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Quantum Theory in Real Hilbert Space III: Fields of the 1st kind (Linear Field Operators)

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Abstract: The method of RHS (real Hilbert space, see^{1) 2)}) is applied to the free scalar and spinor fields. We remark, that *two kinds of fields exist*:

Fields of the 1st kind commute with \check{J} ($\rightarrow i = \sqrt{-1}$ in CHS (complex Hilbert space)). In CHS, they are linear operators.

Fields of the 2nd kind anti-commute with \check{J} . In CHS, they are anti-linear operators.

The general formulas of this article are valid for both cases. In this publication, only the fields of the 1st kind are explicitly discussed. The relation between *statistics* and *strong time reflection* (CT) are clarified. Furthermore, a concise formulation of *contragredient four-spinors* is given. Some well known formulas are explicitly restated in RHS in order to show the difference between fields of the 1st kind and fields of the 2nd kind³⁾.

§ 1. Field Operators

We consider the *scalar field* $w(x)$ ($\neq w^T(x)$) and the *N-component spinor field* $\psi^A(x)$ ($\neq \psi^{TA}(x)$, $AB\dots = 12\dots N$). Field operators satisfy the *wave equation*

$$(\square - M^2) w(x) = (\square - M^2) \psi^A(x) = 0, \quad (1.1)$$

$$\square = - \text{sig}(g^{nn}) g^{\alpha\beta} \partial_\alpha \partial_\beta = \Delta - \partial_t^2,$$

$$\Delta = \sum_1^d (\partial_i)^2; \quad d = n - 1; \quad \partial_n = \partial_t. \quad (1.2)$$

Observables $F^{\alpha\dots}(x)$ are *bi-linear forms* in $w(x)$ and $w^T(x)$ (or $\psi^A(x)$ and $\psi^{TA}(x)$) and their derivatives (involving numbers or \check{J} -dependent operators):

$$w_\alpha(x) = \partial_\alpha w(x); \quad \psi_\alpha^A(x) \equiv \partial_\alpha \psi^A(x). \quad (1.3)$$

There exist two kinds of observables, $F^{(1)\alpha\dots}(x)$ and $F^{(2)\alpha\dots}(x)$, depending on whether the transposed field operator operates *after* ($F^{(1)}$) or *before* ($F^{(2)}$)

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the untransposed operators. Observables are *symmetric operators* in RHS ($F^T = F$), which *commute* with \check{J} . From

$$[A B, C] = A [B, C]_{\mp} \pm [A, C]_{\mp} B \quad (1.4)^*$$

follows, that fields either *commute* (*fields of the 1st kind*)

$$[\check{J}, w(x)] = 0; \quad [\check{J}, \psi^A(x)] = 0 \quad (1.5, 1^{\text{st}}\text{k.})$$

or *anti-commute* (*fields of the 2nd kind*) with \check{J} .

$$(\check{J}, \bar{w}(x)) = 0; \quad (\check{J}, \bar{\psi}^A(x)) = 0. \quad (1.5, 2^{\text{nd}}\text{k.})^* **)$$

Using the operators

$$\check{J} = j \times 1, \quad \bar{K} = k \times 1, \quad \check{L} = l \times 1 \quad (1.6)$$

where j , k and l are the pseudoquaternions (see I A-4.9), one concludes from (we write w , for w and ψ^A)

$$w = 1 \times w_{(r)} + j \times w_{(i)} + k \times w_{(k)} + l \times w_{(l)} \quad (1.7)$$

(a form analogous to (I A-2.3)) and its transposed, that only

$$w = 1 \times w_{(r)} + j \times w_{(i)} \quad (1.8, 1^{\text{st}}\text{k.})$$

or

$$\bar{w} = k \times w_{(k)} + l \times w_{(l)} = (1 \times w_{(k)} + j \times w_{(l)}) (k \times 1) = w \bar{K} \quad (1.8, 2^{\text{nd}}\text{k.})$$

can occur in the bilinear forms defining observables. *The present article is essentially restricted to fields of the 1st kind.* However some formulas, valid for either kind of fields are included. The discussions of fields of the 2nd kind is reserved to a later publication (see ³)).

For either kind of field, a *phase transformation*

$$'w(x) = e^{\lambda \check{J}} w(x); \quad 'w^T(x) = w^T(x) e^{-\lambda \check{J}} \quad (1.9)$$

leaves the observables invariant.

§ 2. Quantization of the scalar field

We look for an observable $\theta^{\alpha\beta}(x)$, from which \check{H}_{μ} and $\check{M}_{\mu\nu}$ may be constructed (see (I 0.25), (I 0.27)). In principle, bilinear observables, for example the scalar

$$F^{(1)}(x) = w^T(x) w(x) \quad (2.1)$$

*) $[A, B] = [A, B]_{-} = AB - BA$; $(A, B) = [A, B]_{+} = AB + BA$.

***) Operators with a $-$ ($\bar{K}, \bar{L}, \bar{w}, \dots$) anti-commute with \check{J} .

transform according to

$$\begin{aligned} 'F^{(1)}('x) &= (O_{(L)}^{-1} w^T('x) O_{(L)}) (O_{(L)}^{-1} w('x) O_{(L)}) = \\ &= F^{(1)}(L^{-1}'x) = F^{(1)}(x) \end{aligned} \tag{2.2}$$

if

$$'w('x) = O_{(L)}^{-1} w('x) O_{(L)} = e^{\lambda \check{J}} w(L^{-1}'x). \tag{2.3}$$

For infinitesimal transformations, the phase λ must be zero. The proper Lorentz-group $\{L_{(\text{cont})}\}$, is therefore generated in RHS by (I 5.6) with

$$[\check{J} \check{H}_\mu, w(x)] = -w_\mu(x), \tag{2.4}$$

$$[\check{J} \check{M}_{\mu\nu}, w(x)] = -[x_\mu, \partial_\nu] w(x). \tag{2.5}$$

Thus, fields of the 1st kind transform like scalar observables. (For fields of the 2nd kind, it is essential that \check{J} is *inside* of the commutators (2.4) and (2.5)!) For pseudochronous and pseudochorous transformations, a phase factor may occur.

We have, for the most general $\theta^{\alpha\beta}$, the form:

$$\theta^{\alpha\beta} = \alpha_1 \theta^{(1)\alpha\beta} + \alpha_2 \theta^{(2)\alpha\beta}, \tag{2.6}$$

$$\theta^{(1)\alpha\beta}(x) = (w^{T\alpha} w^\beta + w^{\beta T} w^\alpha - g^{\alpha\beta}(w_\rho^T w^\rho + M^2 w^T w))(x), \tag{2.7(1)}$$

$$\theta^{(2)\alpha\beta}(x) = (w^\alpha w^{T\beta} + w^\beta w^{T\alpha} - g^{\alpha\beta}(w_\rho w^{T\rho} + M^2 w w^T))(x) \tag{2.7(2)}$$

and (2.4) takes the form

$$\begin{aligned} & - [\check{J} \check{H}_\mu, w(y')] = \\ & = - \int d\check{\sigma}_\alpha(y) \{ \alpha_1 [\check{J}(w^{T\alpha} w_\mu)(y), w(y')] + \alpha_2 [\check{J}(w^\alpha w_\mu^T)(y), w(y')] \\ & \quad + \alpha_1 [\check{J}(w_\mu^T w^\alpha)(y), w(y')] + \alpha_2 [\check{J}(w_\mu w^{T\alpha})(y), w(y')] \} \\ & + \int d\check{\sigma}_\mu(y) \{ \alpha_1 [\check{J}(w_\alpha^T w^\alpha)(y), w(y')] + \alpha_2 [\check{J}(w^\alpha w_\alpha^T)(y), w(y')] \} \\ & + M^2 \int d\check{\sigma}_\mu(y) \{ \alpha_1 [\check{J}(w^T w)(y), w(y')] + \alpha_2 [\check{J}(w w^T)(y), w(y')] \} \\ & \equiv w_\mu(y'). \end{aligned} \tag{2.8}$$

We chose $\tau(y) = \tau(y') = 0$ i.e. y and y' are events on the same hypersurface with a time like normal $\check{\nu}^\alpha(y)$ ($d\check{\sigma}^\alpha(y) = \check{\nu}^\alpha(y) d\sigma(y)$; $(\check{\nu}^\alpha \check{\nu}_\alpha)(y) = \text{sig}(g^{nn})$). w, w_μ, w_μ^T and w^T being linearly independent, all terms, except

the first and fourth terms, have to cancel out. In particular, the last integral proportional to M^2 has to vanish. This leads to:

$$\alpha_1 [\check{J}(w^T w)(y), w(y')] + \alpha_2 [\check{J}(w w^T)(y), w(y')] = 0, \quad (2.9)$$

$$\begin{aligned} -\alpha_1 [\check{J}(w^T \alpha w_\mu)(y), w(y')] - \alpha_2 [\check{J}(w_\mu w^T \alpha)(y), w(y')] = \\ = \check{\delta}^\alpha(y y') w_\mu(y). \end{aligned} \quad (2.10)$$

Where $\check{\delta}^\alpha(y y')$ is the (pseudochronous) δ -function on the surface $\tau(y) = 0$:

$$\check{\delta}^\alpha(y y') = \check{\delta}^\alpha(y' y); \quad \int d\check{\sigma}_\alpha(y) \check{\delta}^\alpha(y' y) f(y) = f(y'). \quad (2.11)$$

In order that the 2nd and 3rd term cancel against the second integral, the symmetry condition, compatible with (2.11)

$$\int d\check{\sigma}_\mu(y) \check{\delta}_\alpha(y y') f(y) = \text{sig}(g^{nn}) (\check{v}_\mu \check{v}_\alpha)(y') f(y') \quad (2.12)$$

must be satisfied. (2.9) and (2.10) are necessary conditions for (2.4). We may integrate these two conditions, using the *pseudochronous invariant number*, defined by

$$\check{D}^0(x y) = \check{D}^0(x - y) = -\check{D}^0(y x), \quad (2.13)$$

$$(\square_x - M^2) \check{D}^0(x y) = 0, \quad (2.14)$$

$$\check{D}^0(y y') = 0; \quad \partial_\alpha^y \check{D}^0(y y') = -\check{\delta}_\alpha(y y') \quad (2.15) *$$

obtaining

$$\alpha_1 [\check{J} w^T(x) w(z), w(y')] + \alpha_2 [\check{J} w(z) w^T(x), w(y')] = \check{D}^0(x y') w(z). \quad (2.16)$$

A somewhat lengthy calculation shows, that (2.16) is a sufficient condition for (2.5). The CR (commutation relation) (2.16) (although more general) is analogous to the CR proposed by GREEN⁴) and VOLKOV⁵) for spinor fields (see (8.8)).

