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Support of a field in momentum space

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The theory of a scalar field $A(x)$, defined under the assumptions of Lorentz invariance, locality, absence of negative energy states, uniqueness and cyclicity of the vacuum, is considered. It is proved that if the Fourier transform of the field in momentum space vanishes in the neighbourhood of a space-like point then the theory describes a generalized free field.

1. Introduction

In recent years local quantum field theory has been extensively studied from an axiomatic viewpoint. This study has led to insight into many of the mathematical difficulties which proliferate in this subject and has also helped to understand some of the connections between the physical postulates upon which the theory has been founded. Many models of local field theories have been examined but unfortunately no model has been found which satisfies all the axioms of the theory and also leads to a scattering matrix different from unity. We mention as an example 'generalized free fields'[1][2].

In this paper the support properties in momentum space of a local field are considered. It is shown that these support properties lead to a very simple criterion for deciding whether a field is a 'generalised free field'[3], and to the result that it is necessary for a field to have support for all space-like momenta in order that it should be an interacting field. It may be deduced from counter examples that this condition is not a sufficient condition for a field to have scattering matrix different from unity.

Section 2 of this paper summarises the axioms and defines the structure of the field theory considered in the following sections. Two theorems

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are also stated. Section 3 contains the statement and proof of the theorem which constitutes the main result of this paper. Section 4 contains a summary of the results with a short discussion.

2. Definitions and Theorems

A neutral scalar field theory is defined, in the manner of WIGHTMAN[4]), by the following assumptions

1. The linear space of states Φ, Ψ is a separable Hilbert space with scalar product denoted by (Φ, Ψ). A neutral scalar field maps each testing function $\varphi(x) \in S$ into a linear operator $A(\varphi)$ of $\mathfrak{H}$. We assume that the intersection of the domains of all $A(\varphi)$ contains a dense manifold D in $\mathfrak{H}$ which is invariant under all $A(\varphi)$. For all $\Phi, \Psi \in D$

$$(\Phi, A(\varphi)\, \Psi) = (A(\overline{\varphi})\, \Phi, \Psi)$$

is assumed to be linear and continuous in φ i.e. a distribution in S'.

Symbolically we write

$$A(\varphi) = \int d^4x\, A(x)\, \varphi(x)$$

and in the following we work with the field $A(x)$. It is therefore necessary to keep in mind that all equations should be interpreted in the sense of distribution theory[5]).

2. Lorentz invariance. There exists a unitary representation $U(a, \Lambda)$ of the proper orthochronous inhomogeneous Lorentz group fulfilling the following conditions

a) $U(a, \Lambda)\, A(x)\, U^{-1}(a, \Lambda) = A\,(\Lambda\, x + a)$,

b) $U(a, \Lambda)\, D \subset D$.

At this point we elaborate a little more fully the properties of the representation $T(a) \equiv U(a, 1)$ of the subgroup of translations. A simple generalization of Stone's theorem allows us to make the spectral decomposition[6])

$$T(a) = \int dE(k)\, e^{i k a}$$

where k is in momentum space. E is a uniquely determined projection valued measure. To each BOREL set Δ in momentum space there corresponds a projection $E(\Delta) = E(\Delta)^2 = E(\Delta)^*$ in $\mathfrak{H}$.

3. Absence of negative energy states. The support of E lies in the closure $\overline{V}_+$ of the forward light cone where we define

$$V_+ = \{p : p_0 > 0 \;\; p^2 > 0\} .$$

4. The vacuum state Ω is unique. The vacuum $\Omega \in D$ is cyclic with respect to successive operations of $A(x)$ on Ω.

5. Locality. The field satisfies the following commutation relation

$$[A(x), A(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0 .$$

In order to fully prepare the ground for the next section we now quote two theorems[7]) which will be extensively used.

