this contradictory statement: Write  $L_{(1)}$  in (A-5.4) and  $\bar{O}^{(1)}$  in (A-5.5).  $L_{(1)}$  transforms from the  $X = \{x \alpha\}$ -frame to the  $'X$ -frame. Let  $L_{(1)}$  be followed by  $L_{(2)}$  transforming  $''X \leftarrow 'X$ :

$$''\bar{F}''^{\alpha}(''x) = L_{(2)}''^{\alpha}_{\alpha} \bar{F}'^{\alpha}(L_{(2)}^{-1} ''x) \tag{A-5.4(2)}$$

$$''\bar{\Psi}''_a = \bar{O}^{(2)}_{a'a} \bar{\Psi}'_a \tag{A-5.3(2)}$$

$$''F''^{\alpha}(''x) = \bar{O}^{(2)} \bar{F}'^{\alpha}(''x) \bar{O}^{(2)-1}. \tag{A-5.5(2)}$$

Now, substitute (5.4<sup>(1)</sup>) into (5.4<sup>(2)</sup>) i. e.

$$''\bar{F}''^{\alpha}(''x) = (L_{(2)} L_{(1)})''^{\alpha}_{\alpha} \bar{F}^{\alpha}((L_{(2)} L_{(1)})^{-1} ''x) \tag{A-5.6}$$

and (A-5.5<sup>(1)</sup>) into (A-5.5<sup>(2)</sup>)

$$''\bar{F}''^{\alpha}(''x) = (\bar{O}^{(2)} \bar{O}^{(1)}) \bar{F}^{\alpha}(''x) (\bar{O}^{(2)} \bar{O}^{(1)})^{-1}. \tag{A-5.7}$$

\*) We denote vectors and operators satisfying (A-5.2) by a bar, in order to distinguish them from the vectors and operators in the text and in (A-5.1).

If we eliminate  ${}^{\prime\prime}\bar{F}^{\alpha}({}^{\prime\prime}x)$  from the last two equations, we find writing  $X = \{\alpha x\}$

$$(L_{(2)} L_{(1)})^{\prime\prime X} \bar{F}_{a^{\prime\prime}b}^X = (\bar{O}^{(2)} \bar{O}^{(1)})_{a^{\prime\prime}a} \bar{F}_{ab}^{\prime\prime X} (\bar{O}^{(2)} \bar{O}^{(1)})_{b^{\prime\prime}b}^{-1}. \quad (\text{A-5.8})$$

Thus, it seems to follow from (A-5.8) that  $\{e^{\int \bar{O}}\}$  is a representation of  $\{L\}$ . This is of course contradictory to the theory presented in the text (§ 3). C. PIRON and H. RUEGG shall publish a note in this journal, which shows how the two contradictory points of view can be understood.

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