Let us consider fields of the 1st kind. Applying (1.4), we find:

$$\begin{aligned} \check{J} \alpha_1 \{w^T(x) [w(z), w(y)]_\mp \pm [w^T(x), w(y)]_\mp w(z)\} + \\ + \check{J} \alpha_2 \{w(z) [w^T(x), w(y)]_\mp \pm [w(z), w(y)]_\mp w^T(x)\} = \\ = \check{D}^0(xy) w(z). \end{aligned} \quad (2.17, 1^{\text{st}} \text{ k.})$$

*) The particular choice $\tau(y) = y^n - y'^n = 0$, $d\check{\sigma}_\alpha(y) = (00\dots 0 d^d y)$; $\check{\delta}^\alpha(y y') = (00\dots 0 \delta(\vec{y} - \vec{y}'))$ leads to the usual definition of $\check{D}^0(x y)$ (metric (6.14)).

The most simple solution is

$$\boxed{[w(x), w(y)]_{\mp} = 0} \tag{2.18, 1^{st} k.}$$

$$\check{J} [w^T(x), w(y)]_{\mp} = \check{D}^0(x y) \tag{2.19, 1^s k.}$$

$$\boxed{\pm \alpha_1 + \alpha_2 = 1} \tag{2.20, 1^{st} k. \mp}$$

It will be more convenient to write (2.19) in the more usual form:

$$\boxed{\check{J} [w(x), w^T(y)]_{\mp} = \pm \check{D}^0(x y)} \tag{2.19, 1^{st} k.}$$

§ 3. The Charge Operator for Scalar Fields

The observables

$$\check{j}^{(1)\alpha}(x) = ((\check{J} w)^T w^\alpha + w^{T\alpha} (\check{J} w)) (x), \tag{3.1^{(1)} *}$$

$$\check{j}^{(2)\alpha}(x) = -(\check{J} w w^{T\alpha} + w^\alpha (\check{J} w)^T) (x) \tag{3.1^{(2)} *}$$

satisfy the continuity equation. We form:

$$\check{j}^\alpha(x) = \beta_1 \check{j}^{(1)\alpha}(x) + \beta_2 \check{j}^{(2)\alpha}(x) \tag{3.2}$$

defining thus a $\tau(y)$ -independent scalar

$$Q = \beta_1 Q^{(1)} + \beta_2 Q^{(2)} = \int d\check{\sigma}_\alpha(y) \check{j}^\alpha(y) \tag{3.3}$$

called the *charge* Q . If we require, that the phase transformation (1.9) is an orthogonal transformation in RHS

$$'w(x) = O^{-1}(\lambda) w(x) O(\lambda) = e^{\lambda \check{J}} w(x), \tag{3.4}$$

$$O(\lambda) = e^{\check{J} \lambda Q}. \tag{3.5}$$

We need the CR:

$$\begin{aligned} & - [\check{J} Q, w(y')] = \\ & = - \int d\check{\sigma}_\alpha(y) \{ \beta_1 [(\check{J} (\check{J} w)^T w^\alpha) (y), w(y')] + \beta_1 [(\check{J} w^{T\alpha} \check{J} w) (y), w(y')] + \\ & + \beta_2 [(w w^{T\alpha}) (y), w(y')] - \beta_2 [(\check{J} w^\alpha (\check{J} w)^T) (y), w(y')] \} \equiv \check{J} w(y'). \end{aligned} \tag{3.6}$$

*) The signes have been chosen so as to give, for fields of the 1st kind:

$$\check{j}^{(1)\alpha}(x) = \check{J}^{-1} (w^T w^\alpha - w^{T\alpha} w) (x), \tag{3.1^{(1)}, 1^{st} k.}$$

$$\check{j}^{(2)\alpha}(x) = \check{J}^{-1} (w w^{\alpha T} - w^\alpha w^T) (x). \tag{3.1^{(2)}, 1^{st} k.}$$

For fields of the 1st kind, this condition reduces to

$$\begin{aligned}
 & - [Q, w(y')] = \\
 & = \int d\check{\sigma}_\alpha(y) \{ \beta_1 [\check{J}(w^T w^\alpha)(y), w(y')] - \beta_2 [\check{J}(w^\alpha w^T)(y), w(y')] - \\
 & \quad - \beta_1 [\check{J}(w^T w^\alpha)(y), w(y')] + \beta_2 [\check{J}(w w^T)(y), w(y')] = \\
 & = \int d\check{\sigma}_\alpha(y) \check{\delta}^\alpha(y, y') w(y) = w(y'). \tag{3.6, 1st k.}
 \end{aligned}$$

Comparison with (2.16) shows, that the first two terms do not contribute if $\beta_1 = \lambda \alpha_1$ and $\beta_2 = -\lambda \alpha_2$, while the second two terms equal the integral in the third member, if

$$\beta_1 = \alpha_1, \quad \beta_2 = -\alpha_2. \tag{3.7, 1st k.}$$

Thus, even for the general CR (2.16), Q defined by (3.7) is the generator of the infinitesimal phase transformation for fields of the 1st kind.

§ 4. Charge Conjugation for the Scalar Field of the 1st kind

For fields of the 1st kind and for the most simple CR's (2.18₋) and (2.19₋), follows that:

$$'w(x) = O_{(C)}^{-1} w(x) O_{(C)} = w^T(x), \tag{4.1}$$

$$'w^T(x) = O_{(C)}^{-1} w^T(x) O_{(C)} = w(x) \tag{4.1}^T$$

is an orthogonal transformation in RHS. The CR's lead to BE-statistics. In other words: $O_{(C)}$ -covariance requires BE-statistics. We have further:

$$O_{(C)}^{-1} \theta^{(1)\alpha\beta}(x) O_{(C)} = \theta^{(2)\alpha\beta}(x), \tag{4.2(1)}$$

$$O_{(C)}^{-1} \theta^{(2)\alpha\beta}(x) O_{(C)} = \theta^{(1)\alpha\beta}(x). \tag{4.2(2)}$$

Thus, choosing $\alpha_1 = \alpha_2 = 1/2$ in (2.20₋), we find, that

$$\theta^{\alpha\beta}(x) = \frac{1}{2} (\theta^{(1)\alpha\beta} + \theta^{(2)\alpha\beta})(x) \tag{4.3}$$

is invariant with respect to $O_{(C)}$. Furthermore, from

$$O_{(C)}^{-1} \check{j}^{(1)\alpha}(x) O_{(C)} = \check{j}^{(2)\alpha}(x), \tag{4.4(1)}$$

$$O_{(C)}^{-1} \check{j}^{(2)\alpha}(x) O_{(C)} = \check{j}^{(1)\alpha}(x) \tag{4.4(2)}$$

follows, that on account of (3.7) and (3.3)

$$O_{(C)}^{-1} \check{Q} O_{(C)} = -\check{Q} \tag{4.5}^*$$

i.e. \check{j}^α and \check{Q} change sign under $O_{(C)}$.

We may now define two kinds of time reversal

(1) $T \rightarrow O_{(T)}$, *weak time reversal* (T)

$$'w('x) = O_{(T)}^{-1} w('x) O_{(T)} = w(T^{-1} 'x) \tag{4.7}^{**}$$

with respect to which $\check{j}^\alpha(x)$ is a pseudochronous vector, and \check{Q} a scalar;

(2) $T \rightarrow O_{(C)} O_{(T)} \equiv O_{(CT)}$, *strong time reversal* (CT)

$$'w('x) = O_{(CT)}^{-1} w('x) O_{(CT)} = w^T(T^{-1} 'x) \tag{4.8}$$

with respect to which $\check{j}^\alpha(x)$ is an (ortho)vector and \check{Q} a pseudochronous scalar.

The second definition seems more appropriate because *classical particle theory* (see STUECKELBERG⁸) defines

$$\begin{aligned} \check{j}^\alpha(x) &= \int_{-\infty}^{+\infty} d\lambda \dot{z}^\alpha(\lambda) \delta(x - z(\lambda)), \\ \dot{z}^\alpha(\lambda) &= \frac{d}{d\lambda} z^\alpha(\lambda); \quad (\dot{z}_\alpha \dot{z}^\alpha)(\lambda) = \text{sig}(g^{nn}), \\ \check{Q} &= \int d\sigma_\alpha(y) \check{j}^\alpha(y) = \text{sig}(\dot{z}^n(\lambda)). \end{aligned}$$

Thus $O_{(C)}$ -covariance or strong time reversal ($O_{(CT)}$)-covariance decide for BE-statistics in the case of scalar fields (see SCHWINGER⁶) and PAULI⁷).