Theorem A: If $\mathfrak{F}$ is the set of all tempered distributions $F(y)$ which vanish for $y^2 < 0$, and if $\mathfrak{G}$ is the set of all tempered distributions $G(q, \sigma)$ on five dimensional space, labeled by $(q_0, q_1, q_2, q_3, \sigma)$ satisfying

$$\left(\frac{\partial^2}{\partial q_0^2} - \frac{\partial^2}{\partial q_1^2} - \frac{\partial^2}{\partial q_2^2} - \frac{\partial^2}{\partial q_3^2} - \frac{\partial^2}{\partial \sigma^2}\right) G(q, \sigma) = 0 ,$$

$$G(q, \sigma) = G(q, -\sigma)$$

then the transformation

$$G(q, \sigma) = \frac{1}{(2\pi)^2} \int d^4y \cos\left(\sigma \sqrt{y^2}\right) e^{-iqy} F(y)$$

maps $\mathfrak{F}$ one-to-one on $\mathfrak{G}$, and $G(q, \sigma)$ defined by the transformation has the property that $G(q, 0)$ is a tempered distribution and is the Fourier transform of $F(y)$.

Theorem B (Uniqueness theorem): If $G(q, \sigma)$ is a tempered distribution in the set G then $G(q, \sigma)$ has the following support properties

a) If $G(q, \sigma)$ vanishes of infinite order on a time like segment, it vanishes in the double cone subtended by that segment.

b) If $G(q, \sigma)$ vanishes in a slab bounded by two space like surfaces, it vanishes in the dependence domain of the slab.

The proofs of these theorems and further discussions thereof may be found in the reference cited. The uniqueness theorem is a consequence of Huyghens principle and although it is not the strongest theorem that may be derived from this principle it suffices for our purpose.

3. Support properties of the field

The essential content of the present article is contained in the theorem which we now quote and prove.

Theorem: If a neutral scalar field $A(x)$ is cyclic in $\mathfrak{H}$ with respect to the vacuum Ω and if the following conditions are satisfied

1. Lorentz invariance.
2. Absence of negative energy states.
3. Locality
4. The support of the Fourier transform of the field

$$\tilde{A}(p) = \frac{1}{(2\pi)^2} \int d^4x \, e^{-ipx} A(x)$$

excludes the neighbourhood of a space like point p in momentum space then $A(x)$ is a generalized free field.

The support of the field in momentum space is a Lorentz invariant set and thus condition 4 of the theorem is equivalent to the statement that $\tilde{A}(p)$ has support R defined by

$$R = \{p : p^2 \leqq -a \quad \text{cr} \quad p^2 \geqq -b\} .$$

The real positive numbers a and b satisfy the condition $a > b$ but are otherwise arbitrary. It is not necessary that b is positive but we make this assumption in order to avoid placing restrictions on the physical spectrum of the field.

In order to prove the theorem we consider the tempered distribution in two variables $f(x_1, x_2)$ defined by

$$f(x_1, x_2) = (\Psi_1 [A(x_1), A(x_2)] \, \Omega) \tag{1}$$

where Ψ is a vector in $\mathfrak{H}$. We derive from this distribution the tempered distribution in one variable

$$F(y) = \frac{1}{(2\pi)^2} \int d^4x \, \varphi(x) \, f(x+y, x-y) \tag{2}$$

where $\varphi(x)$ is a test function in S. The Fourier transform of $f(x_1, x_2)$ is defined by

$$\tilde{f}(p_1, p_2) = \frac{1}{(2\pi)^4} \int d^4x_1 \, d^4x_2 \, e^{-i(p_1 x_1 + p_2 x_2)} f(x_1, x_2)$$

and we find that the Fourier transform of $F(y)$ is given by

$$\tilde{F}(q) = \frac{1}{2^4} \int d^2p \, \tilde{\varphi}(p) \, \tilde{f}\left(\frac{p+q}{2}, \frac{p-q}{2}\right) . \tag{3}$$