§ 5. The Development of the Scalar Field (1st kind) in Terms of Positive Frequency Wave Packet Operators

Let us define integral operators Ω and $\Omega^{1/2}$, operating on space functions $f(\vec{x})$ which vanish sufficiently strong for $|\vec{x}| \rightarrow \infty$:

$$\Omega f(\vec{x}) = (M^2 - \Delta)^{1/2} f(\vec{x}) = \int d^d y \Omega(|\vec{x} - \vec{y}|) f(\vec{y}), \tag{5.1}$$

$$\Omega^{1/2} f(\vec{x}) = (M^2 - \Delta)^{1/4} f(\vec{x}) = \int d^d y \Omega^{1/2}(|\vec{x} - \vec{y}|) f(\vec{y}). \tag{5.2}$$

$$*) \quad O_{(C)}^{-1} \check{j}^\alpha(x) O_{(C)} = -\check{j}^\alpha(x). \tag{4.6}$$

***) $'x = Tx \rightarrow \{ 'x^i = x^i; 'x^n = -x^n \}$. $T^{-1} = T$. The arbitrary phase factor may be left out.

The kernels $\Omega(|\vec{z}|)$ are essentially Hankel functions decreasing $\propto \exp(-|M||\vec{z}|)$ for $|\vec{z}| \gg |M|^{-1}$. We note especially

$$\int d^d x g(\vec{x}) \cdot \Omega f(\vec{x}) = \int d^d x g(\vec{x}) \Omega \cdot f(\vec{x}) = \int d^d x (\Omega^{1/2} g) (\Omega^{1/2} f) (\vec{x}). \quad (5.3)^*$$

In terms of Ω , we define *two denumerable sets* $\{u', u'', \dots, u^{(e)}, \dots\}$ and $\{v', v'', \dots, v^{(e)}, \dots\}$ of *positive frequency wave packet operators* (PFWP's) depending but on the operator \check{J} and vanishing for $|\vec{x}| \rightarrow \infty$. They satisfy:

$$-\partial_t u'(\vec{x}, t) = \partial^n u'(x) = \check{J} \Omega u'(\vec{x}, t). \quad (5.4)$$

These sets are *normalised* in terms of 'matrix elements'

$$j^\alpha(u', v')(x) = -j^\alpha(v', u')(x) = \frac{1}{2} \check{J}^{-1} (u' (\cdot \partial^\alpha - \partial^\alpha \cdot) v')(x), \quad (5.5)^*$$

$$\begin{aligned} Q(u', v') &= \int (d\check{\sigma}_\alpha j^\alpha(u', v'))(y) = \frac{1}{2} \check{J}^{-1} \int d^d y (u' (\cdot \partial^n - \partial^n \cdot) v')(y) = \\ &= \frac{1}{2} \int d^d y (u' (\cdot \Omega - \Omega \cdot) v')(\vec{y}, t) = 0, \end{aligned} \quad (5.6)$$

$$\begin{aligned} Q(u''^T, u') &= -Q(u', u''^T) = \frac{1}{2} \int d^d y (u''^T (\cdot \Omega + \Omega \cdot) u')(\vec{y}, t) = \\ &= \int d^d y ((\Omega^{1/2} u''^T) (\Omega^{1/2} u'))(\vec{y}, t) \equiv \delta_{u'' u'} \geq 0. \end{aligned} \quad (5.7)$$

Furthermore, each set is *complete* if the \check{J} dependent operators:

$$\mathbf{S}_{u'} u'(x) u'^T(y) = D^+(x y) = D^+(x - y), \quad (5.8^+)$$

$$\mathbf{S}_{u'} u'^T(x) u'(y) = D^-(x y) = D^-(x - y) = D^+(x y) = D^{+T}(x y) \quad (5.8^-)$$

depend but on $x - y$ and are invariant with respect to the subgroup $\{L_{(\text{ochr})}\}$. A PF-solution $f^+(x)$ of the wave equation may be expanded in terms of one of the sets:

$$f^+(x) = \mathbf{S}_{u'} u'(x) f_u^+ = \frac{1}{2} \check{J}^{-1} \int d\check{\sigma}_\alpha(y) D^+(x y) (\cdot \partial_y^\alpha - \partial_y^\alpha \cdot) f^+(y). \quad (5.9^+)$$

For NF-solutions:

$$f^-(x) = \mathbf{S}_{v'} v'^T(x) f_v^- = -\frac{1}{2} \check{J}^{-1} \int d\check{\sigma}_\alpha(y) D^-(x y) (\cdot \partial_y^\alpha - \partial_y^\alpha \cdot) f^-(y) \quad (5.9^-)$$

*) Operators $\cdot \partial^\alpha$, $\cdot \Omega$ (with point on the left) operate, in the usual way, *to the right*. Operators $\partial^\alpha \cdot$, $\Omega \cdot$ (with point on the right) operate *to the left*.

holds. The general solution of the wave equation may be written as

$$\begin{aligned}
 w(x) &= 2^{-1/2} \left(\mathbf{S}_{u'} a_{u'} u'(x) + \mathbf{S}_{v'} b_{v'}^T v'^T(x) \right) = \\
 &= - \int \overset{\circ}{d}\sigma_\alpha(y) \overset{\circ}{D}^0(x y) (\cdot \partial_y^\alpha - \partial_y^\alpha \cdot) w(y) = \\
 &= \int \overset{\circ}{d}\sigma_\alpha(y) (-\partial_x^\alpha \overset{\circ}{D}^0(x y) \cdot w(y) - \overset{\circ}{D}^0(x y) w^\alpha(y)) \tag{5.10}
 \end{aligned}$$

with

$$\overset{\circ}{D}^0(x y) = \frac{1}{2} \overset{\circ}{J} (D^+ - D^-) (x y) = - \overset{\circ}{D}^0(y x) = \overset{\circ}{D}^0(x y). \tag{5.11}$$

A comparison, for $x = y'$, shows, that the *pseudochronous number* (5.11) is identical with the number defined by (2.13), (2.14) and (2.15).

The CR's, resp. ACR's (2.18) and (2.19) imply

$$[a_{u'}, a_{u''}]_{\mp} = [b_{v'}^T, b_{v''}^T]_{\mp} = [a_{u'}, b_{v''}^T]_{\mp} = 0, \tag{5.12}$$

$$\begin{aligned}
 [a_{u'}, a_{u''}^T]_{\mp} &= \pm \delta_{u' u''}; & [b_{v'}^T, b_{v''}]_{\mp} &= \mp \delta_{v' v''}; \\
 [a_{u'}, b_{v'}]_{\mp} &= 0. \tag{5.13}
 \end{aligned}$$

As $a_{u'}$, $a_{u'}^T$ and $a_{u'}^T$, $a_{u'}$ are positive operators, (5.13₊) contains an algebraic contradiction. This is PAULI's⁹⁾ argument for excluding ACR's and FD-statistics for scalar fields.

The CR's can be satisfied in terms of the creation- (a^T) and annihilation-(a)-operators:

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & \sqrt{2} & 0 & \cdot \\ 0 & 0 & 0 & \sqrt{3} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}; \quad a^T = \begin{pmatrix} 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & \cdot \\ 0 & \sqrt{2} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}; \quad N = a^T a = \begin{pmatrix} 0 & 0 & 0 & \cdot \\ 0 & 1 & 0 & \cdot \\ 0 & 0 & 2 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{5.14}$$

writing

$$\left. \begin{aligned}
 a_{u'(e)} &= 1 \times (1 \times 1) \times (1 \times 1) \times \cdots \times (a \times 1) \times (1 \times 1) \times \cdots \\
 b_{v'(e)} &= 1 \times (1 \times 1) \times (1 \times 1) \times \cdots \times (1 \times a) \times (1 \times 1) \times \cdots
 \end{aligned} \right\} \tag{5.15}$$

The eigenvalues of $N_{u'} = a_{u'}^T a_{u'}$ are the non-negative integers and the charge has the form

$$Q = \mathbf{S}_{u'} \left(N_{u'} + \frac{1}{2} \right) - \mathbf{S}_{v'} \left(N_{v'} + \frac{1}{2} \right) = \mathbf{S}_{u'} N_{u'} - \mathbf{S}_{v'} N_{v'}. \tag{5.16}$$

No 'zero-point charge' appears because a 1 to 1 correspondance between the sets $\{u'\}$ and $\{v'\}$ can be established.

Charge conjugation can be written explicitly, if the sets $\{u'\}$ and $\{v'\}$ are chosen *identical*:

$$\begin{aligned} 'a_{u'} &= O_{(C)}^{-1} a_{u'} \quad O_{(C)} = b_{v'} , \\ 'b_{v'} &= O_{(C)}^{-1} b_{v'} \quad O_{(C)} = a_{u'} \end{aligned} \tag{5.17}$$

from which:

$$O_{(C)} = O_{(C)}^{-1} = O_{(C)}^T \tag{5.18}$$

follows. The operator is

$$O_{(C)} = \prod_{u'} e^{-\pi/2 (a_{u'}^T b_{v'} - b_{v'}^T a_{u'})} e^{\pm \check{J} \pi N_{u'}} . \tag{5.19}$$

In order to quantize explicitly $\check{\Pi}_\mu$, denumerable sets, satisfying

$$\begin{aligned} \partial^\alpha u'(x) &\cong \check{J} \check{k}'^\alpha u'(x) , \\ \check{k}'_\alpha \check{k}'^\alpha + M^2 &= 0 ; \quad \check{k}'^n \geq |M| \end{aligned} \tag{5.20}$$

have to be chosen. Taking in account some additional orthogonality relations, one finds

$$\check{\Pi}^\mu \cong \mathbf{S}_{u'} \left(N_{u'} + \frac{1}{2} \right) \check{k}'^\mu + \mathbf{S}_{v'} \left(N_{v'} + \frac{1}{2} \right) \check{k}'^\mu . \tag{5.21}$$

(5.21) is symmetric for particle and antiparticle states. Aside from the infinite 'zero-point contribution', the energy spectrum has a lower limit: Thus, thermo-statistics with a positive absolute temperature can be applied¹⁰).

§ 6. Dual Spin-Spaces

In this section we deal exclusively with real and complex numbers: we distinguish between φ^A *contravariant* (ctr) and χ_A *covariant* (cov) *spinors*.

φ^A and χ_A are two dual, N-dimensional *spin spaces* (SS). We shall distinguish between (real) RSS and (complex) CSS. The general case being CSS, all symbols φ^A, χ_A, \dots should be written as $\widehat{\varphi}^A, \widehat{\chi}_A, \dots$. For any complex number

$$\widehat{\varrho} = \varrho_{(r)} + i \varrho_{(i)} \Leftrightarrow \varrho = \varrho(\check{J}) = \varrho_{(r)} + \check{J} \varrho_{(i)} \tag{6.1)*}$$

gives the relation between the complex number $\widehat{\varrho}$ and the \check{J} -dependent operator $\varrho = \varrho(\check{J})$ (see I, Annex 2).

*) We shall use $\widehat{\varrho}, \widehat{\sigma}, \dots$ for complex numbers (resp. for \check{J} -dependent operators) and λ, μ, \dots for real numbers.

SS or A -space is related to physical space-time or α -space by a *mixed A -space bi-spinor α -space vector* $\gamma^{\alpha A}_B$, satisfying

$$\gamma^{\alpha A}_C \gamma^{\beta C}_B + \gamma^{\beta A}_C \gamma^{\alpha C}_B = 2 g^{\alpha\beta} \gamma^{0A}_B \quad (6.2)$$

or, in matrix notation

$$(\gamma^\alpha, \gamma^\beta) = 2 g^{\alpha\beta} \gamma^0, \quad (6.2 M) *$$

where $\gamma^{0A}_B = \delta^A_B$ is the identity bi-spinor. (6.2) with $\widehat{\gamma}^{\alpha A}_B$ is an *algebraic equation* and, with $\gamma^{\alpha A}_B$, it is an *operator equation* depending but on J . Two contragredient spinors allow to define a scalar $\chi \varphi = \chi_A \varphi^A$ and a vector $\chi \gamma^\alpha \varphi = \chi_A \gamma^{\alpha A}_B \varphi^B$ in α -space.

We resume, without proof and references, a number of well known theorems:

(1) The ring $\Gamma = \{\gamma^r\}$, $r = 0, \alpha, [\alpha_1 \alpha_2], \dots, [\alpha_1 \alpha_2 \dots \alpha_n] = 1, 2, \dots, 2^n$ is formed from the totally antisymmetric α -space tensors

$$\begin{aligned} \gamma^{\alpha_1 \dots \alpha_\nu} &= \gamma^{[\alpha_1 \dots \alpha_\nu]} = (\nu!)^{-1} \sum_P (-1)^P P(\gamma^{\alpha_1} \dots \gamma^{\alpha_\nu}) = \\ &= \gamma^{\alpha_1} \dots \gamma^{\alpha_\nu} \text{ if all } \alpha_1 \dots \alpha_\nu \text{ are different.} \end{aligned} \quad (6.3 M) **$$

The subsets $\Gamma^{(\nu)} = \{\gamma^{\alpha_1 \dots \alpha_\nu}\}$ contain $\binom{n}{\nu}$ independent elements.

(2) Two irreducible representations of (6.2) $'\gamma^\alpha$ and γ^α are related by

$$'\gamma^\alpha S = S \gamma^\alpha \quad (6.4 M)$$

where S is either the zero matrix or a non-singular square matrix. In particular, for $n = 2m$ (even), all irreducible representations are *equivalent* i.e. we have

$$'\gamma^\alpha = S \gamma^\alpha S^{-1}; \quad S = \{S'^A_A\}; \quad 'N = N. \quad (6.5 M)$$

For $n = 2m + 1$, two non equivalent representations exist. A matrix S commuting with the $n \gamma^\alpha$'s of an irreducible representation is always of the form:

$$[S, \gamma^\alpha] = 0; \quad S = \varrho \gamma^0. \quad (6.6 M)$$

Furthermore if S and S' satisfy (6.5 M), we have

$$S' = \varrho S. \quad (6.7 M)$$

*) Matrix multiplication is always a *contraction over contragredient spinor indices*. Formulas with (... M) are matrix identities.

**) \sum_P implies summation over all $\nu!$ permutations P of $\alpha_1 \dots \alpha_\nu$ with $(-1)^P = +(-)1$ for even (odd) permutations.

(3) For $n = 2m$, the 2^n elements of the ring $\Gamma = \{\gamma^r\}$ are linearly independent.

(4) The relation

$$\gamma^r \gamma^s = \varepsilon^{rs} \gamma^t; \quad t = t(rs); \quad \varepsilon^{rs} = \pm 1 \quad (6.8 M)$$

holds.

(5) If an irreducible representation of (6.2) for $n = 2m$ is found, we may form

$$\gamma_{n+1} = \varrho \gamma^{12\dots n}, \quad \varrho = 1 \text{ or } \check{J} \quad (6.9 M)$$

and thus obtain an irreducible representation for $'n = n + 1 = 2m + 1$, because

$$(\gamma_{n+1}, \gamma^\alpha) = 0; \quad \alpha = 12 \dots 2m; \quad (\gamma_{n+1})^2 = \gamma^0. \quad (6.10 M)$$

(6) The number ξ^r in

$$(w. s.) \gamma_r \gamma^r = \xi^r \gamma^0 = \xi^{(v)} \gamma^0 \quad (6.11 M) *$$

depends but on the set $\Gamma^{(v)}$ and takes the values

$$\xi^{(v)} = (-1)^{v/2} \quad \text{for } v = 2\mu, \quad (6.12)$$

$$\xi^{(v)} = (-1)^{(v-1)/2} \quad \text{for } v = 2\mu + 1 \quad (6.13)$$

if

$$\text{signat}(g^{\alpha\beta}) = + (11 \dots 1 - 1). \quad (6.14)$$

(7) If the representation is given, to each L corresponds a transformation $S_{(L)}$

$$L \rightarrow \varrho S_{(L)}; \quad S \rightarrow L_{(L)} \quad (6.15)$$

(defined up to a factor ϱ) leaving $\gamma^{\alpha A}{}_B$ invariant:

$$\gamma'^{\alpha A}{}_B = L'^{\alpha}{}_{\alpha} S'^A{}_{(L)A} \gamma^{\alpha A}{}_B S^{-1B}{}_{(L)B}, \quad (6.16)$$

$$\gamma'^{\alpha} = L'^{\alpha}{}_{\alpha} S_{(L)} \gamma^{\alpha} S_{(L)}^{-1}. \quad (6.16 M)$$

This relation in SS is perfectly analogous to (I, 3.9) in RHS; instead of the arbitrary phase factor in (I, 3.12), the arbitrary $\hat{\varrho}$ appears in (6.15).

(8) To the infinitesimal transformation

$$L'^{\alpha}{}_{\alpha} = \delta'^{\alpha}{}_{\alpha} + \frac{1}{2} \delta \omega^{\mu\nu} \Sigma_{\mu\nu}{}^{\alpha}{}_{\alpha} \quad (6.17)$$

*) (w. s.) means 'without summation over indices in contragredient positions'.

corresponds

$$S_{(L)} = \gamma^0 + \frac{1}{4} \delta \omega^{\mu\nu} \gamma_{\mu\nu}. \quad (6.18 M)$$

(9) To the transformations $L = (P, T, PT)^*$ corresponds, for $n = 2m$ ($d = n - 1$)**)

$$\begin{aligned} S_P &= \varrho \gamma^n; & S_T &= \varrho \gamma^{12\dots d}; & S_{PT} &= \varrho \gamma_{n+1}; \\ S_P^{-1} &= \varrho^{-1} \gamma_n; & S_T^{-1} &= \varrho^{-1} \gamma_{d\dots 21}; & S_{PT}^{-1} &= \varrho^{-1} \gamma^{n+1}. \end{aligned} \quad (6.19 M)$$

(10) If the Dirac-equations are written as

$$(\gamma^{\alpha A}{}_B \partial_\alpha + M \gamma^{0A}{}_B) \varphi^B(x) = (\gamma^\alpha \partial_\alpha + M \gamma^0) \varphi(x) = 0, \quad (6.20 \text{ ctr})$$

$$\chi_A(x) (\gamma^{\alpha A}{}_B \partial_\alpha - M \gamma^{0A}{}_B) = \chi(x) (\gamma^\alpha \partial_\alpha - M \gamma^0) = 0, \quad (6.20 \text{ cov})$$

the momentum-energy-density tensor is

$$\theta^{\alpha\beta}(x) = T^{\alpha\beta}(x) + \frac{1}{2} \partial_\sigma s^{\sigma\alpha\beta}(x), \quad (6.21)$$

$$\partial_\sigma s^{\sigma\alpha\beta}(x) = (T^{\beta\alpha} - T^{\alpha\beta})(x), \quad (6.22)$$

$$T^\alpha{}_\beta(x) = \frac{1}{2} (\chi \gamma^\alpha (\cdot \partial_\beta - \partial_\beta \cdot) \varphi)(x), \quad (6.23)$$

$$\partial_\alpha T^\alpha{}_\beta(x) = 0, \quad (6.24)$$

$$s^{\alpha\beta\gamma}(x) = \frac{1}{2} (\chi \gamma^{\alpha\beta\gamma} \varphi)(x) \quad (6.25)$$

and the current-charge-density vector

$$j^\alpha(x) = (\chi \gamma^\alpha \varphi)(x), \quad (6.26)$$

$$j^\alpha(x) = j^\alpha_{(\text{convection})}(x) - \partial_\sigma m^{\sigma\alpha}(x), \quad (6.27)$$

$$j^\alpha_{(\text{convection})}(x) = - (2M)^{-1} (\chi (\cdot \partial^\alpha - \partial^\alpha \cdot) \varphi)(x), \quad (6.28)$$

$$m^{\alpha\beta}(x) = - (2M)^{-1} (\chi \gamma^{\alpha\beta} \varphi)(x). \quad (6.29)$$

In order that $\varphi^A(x)$ and $\chi_A(x)$ satisfy the wave equation (1.1), M must be a *real number* and the signature must have the form (6.14).