The Fourier transform of $\varphi(x)$ has been introduced

$$\tilde{\varphi}(p) = \frac{1}{(2\pi)^2} \int d^4x \, e^{ipx} \varphi(x) .$$

We next consider the measure E determined by the spectral decomposition of the translation operator. We make a decomposition of unity

$$\sum_i E(D_i) = 1 \tag{4}$$

where the Borel sets D_i are chosen such that

$$\overline{V}_+ = \bigcup_i D_i$$

and

$$D_i \cap D_j = \phi \quad \text{if} \quad i \neq j \, . \tag{5}$$

Corresponding to this partition of unity we may decompose $f(x_1, x_2)$

$$f(x_1, x_2) = \sum_i f_i(x_1, x_2) \tag{6}$$

where

$$f_i(x_1, x_2) = \big(E(D_i)\, \Psi, [A(x_1), A(x_2)]\, \Omega\big) \, .$$

Similarly

$$F(y) = \sum_i F_i(y)$$

where

$$F_i(y) = \frac{1}{(2\,\pi)^2} \int d^4x \; \varphi(x)\; f_i\,(x + y, x - y) \, .$$

The distributions $F_i(y)$ and $\tilde{F}_i(q)$ have support properties determined by conditions 3 and 4 of the theorem. These support properties are the following

1. $F_i(y) = 0 \quad \text{if} \quad y^2 < 0$

and

2. $\tilde{F}_i(q) = 0 \quad \text{unless} \quad q \in C = C_1 \cup C_2$ (7)

where

$$C_1 = \left\{q : p \in D_i \, \frac{p-q}{2} \in \overline{V}_+ \, \frac{p+q}{2} \in R\right\} ,$$

$$C_2 = \left\{q : p \in D_i \, \frac{p+q}{2} \in \overline{V}_+ \, \frac{p-q}{2} \in R\right\} .$$

The support $\tilde{F}_i(q)$ for D_i a neighbourhood of P_i is depicted in figure 1. We have defined

$$D_{-i} = \{q : -q \in D_i\} \, .$$

The support of $\tilde{F}_i(q)$ may be seen from figure 1 to be the union of two cones minus two 'grooves' which are both bounded by sections of time-like surfaces (one-sheeted hyperboloids). These grooves arise from the assumed properties of the support of the field $A(x)$ in momentum space. The grooves are absent if the diameter of the set D_i is larger than some number $n(\varepsilon)$ determined by $\varepsilon = a - b$, the real number determining the gap in the support of $\tilde{A}(p)$. However, we have the freedom to choose the diameter of D_i arbitrarily small, thus we may assume the grooves to be present.