*) $P \rightarrow \{x^i = -x^i, x^n = x^n\}$, $T \rightarrow \{x^i = x^i, x^n = -x^n\}$.

$PT \rightarrow \{x^\alpha = -x^\alpha\}$ for $n = 2m$.

***) For $n = 2m + 1$, only $S_{(PT)}$ can be defined. As we require the full L -group, our discussion is limited to $n = 2m = \text{even}$, $d = n - 1 = \text{odd}$.

In addition to these well known theorems, we add the two following theorems which follow from a theorem of FROBENIUS and SCHUR^{11) 12)} (given in Annex):

(11) For $n = 2m$, the number of dimensions of an irreducible representation is

$$N = 2^{n/2} = 2^m. \tag{6.30}$$

(12) For

$$n = 2, 4 \pmod{8} = 2, 4; 10, 12; \dots \tag{6.31}$$

all representations are equivalent to a *real representation*. For

$$n = 6, 8 \pmod{8} = 6, 8; 14, 16; \dots \tag{6.32}$$

all representations are *necessarily complex*. However a matrix C , the *Pauli-Matrix*⁹⁾, exists, relating

$$\widehat{\gamma}^{*r} = \widehat{C} \widehat{\gamma}^r \widehat{C}^{-1} \quad \text{or} \quad \gamma^{Tr} = C \gamma^r C^{-1}. \tag{6.33}$$

(13) Every IMG (irreducible matrix group, see Annex) $\{\pm \widehat{\gamma}^r\}$ is equivalent to a unitary representation $\{\pm \widehat{\gamma}^r\}$. In RHS, this implies for every $\{\pm \widehat{\gamma}^r\}$, there exists an S

$$\widehat{\gamma}^r = S \gamma^r S^{-1} \tag{6.34 M}$$

so as to have

$$\gamma^{\sim Tr} \equiv \gamma^{\times r} \stackrel{*}{=} (\gamma^r)^{-1} = (w. s.) \xi^r \gamma_r. \tag{6.35 M) *}$$

(14) For $n = 2, 4 \pmod{8}$, where $\{\widehat{\gamma}^r\}$ may be chosen real, the unitary representation may be chosen *real and orthogonal*:

$$\gamma^{\sim r} \stackrel{*}{=} (\gamma^r)^{-1} = (w. s.) \xi^r \gamma_r. \tag{6.36 M) *}$$

) $\stackrel{}{=}$ signifies 'equal in a particular representation'.

\sim is the *transposed matrix* in SS (T being reserved for transposed operator in RHS, see I).

$$\gamma_A^{\sim r B} = \gamma^r B_A. \tag{6.37}$$

\sim interchanges the spinor-indices 'left \rightleftharpoons right', leaving their covariance unchanged!

\times is the *hermitian conjugate matrix* in CSS

$$\widehat{\gamma}_A^{\times r B} = \widehat{\gamma}^{*r B}_A. \tag{6.38}$$

To it corresponds, in a \check{J} -dependent representation, the operator relation in RHS

$$\gamma_A^{\times r B} = \gamma^{Tr B}_A. \tag{6.39}$$

§ 7. The fundamental spinors $\overset{\smile}{\eta}_{AB}$ and $\widehat{\eta}_{AB}$

In order to form observables $\theta^{\alpha\beta}$, T^α_β , $s^{\alpha\beta\gamma}$ and j^α ((6.21)–(6.25)) from a contravariant field operator $\psi^A(x)$, we need a *non-singular fundamental spinor* η_{AB} from which we may form a *covariant operator* $\psi^T_A(x) = \eta_{AB}\psi^{TB}(x)$

$$\eta^{-1 A C} \eta_{CB} = \eta_{BC} \eta^{-1 CA} = \gamma^{0A}{}_B, \tag{7.1}$$

$$\eta^{-1} \eta = \eta \eta^{-1} = \gamma^0. \tag{7.1 M}$$

We shall assume a *real representation* of $\gamma^{\alpha A}{}_B$ i.e. restrict our considerations to $n = 2, 4 \pmod{8}$. However the theory can also be written for $\overset{\smile}{J}$ -dependent $\gamma^{\alpha A}{}_B$ (thus for $n = 6, 8 \pmod{8}$, but the calculations are much longer*).

Analogous to (6.16), we require invariance of η with respect to the group $S_{(L)}$ (i.e. $c(L) = 1$ or $= \text{sig}(\det(L'^i_i))$ or $= \text{sig}(L'^n_n)$)

$$\eta'_{A'B} = c(L) \eta_{AB} S_{(L)}^{-1 A}{}_{A'} S_{(L)}^{-1 B}{}_{B'}, \tag{7.2}$$

$$\eta = c(L) S_{(L)}^{-1} \sim \eta S_{(L)}^{-1}. \tag{7.2 M}$$

For $L_{(\text{cont})}$ (cf. (6.18)) we have

$$\gamma^{\sim\mu\nu} \eta + \eta \gamma^{\mu\nu} = 0, \tag{7.3 M}$$

which allows the *two possibilities*:

$$\gamma^{\sim\alpha} = \mp \overset{\smile}{\eta} \gamma^\alpha \overset{\smile}{\eta}^{-1} \quad \text{i. e.} \quad \gamma^{\sim\alpha} \overset{\smile}{\eta} = \mp \overset{\smile}{\eta} \gamma^\alpha. \tag{7.4 M \smile}$$

We shall now show that $\overset{\smile}{\eta}$ is pseudochronous while $\widehat{\eta}$ is pseudochorous. From (7.2), with (6.19 M) ($\varrho = \lambda$), follows

$$\overset{\smile}{\eta} = c(P) \lambda^{-2} \gamma_n^{\sim} \overset{\smile}{\eta}_n \gamma_n = \mp c(P) \lambda^{-2} \overset{\smile}{\eta} (\gamma_n)^2 = \pm c(P) \lambda^{-2} \overset{\smile}{\eta}. \tag{7.5 M}$$

Thus we have $\lambda^2 = 1$ and $c(P) = + (-) 1$ for $\overset{\smile}{\eta}$ ($\widehat{\eta}$). $\overset{\smile}{\eta}$ is thus ortho-chorous and $\widehat{\eta}$ is pseudochorous. Further

$$\begin{aligned} \overset{\smile}{\eta} &= c(T) \lambda^{-2} \gamma_{d\dots 1}^{\sim} \overset{\smile}{\eta} \gamma_{d\dots 1} = \\ &= \mp c(T) \lambda^{-2} \overset{\smile}{\eta} (\gamma_d)^2 \dots (\gamma_1)^2 = \mp c(T) \lambda^{-2} \overset{\smile}{\eta} \end{aligned} \tag{7.6 M}$$

leads again to $\lambda^2 = 1$, and $c(T) = - (+) 1$ for $\overset{\smile}{\eta}$ ($\widehat{\eta}$). $\overset{\smile}{\eta}$ is thus pseudo-chronous and $\widehat{\eta}$ orthochronous.

*) The statement given in a previous communication¹³), that only real representations can be used, is therefore erroneous!

Transposing (7.4), we find

$$\gamma^\alpha = \mp \overset{\smile}{\eta}^{-1} \sim \gamma^{\sim\alpha} \overset{\smile}{\eta}; \quad \gamma^{\sim\alpha} = \mp \overset{\smile}{\eta} \sim \gamma^\alpha \overset{\smile}{\eta}^{-1}. \quad (7.7 \text{ M})$$

(7.4) and (7.7 M) are both changes of representation of the type (6.5 M) (because $\gamma^\alpha = -\gamma^{\sim\alpha}$ is a change of representation). Thus, on account of (6.7 M), we have

$$\overset{\smile}{\eta} = \lambda \overset{\smile}{\eta}. \quad (7.8 \text{ M})$$

Transposing this equation, we find $\lambda^2 = 1$; $\overset{\smile}{\eta}$ and $\overset{\smile}{\eta}$ are either symmetric or antisymmetric matrices. Furthermore we have

$$\overset{\smile}{\eta}^{-1} \overset{\smile}{\eta} = S_{(PT)} = \pm \gamma^{12\dots n} = \pm \gamma_{n+1}. \quad (7.9 \text{ M})$$

In order to find the symmetry of $\overset{\smile}{\eta}$, we choose an orthogonal representation (6.36):

$$\gamma^{\sim i} \stackrel{*}{=} (\gamma^i)^{-1} = \gamma_i = \gamma^i, \quad (7.10 \text{ M})$$

$$\gamma^{\sim n} \stackrel{*}{=} (\gamma^n)^{-1} = \gamma_n = -\gamma^n. \quad (7.11 \text{ M})$$

From the ACR's (6.2 M) follows that $\overset{\smile}{\eta} \stackrel{*}{=} \lambda \gamma^n$ satisfies (7.4 M[∘]), $\overset{\smile}{\eta}$ is therefore an *anti-symmetric matrix* in this particular representation. Now, in analogy to (6.34 M), we have in a general representation

$$\overset{\smile}{\eta}'_{A'B} = \overset{\smile}{\eta}_{AB} S^{-1A}{}_{,A} S^{-1B}{}_{,B} \quad (7.12)$$

conservation of this antisymmetry: $\overset{\smile}{\eta}_{AB}$ is, in all (real) representations, *antisymmetric*:

$$\overset{\smile}{\eta}'_{AB} = \overset{\smile}{\eta}'_{BA} = -\overset{\smile}{\eta}'_{AB}; \quad \overset{\smile}{\eta}' \sim = -\overset{\smile}{\eta}'. \quad (7.13); (7.13 \text{ M})$$

We choose, in particular

$$\overset{\smile}{\eta}_{AB} \stackrel{*}{=} \gamma_B^{nA} \quad (7.14)$$

in order to have for the charge-density

$$\begin{aligned} \overset{\smile}{j}^{(1)n} &= \overset{\smile}{\psi}^T \gamma^n \psi = \overset{\smile}{\psi}_C^T \gamma^{nC} \psi^B = \psi^{TA} (-\overset{\smile}{\eta}_{AC} \gamma^{nC}) \psi^B \equiv \\ &\equiv \psi^{TA} \psi^B (-\overset{\smile}{\gamma}_{AB}^n) \stackrel{*}{=} \sum_A \psi^{TA} \psi^A > 0; \quad -\overset{\smile}{\gamma}_{AB}^n \stackrel{*}{=} \delta_B^A \end{aligned} \quad (7.15)$$

a *positive operator*. We may now determine the symmetries of the *pseudo-chronous covariant bi-spinors*:

$$\overset{\smile}{\gamma}^r{}_{AB} = \overset{\smile}{\eta}_{AC} \gamma^{rC}{}_{B}; \quad \overset{\smile}{\gamma}^{r(\text{cov})} = \overset{\smile}{\eta} \gamma^r. \quad (7.16); (7.16 \text{ M})$$

We have, on account of (7.4 M),

$$\overset{\smile}{\gamma^{\alpha_1 \dots \alpha_n}} = \overset{\smile}{\gamma^{\alpha_n}} \dots \overset{\smile}{\gamma^{\alpha_1}} \overset{\smile}{\eta} = - \overset{\smile}{\gamma^{\alpha_n}} \dots \overset{\smile}{\gamma^{\alpha_1}} \overset{\smile}{\eta} = - (-1)^n \overset{\smile}{\eta} \overset{\smile}{\gamma^{\alpha_n}} \dots \overset{\smile}{\gamma^{\alpha_1}}. \tag{7.17}$$

This gives the following table:

sym.:		$\overset{\smile}{\gamma^\alpha}$	$\overset{\smile}{\gamma^{\alpha_1 \alpha_2}}$		$\overset{\smile}{\gamma^{\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5}}$	(7.18)
antisym.:	$\overset{\smile}{\gamma^0} = \overset{\smile}{\eta}$			$\overset{\smile}{\gamma^{\alpha_1 \alpha_2 \alpha_3}}$	$\overset{\smile}{\gamma^{\alpha_1 \alpha_2 \alpha_3 \alpha_4}}$	

From (7.9 M) and (7.17) follows:

$$\overset{\smile}{\eta} = \pm \overset{\smile}{\eta} \quad \text{for} \quad n = \begin{cases} 2 \pmod{8} \\ 4 \pmod{8} \end{cases}. \tag{7.19}$$

§ 8. Quantization of the Spinor Field

If $\psi^A(x)$ satisfies (6.20 ctr), it follows from the symmetries of $\overset{\smile}{\gamma^{\alpha}_{AB}} = \overset{\smile}{\gamma^{\alpha}_{(AB)}}$ and $\overset{\smile}{\gamma^0_{AB}} = \overset{\smile}{\gamma^0_{[AB]}}$, that

$$\overset{\smile}{\psi}_A(x) = \overset{\smile}{\eta}_{AB} \overset{\smile}{\psi}^B(x) = - \overset{\smile}{\psi}^B(x) \overset{\smile}{\eta}_{BA} \tag{8.1}$$

satisfies (6.20 cov). The same is true for $\psi^{TA}(x)$ and $\psi^T_A(x)$, because we consider but real representations ($n = 2, 4 \pmod{8}$). We form the tensors $T^\alpha_\beta(x)$ and $s^{\alpha\beta\gamma}(x)$, either posing

$$\begin{aligned} \chi_A &= (\overset{\smile}{J} \overset{\smile}{\psi}_A)^T; & \varphi^A &= \psi^A: \\ T^{(1)\alpha}_\beta(x) &= \frac{1}{2} ((\overset{\smile}{J} \overset{\smile}{\psi})^T \overset{\smile}{\gamma}^\alpha \overset{\smile}{\psi}_\beta - (\overset{\smile}{J} \overset{\smile}{\psi}_\beta)^T \overset{\smile}{\gamma}^\alpha \psi) = \\ &= \frac{1}{2} (\psi^{TA} \overset{\smile}{J}^T \overset{\smile}{\psi}_\beta^B + \overset{\smile}{\psi}_\beta^{TB} \overset{\smile}{J} \psi^A) (x) (-\overset{\smile}{\gamma}^{\alpha}_{AB}) \end{aligned} \tag{8.2(1)}$$

or, posing

$$\begin{aligned} \chi_A &= \overset{\smile}{\psi}_A; & \psi^A &= (\overset{\smile}{J} \varphi^A)^T: \\ T^{(2)\alpha}_\beta(x) &= \frac{1}{2} (\overset{\smile}{\psi} \overset{\smile}{\gamma}^\alpha (\overset{\smile}{J} \overset{\smile}{\psi}_\beta)^T - \overset{\smile}{\psi}_\beta \overset{\smile}{\gamma}^\alpha (\overset{\smile}{J} \psi)^T) = \\ &= \frac{1}{2} (\psi^A \overset{\smile}{\psi}_\beta^{TB} \overset{\smile}{J}^T + \overset{\smile}{J} \overset{\smile}{\psi}_\beta^B \psi^{AT}) (x) (-\overset{\smile}{\gamma}^{\alpha}_{(AB)}). \end{aligned} \tag{8.2(2)}$$

The second form of both equations shows a) that $T^{(1)\alpha}_\beta$ and $T^{(2)\alpha}_\beta$ are *observables* (on account of the *symmetry* of $\overset{\smile}{\gamma}^{\alpha}_{(AB)}$) and b) that they are *orthochronous*, if we *define*:

$$\psi'^A(x) = O_{(L)}^{-1} \psi^A(x) O_{(L)} = e^{\check{J}\lambda} S'_{(L)A}{}^A \psi^A(L^{-1}x). \tag{8.3}^*$$

The same is true for $s^{(1)\alpha\beta\gamma}$ and $s^{(2)\alpha\beta\gamma}$, on account of the antisymmetry of $\check{\gamma}_{AB}^{\alpha\beta\gamma} = \check{\gamma}_{[A}^{\alpha\beta\gamma]}_{B]}$. Thus, $\theta^{(1)\alpha\beta}$ and $\theta^{(2)\alpha\beta}$ are *orthochronous observables*.

It can be shown, using the Dirac-equations and partial integrations, that one may write

$$\check{\check{H}}_{\mu}^{(1)} = \int d\check{\sigma}_{\alpha}(y) ((\check{J}\check{\psi})^T \gamma^{\alpha} \psi_{\mu})(y), \tag{8.4(1)}$$

$$\check{\check{H}}_{\mu}^{(2)} = - \int d\check{\sigma}_{\alpha}(y) (\check{\psi}_{\mu} \gamma^{\alpha} (\check{J}\check{\psi})^T)(y). \tag{8.4(2)}$$

With

$$\check{\check{H}}_{\mu} = \alpha_1 \check{\check{H}}_{\mu}^{(1)} + \alpha_2 \check{\check{H}}_{\mu}^{(2)},$$

the relation

$$- [\check{J} \check{\check{H}}_{\mu}, \psi^{B'}(y')] = \psi^{B'}(y') \tag{8.5}$$

is satisfied, if

$$\begin{aligned} & - \alpha_1 [(\check{J} (\check{J}\check{\psi})^T \gamma^{\alpha} \psi_{\mu})(y), \psi^{B'}(y')] + \alpha_2 [(\check{J} \check{\psi}_{\mu} \gamma^{\alpha} (\check{J}\check{\psi})^T)(y), \psi^{B'}(y')] \\ & \equiv \check{\delta}^{\alpha}(y' y) \psi^{B'}(y). \end{aligned} \tag{8.6}$$

This relation can be integrated, using the *invariant pseudochronous (real) number (function)*

$$\check{S}^{0A}{}_B(x y) = (-\gamma^{\alpha A}{}_B \partial_{\alpha}^x + \gamma^{0A}{}_B M) \check{D}^0(x y) \tag{8.7}^{**}$$

in the form

$$\begin{aligned} & - \alpha_1 [\check{J} (\check{J}\check{\psi}_A)^T(x) \gamma^{\alpha A}{}_B \psi^B(z), \psi^{B'}(y')] + \\ & + \alpha_2 [\check{J} \check{\psi}_A(z) \gamma^{\alpha A}{}_B (\check{J}\check{\psi}^B)^T(x), \psi^{B'}(y')] \equiv \check{S}^{0B'}{}_A(y' x) \gamma^{\alpha A}{}_B \psi^B(z). \end{aligned} \tag{8.8}$$

In order to verify (8.6), we operate on (8.8) with ∂_{μ}^z and pose $x = z = y$. Further we use (2.15)

$$\check{S}^{0B'}{}_A(y' y) = \gamma^{\beta B'}{}_A \check{\delta}_{\beta}(y' y) \tag{8.9}$$

and (2.12)

$$\begin{aligned} & \int d\check{\sigma}_{\alpha}(y) \check{\delta}_{\beta}(y' y) (\gamma^{\beta} \gamma^{\alpha})^{B'}{}_B f^B(y) = - \text{sig}(g_{nn}) (\check{\nu}_{\alpha} \check{\nu}_{\beta})(y') (\gamma^{\beta} \gamma^{\alpha})^{B'}{}_B f^B(y') = \\ & = f^{B'}(y') = \int d\check{\sigma}_{\alpha}(y) \check{\delta}^{\alpha}(y' y) f^{B'}(y). \end{aligned} \tag{8.10}^{***}$$

*) We may, as in (2.3), introduce the arbitrary phase factor for pseudochorous and for pseudochronous transformations.