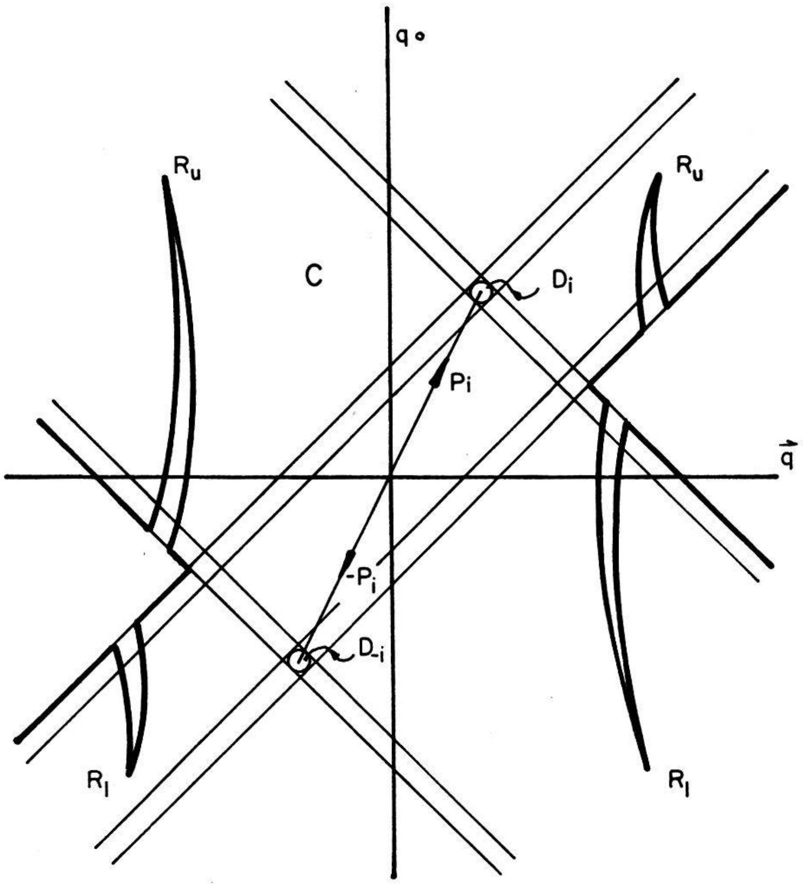

Fig. 1
The support C of $\tilde{F}_i(q)$ when D_i is a neighbourhood of P_i.

In order to examine the support of $\tilde{F}_i(q)$ we use theorems A and B of section 2. We define

$$G_i(q, \sigma) = \frac{1}{(2\pi)^2} \int d^4y \cos\left(\sigma \sqrt{y^2}\right) e^{-iqy} F_i(y) \tag{8}$$

and then from Theorem A and the support property of $F_i(x)$ we are ensured that

$$G_i(q, \sigma) \in \mathfrak{G}$$

and

$$G_i(q, 0) = \tilde{F}_i(q) .$$

The ground is now prepared to apply Theorem B. The initial application of Theorem B makes use of the first part of the theorem only. The procedure is to take time-like segments which lie completely outside of the support of the distribution $G(q, \sigma)$. The support of $G(q, \sigma)$ must then lie outside of the double cones subtended by these segments and therefore the support of $\tilde{F}_i(q)$ lies in the plane $\sigma = 0$ but outside of the four

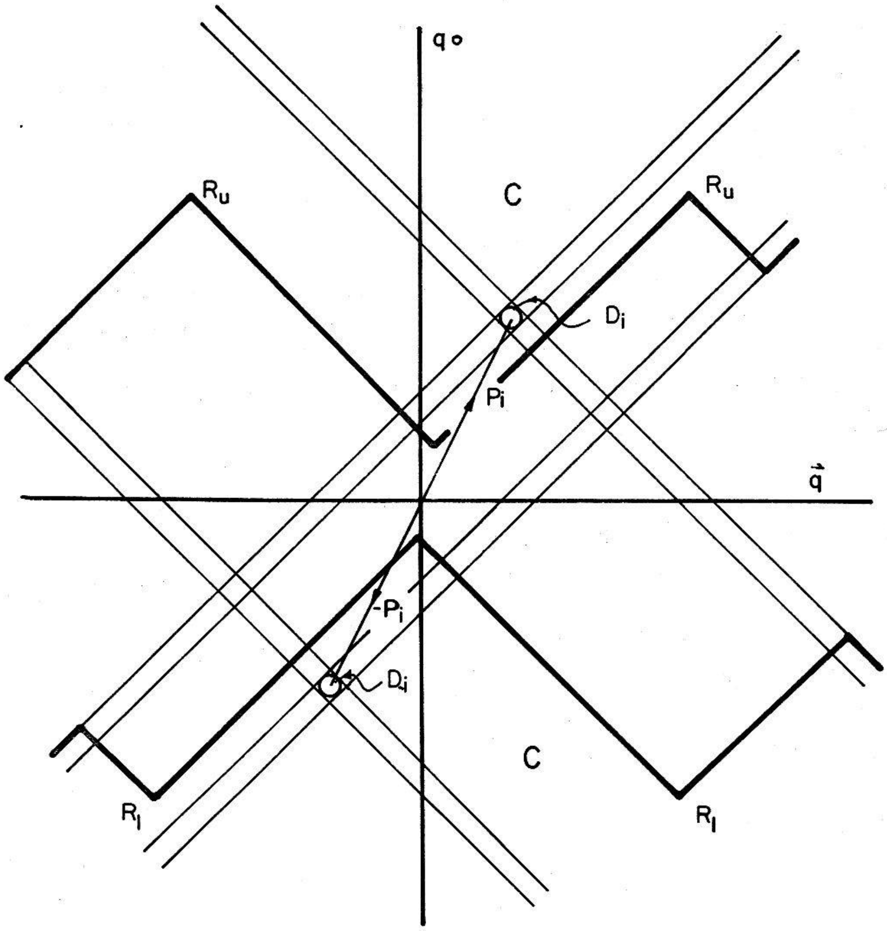

Fig. 2
The support C of $\tilde{F}_i(q)$ after the first application of theorem B.

dimensional intersection of these double cones with the plane $\sigma = 0$. It is now important to note that the bounding surfaces of the grooves are time-like surfaces and hence it is possible to enlarge these grooves by taking time-like segments lying in the grooves. If the locus of the vertex points of the upper groove is denoted by R_u and that of the lower groove by R_l (see figure 1), it is possible to choose a series of time-like segments connecting points of R_u to points of R_l. After removing the subset of the support contained in the double cones subtended by these segments we are left with the support of $\tilde{F}_i(q)$ depicted in figure 2. The figure is a little misleading insofar that it depicts the support C of $\tilde{F}_i(q)$ as the union of two sets of empty intersection. The intersection of these sets may be non-empty, this depends critically on the neighbourhood D_i, but this does not affect the next part of the argument.

To continue further it is necessary to clarify the properties of the grooves and in particular the loci, R_u and R_l, of their vertex points. The boundary surfaces of the grooves are one-sheeted hyperboloids having as asymptotic surfaces light cones which envelope D_i(or D_{-i}). If the domain D_i is decreased in diameter the cones approach one another, the grooves deepen, and the points of R_u and R_l recede from the origin. If it were possible to decrease D_i in diameter to the limit of being the point P_i the two asymptotic surfaces would be identical with the light cone having vertex at P_i. In this limit the upper (lower) groove would be bounded on both sides by hyperboloids asymptotic to the light cone with vertex $P_i(-P_i)$ and the points of $R_u(R_l)$ would lie on the cone at infinity.

We now define $\overline{V}^i_+(\overline{V}^i_-)$ as the closure of the smallest forward (backward) light cone with the property $D_i \subset \overline{V}^i_+$ $(D_i \subset \overline{V}^i_-)$. The next stage in the argument is to show that the support C of $F_i(q)$ can be reduced to

$$C = \overline{V}^i_+ \cup \overline{V}^{-i}_- . \tag{9}$$

This follows from the properties of the partition of the unity. If $q' \notin C$ we choose an open covering $\{D_{i,a}\}$ of the set D_i, with corresponding partition of the distribution $\tilde{F}_i(q)$, such that

$$\tilde{F}_{i,a}(q') = 0 \quad \text{for all } a .$$

and hence

$$\tilde{F}_i(q') = \sum_a \tilde{F}_{i,a}(q') = 0 .$$

That this choice is possible is clear from the discussion of the last paragraph. By choosing the diameters of the covering sets $D_{i,a}$ sufficiently small we may ensure that q' lies outside of the support of each $\tilde{F}_{i,a}(q)$. The support C given by (9) is depicted in figure 3.

Figure 3 shows the support as the union of two sets of empty intersection but as mentioned before this is not necessarily the case. The intersection $\overline{V}^i_+ \cap \overline{V}^{-i}_-$ is certainly non-empty if the domain D_i contains the origin. If D_i does not contain the origin it may be assumed that the intersection is empty. The justification of this assumption follows from a similar argument to that given above, namely D_i may be decomposed into a union of sets having the required property for each of which the following application of Theorem B is valid. In the next part of the discussion we consider a set D_i not containing the origin and assume that

$$\overline{V}^i_+ \cap \overline{V}^{-i}_- = \phi .$$

To reduce the support of $\tilde{F}_i(q)$ further it is necessary to use both parts of theorem B applied to the distribution $G_i(q, \sigma)$. First construct double cones on time like segments not contained in the support C. The intersections of these double cones with the plane $q_0 = 0$ are spheres in four dimensional space and by the first part of Theorem B we have the result that $G_i(q, \sigma)$ vanishes in the interior of these spheres. These spheres have their centres in the plane $\sigma = 0$, and by assumption the subspace $q_0 = 0$ $\sigma = 0$ of five dimensional space is not contained in the support of $G_i(q, \sigma)$.