***) The function $\check{S}^A{}_B(x y)$ is not to be confounded with the transformation matrix $S'^A{}_A$ or $S'_{(L)A}{}^A$.

***) $(\gamma^{\beta} \gamma^{\alpha})^{B'}{}_B \check{\delta}_{\beta}(y' y) = \gamma^{0B'}{}_B \check{\delta}^{\alpha}(y' y).$ (8.10 a)

So far, all relations are valid for fields of the 1st and of the 2nd kind. Leaving the discussion of fields of the 2nd kind for a later publication³⁾, we rewrite (8.3) for fields of the 1st kind ($[J, \psi^A] = 0$) using (1.4):

$$\begin{aligned}
 & -\alpha_1 \{ \overset{\smile}{\psi}_A^T(x) \gamma^{\alpha A}{}_B [\psi^B(z), \psi^{B'}(y')]_{\mp} \pm [\overset{\smile}{\psi}_A^T(x), \psi^{B'}(y')]_{\mp} \gamma^{\alpha A}{}_B \psi^B(z) \} + \\
 & + \alpha_2 \{ \overset{\smile}{\psi}_A(z) \gamma^{\alpha A}{}_B [\psi^{TB}(x), \psi^{B'}(y')]_{\mp} \pm [\overset{\smile}{\psi}_A(z), \psi^{B'}(y')]_{\mp} \gamma^{\alpha A}{}_B \psi^{TB}(x) \} = \\
 & = \overset{\smile}{S}^{0B'}{}_A(y' x) \gamma^{\alpha A}{}_B \psi^B(z). \tag{8.11, 1st k.}
 \end{aligned}$$

This is a generalised form of the CR's discussed by GREEN⁴⁾ and VOLKOV⁵⁾. The most simple solution is

$$\boxed{[\psi^A(x), \psi^B(y)]_{\mp} = 0} \tag{8.12, 1st k.}$$

and

$$\mp [\overset{\smile}{\psi}_A^T(x), \psi^{B'}(y')]_{\mp} = \overset{\smile}{S}^{0B'}{}_A(y' x) \tag{8.13, 1st k.}$$

which give to the 1st term the right kind of structure. In the 2nd term, we use the identity, following from (8.1) and (7.4 M)

$$\overset{\smile}{\psi}_A(z) \gamma^{\alpha A}{}_B \psi^{TB}(x) = \psi^B(z) \gamma^{\alpha A}{}_B \overset{\smile}{\psi}_A^T(x).$$

Using again (8.13) and (8.12), we find that (8.11) holds, provided

$$\boxed{\alpha_1 \mp \alpha_2 = 1.} \tag{8.14, 1st k. \mp}$$

We may rewrite (8.13) in the more usual form

$$\boxed{[\psi^A(x), \overset{\smile}{\psi}_B(y)]_{\mp} = \overset{\smile}{S}^{0A}{}_B(x y).} \tag{8.13, 1st k.}$$

§ 9. The Charge Operator for Spinor Fields

The two expressions for j^α (6.26) are

$$\overset{\smile}{j}^{(1)\alpha}(x) = (\overset{\smile}{\psi}^T \gamma^\alpha \psi)(x) = (\psi^{TA} \psi^B)(x) (-\overset{\smile}{\gamma}^{\alpha}{}_{AB}), \tag{9.1(1)}$$

$$\overset{\smile}{j}^{(2)\alpha}(x) = (\psi \gamma^\alpha \psi^T)(x) = (\psi^A \psi^{TB})(x) (-\overset{\smile}{\gamma}^{\alpha}{}_{AB}). \tag{9.1(2)}$$

Forming again $Q = \beta_1 Q^{(1)} + \beta_2 Q^{(2)}$ (see (3.2)), we have, for fields of the 1st kind, analogous to (3.6, 1st k.),

$$\begin{aligned} & - [Q, \psi^{B'}(y')] = \\ & = - \int d\overset{\smile}{\sigma}_\alpha(y) \{ \beta_1 [(\overset{\smile}{\psi}^T \gamma^\alpha \psi)(y), \psi^{B'}(y')] + \beta_2 [(\overset{\smile}{\psi} \gamma^\alpha \psi^T)(y), \psi^{B'}(y')] \} \equiv \\ & \equiv \int d\overset{\smile}{\sigma}_\alpha(y) \overset{\smile}{\delta}^\alpha(y' y) \psi^{B'}(y) = \psi^{B'}(y'). \end{aligned} \quad (9.2, 1^{\text{st}} \text{ k.})$$

For fields of the 1st kind, we compare (9.2) with (8.6), (the J 's cancel out and we omit the index μ of differentiation*). (9.2) results, if we pose again (3.7, 1st k.) i.e.

$$\beta_1 = \alpha_1; \quad \beta_2 = -\alpha_2. \quad (9.3, 1^{\text{st}} \text{ k.})$$

§ 10. Charge Conjugation for the Spinor Field (1st kind)

We see at once, rising the index of (8.13)

$$[\psi^A(x), \psi^{TB}(y)]_{\mp} = S^{0AB}(x y) \quad (10.1)$$

that the invariant number (function)

$$\begin{aligned} S^{0AB}(x y) & = (-\overset{\smile}{\gamma}^{\alpha AB} \partial_\alpha^x + \overset{\smile}{\gamma}^{0AB} M) \overset{\smile}{D}^0(x y) = \\ & = S^{0BA}(y x) = \overset{\smile}{\eta}^{-1BC} \overset{\smile}{S}^{0A}_C(x y) \end{aligned} \quad (10.2) **$$

is symmetric with respect to $A, x \rightleftharpoons B, y$. Therefore the substitution

$$\begin{aligned} \psi^A(x) & = O_{(C)}^{-1} \psi^A(x) O_{(C)} = \psi^{TA}(x), \\ \psi^{TA}(x) & = O_{(C)}^{-1} \psi^{TA}(x) O_{(C)} = \psi^A(x) \end{aligned} \quad (10.3)$$

is again an orthogonal transformation, if we choose the ACR's (8.12₊), (8.13₊), (10.1₊). Thus *FD-statistics for spinor fields is a consequence of $O_{(C)}$ -covariance or of $O_{(CT)}$ -covariance* (in perfect analogy to § 4). Furthermore we have, on account of (8.2), again the relations (4.2). Using (8.14₊), we have again, with $\alpha_1 = \alpha_2 = 1/2$, the expression (4.3): $\theta^{\alpha\beta}(x)$ is in-

*) Omitting the index μ means, that we use the integrated equation (8.8)!

$$**) \quad \overset{\smile}{\gamma}^{rAB} = \overset{\smile}{\eta}^{-1BD} \overset{\smile}{\gamma}^{rA}_D = \overset{\smile}{\eta}^{-1AC} \overset{\smile}{\eta}^{-1BD} \overset{\smile}{\gamma}^{r}_{CD}, \quad (10.4)$$

$$\overset{\smile}{\gamma}^{r(\text{ctr})} = -\overset{\smile}{\gamma}^r \overset{\smile}{\eta}^{-1} = -\overset{\smile}{\eta}^{-1} \overset{\smile}{\gamma}^{r(\text{cov})} \overset{\smile}{\eta}^{-1}. \quad (10.4 \text{ M})$$

variant with respect to $O_{(C)}$). The equations (4.4) are also valid. Therefore, using again (3.7, 1st k.) (= (9.3, 1st k.)) we have again (4.5) and (4.6)*: j^α and Q change sign under $O_{(C)}$. The rest of the discussion is analogous to § 4.

§ 11. The Development of the Spinor Field in Terms of Positive Frequency Wave Packet Operators

We introduce again two denumerable sets of PFWP's $\{\varphi'^A, \varphi''^A, \dots, \varphi^{(\varrho)A} \dots\}$ and $\{\chi'^A, \chi''^A \dots, \chi^{(\varrho)A} \dots\}$ satisfying the Dirac equation (6.20) and (5.4). The sets are normalised in terms of

$$j^\alpha(\varphi', \chi')(x) = j^\alpha(\chi', \varphi')(x) = (\overset{\smile}{\varphi'} \gamma^\alpha \overset{\smile}{\chi'})(x) = (\varphi'^A \chi'^B)(x) (-\overset{\smile}{\gamma}_{AB}^\alpha). \quad (11.1)**$$

On account of the decomposition (6.27), one verifies that the normalisation

$$Q(\varphi', \chi') = 0; \quad Q(\varphi'^T, \varphi'') = Q(\varphi'', \varphi'^T) = \delta_{\varphi' \varphi''} > 0 \quad (11.2)$$

is possible. Completeness is assured, if the $\overset{\smile}{J}$ -dependent operators

$$\overset{\smile}{S}^{+A}_B(x y) \equiv \mathbf{S}_{\varphi'} \varphi'^A(x) \overset{\smile}{\varphi}'^T_B(y) = \overset{\smile}{S}^{+A}_B(x - y), \quad (11.3+)**$$

$$\overset{\smile}{S}^{-A}_B(x y) \equiv \mathbf{S}_{\varphi'} \varphi'^{TA}(x) \overset{\smile}{\varphi}'_B(y) = \overset{\smile}{S}^{-A}_B(x - y), \quad (11.3-)**$$

$$S^{-AB}(x y) = S^{+TAB}(x y) = S^{+BA}(y x) \quad (11.4)$$

are invariant with respect to $\{L_{(\text{ochr})}\}$. Again a PF-solution $f^{+A}(x)$ of (6.20) may be written in the form

$$f^{+A}(x) = \mathbf{S}_{\varphi'} \varphi'^A(x) f_{\varphi'}^+ = \int d\overset{\smile}{\sigma}_\alpha(y) \overset{\smile}{S}^{+A}_B(x y) \gamma^{\alpha B}_C f^{+C}(y) \quad (11.5+)$$

and

$$f^{-A}(x) = \mathbf{S}_{\chi'} \chi'^{TA}(x) f_{\chi'}^- = \int d\overset{\smile}{\sigma}_\alpha(y) \overset{\smile}{S}^{-A}_B(x y) \gamma^{\alpha B}_C f^{-C}(y). \quad (11.5-)$$

*) $\theta^{\alpha\beta}$ and $\overset{\smile}{J}^\alpha$ can be expressed in terms of commutators:

$$T^\alpha_\beta(x) = \frac{1}{4} \overset{\smile}{J}^{-1} ([\psi^{TA}, \psi^B] + [\psi^A, \psi^{TB}]) (x) (-\overset{\smile}{\gamma}_{(AB)}^\alpha), \quad (10.6)$$

$$s^{\alpha\beta\gamma}(x) = \frac{1}{4} \overset{\smile}{J}^{-1} [\psi^{TA}, \psi^B] (x) (-\overset{\smile}{\gamma}_{[AB]}^{\alpha\beta\gamma}). \quad (10.7)$$

$$\overset{\smile}{j}^\alpha(x) = \frac{1}{2} [\psi^{TA}, \psi^B] (x) (-\overset{\smile}{\gamma}_{(AB)}^\alpha). \quad (10.8)$$

**) The pseudochronous sign in $\overset{\smile}{\varphi}'_A$ and $\overset{\smile}{S}^{+A}_B$ does not imply any covariance property but only indicates lowering of the index.

The general solution of (6.20) may be written as

$$\begin{aligned} \psi^A(k) &= \mathbf{S}_{\varphi'} a_{\varphi'} \varphi'^A(x) + \mathbf{S}_{\chi'} b_{\chi'}^T \chi'^{TA}(x) = \\ &= \int d\check{\sigma}_\alpha(y) \check{S}^{0A}_B(x y) \gamma^{\alpha B}_C \psi^C(y) \end{aligned} \tag{11.6}$$

with

$$\check{S}^{0A}_B(x y) = (\check{S}^{+A}_B + \check{S}^{-A}_B)(x y). \tag{11.8}$$

If $x = y'$ is chosen on $\tau(y') = \tau(y) = 0$, it follows that

$$\check{S}^{0B'}_C(y' y) \gamma^{\alpha C}_B = \gamma^{0B'}_B \check{\delta}^\alpha(y' y). \tag{11.9}$$

Now, this is exactly the condition (8.9), (8.10). Thus $\check{S}^{0A}_B(x y)$, defined by (11.8) is identical with (8.7).

From the development (11.6) and (8.12), (8.13), follow the CR's or ACR's:

$$[a_{\varphi'}, a_{\varphi''}]_{\mp} = [b_{\chi'}^T, b_{\chi''}^T]_{\mp} = [a_{\varphi'}, b_{\chi'}^T]_{\mp} = 0, \tag{11.10_{\mp}}$$

$$[a_{\varphi'}, a_{\varphi''}^T]_{\mp} = \delta_{\varphi' \varphi''}; \quad [b_{\chi'}^T, b_{\chi''}^T]_{\mp} = \delta_{\chi' \chi''}; \quad [a_{\varphi'}, b_{\chi'}^T]_{\mp} = 0. \tag{11.11_{\mp}}$$

Taking the ACR's (on account of $O_{(C)}$ -covariance), (10.8) leads to

$$Q = \mathbf{S}_{\varphi'} \left(N_{\varphi'} - \frac{1}{2} \right) - \mathbf{S}_{\chi'} \left(N_{\chi'} - \frac{1}{2} \right) = \mathbf{S}_{\varphi'} N_{\varphi'} - \mathbf{S}_{\chi'} N_{\chi'} \tag{11.12}$$

with $N_{\varphi'} = a_{\varphi'}^T a_{\varphi'}$. The ACR's (11.10) and (11.11) are satisfied in terms of the pseudo-quaternions (I, A-4.8)

$$\begin{aligned} a^T &= \frac{1}{2} (l + j) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad a = \frac{1}{2} (l - j) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \\ N &= a^T a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2} (1 - k) \end{aligned} \tag{11.13}$$

with

$$a_{\varphi^{(e)}} = 1 \times (k \times k) \times (k \times k) \times \dots \times (a \times 1) \times (1 \times 1) \times \dots \tag{11.14}$$

$$b_{\chi^{(e)}} = 1 \times (k \times k) \times (k \times k) \times \dots \times (k \times a) \times (1 \times 1) \times \dots$$

The eigenvalues are $\{N'_{\varphi'}\} = \{0, 1\}$ i.e. FD-statistics holds. The explicit form of $O_{(C)}$ is again given by (5.19), if the sets $\{\varphi'\}$ and $\{\chi'\}$ are chosen to be identical. Quantization of $\check{\Pi}_\mu$ leads to

$$\check{\Pi}^\mu = \mathbf{S}_{\varphi'} \left(N_{\varphi'} - \frac{1}{2} \right) \check{k}'^\mu + \mathbf{S}_{\chi'} \left(N_{\chi'} - \frac{1}{2} \right) \check{k}'^\mu, \tag{11.15}$$

if the PFWP's satisfy (5.20). Thus, the energy is, apart from the negative infinite 'zero point contribution' (which is again symmetric for particle and anti-particle states), a positive definit operator: the spectrum has a lower limit, and statistical thermodynamics with a positive absolute temperature may be applied¹⁰).

The CR's (11.10₋) and (11.11₋) lead, whatever α_1 and α_2 (satisfying (8.14, 1st k.₋)) we choose, to an energy spectrum without lower or upper limit: thus statistical thermodynamics cannot be applied. This is PAULI's⁹) reason for excluding BE-statistics for spinor fields.

Annex I: The Theorem of Frobenius and Schur^{11) 12)},

states that an *irreducible matrix group* (IMG) $\{\widehat{D}_i\}$ ($ik. \dots = 12 \dots h$) of order h belongs to one of the three kinds:

An IMG is of the 1st, 2nd or 3rd kind, depending on wether

$$h^{-1} \sum_i \text{tr}(\widehat{D}_i^2) = \begin{cases} -1 \\ +1 \\ 0 \end{cases} \text{ for an IMG of the } \begin{cases} 1^{\text{st}} \\ 2^{\text{nd}} \\ 3^{\text{rd}} \end{cases} \text{ kind.} \quad (\text{A-1.1})$$

(1) An IMG is of the 1st kind, if it is equivalent to a group of real matrices, i.e. if for all representations $\{\widehat{D}_i\}$ a matrix \widehat{S} exists, satisfying

$$\{\widehat{D}_i\} = \{\widehat{S} \widehat{D}_i \widehat{S}^{-1}\} \quad \text{with} \quad \widehat{D}_i^* = \widehat{D}_i. \quad (\text{A-1.2})$$

(2) An IMG is of the 2nd kind, if it is equivalent to its conjugate complex group, i.e. if for all representations a matrix \widehat{C} exists, satisfying

$$\{\widehat{D}_i\} = \{\widehat{C} \widehat{D}_i \widehat{C}^{-1}\} \quad \text{with} \quad \widehat{D}_i = \widehat{D}_i^*. \quad (\text{A-1.3})$$

NB.: For an IMG of the 1st kind, a matrix \widehat{C} exists a fortiori ($\widehat{C} = \widehat{S}^{*-1} \widehat{S}$).

(3) An IMG is of the 3rd kind, if it is not equivalent to its complex conjugate group i.e. if no matrix \widehat{C} satisfying (A-1.3) exists.

Now the set $\{\pm \Gamma\} = \{\pm \gamma^r\}$ forms, for $n = 2m$, on account of (6.8 M) an IMG of order 2×2^n . In order to find out to which kind it belongs, we have to evaluate

$$(2 \times 2^n)^{-1} \sum_r \sum_{(\pm)} \text{tr}((\pm \gamma^r)^2) = 2^{-n} \sum_r \text{tr}((\gamma^r)^2). \quad (\text{A-1.4})$$

To evaluate this sum, we decompose each subset $\Gamma^{(v)}$ into two parts

$$\{\gamma^{\alpha_1 \dots \alpha_n}\} = \{\gamma^{i_1 \dots i_n}\} + \{\gamma^{i_1 \dots i_{n-1} n}\}, \quad (\text{A-1.5})$$

each having ($d = n - 1$)

$$\binom{n}{\nu} = \binom{d}{\nu} + \binom{d}{\nu-1} \quad (\text{A-1.6})$$

elements. Using (6.12) and (6.13), we have

$$\begin{aligned} \sum_r \text{tr} [(\gamma^r)^2] &= \sum_{\nu} \left[\binom{d}{\nu} - \binom{d}{\nu-1} \right] \begin{Bmatrix} (-1)^{\nu/2} \\ (-1)^{(\nu-1)/2} \end{Bmatrix} \text{tr}(\gamma^0) = \\ &= N \text{Re} \sum_{\nu} \left[\binom{d}{\nu} - \binom{d}{\nu-1} \right] (1 - i) i^{\nu} = \\ &= 4 N \text{Re} [(1 + i)^{n-3}] = 2^{(n+1)/2} N \cos\left((n-3) \frac{\pi}{4}\right). \quad (\text{A-1.7}) \end{aligned}$$

We remark first, that our condition (A-1.1), (A-1.4) has the periodicity ' $n = n \pmod{8} = \dots, n - 8, n, n + 8, \dots$ '. In particular, we find, on account of $\cos(\pi/4) = 2^{-1/2}$, $N = 2^{n/2}$ i.e. (6.30) and furthermore that our IMG is of the 1st kind for $n = 2, 4 \pmod{8}$ (6.31) and of the 2nd kind for $n = 6, 8 \pmod{8}$ (6.32).

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