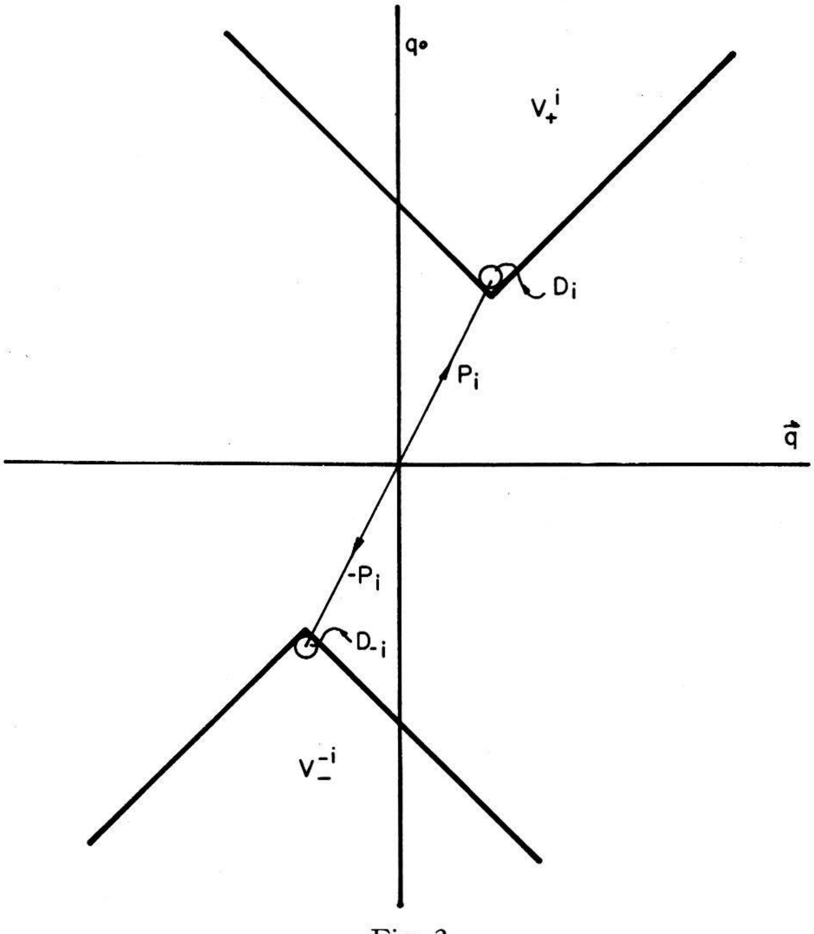

Fig. 3

The support $C = \overline{V}_+^i \cup \overline{V}_-^{-i}$ of $F_i(q)$ after the second application of theorem B.

The union of the interiors of all possible four dimensional spheres obtained in this way is the complete four dimensional subspace $q_0 = 0$ of (q, σ) space. Again there is by assumption a real positive ε such than the subspace $q_0 = \varepsilon$ $\sigma = 0$ lies outside of the support of $G_i(q, \sigma)$. Thus by repeating the above argument we arrive at the conclusion that $G_i(q, \sigma)$ vanishes in the slab bounded by the space-like surfaces $q_0 = 0$ and $q_0 = \varepsilon$. The second part of Theorem B now ensures that $G_i(q, \sigma)$ vanishes in the complete five dimensional (q, σ) space and hence $\tilde{F}_i(q)$ vanishes in the four dimensional $\sigma = 0$ subspace. Thus we have proved that $\tilde{F}_i(q)$ vanishes unless D_i is a neighbourhood of the origin.

Using this result we now see from (3) that

$$f_i(x_1, x_2) = 0$$

unless D_i is a neighbourhood of the origin.

Thus from (6)

$$f(x_1, x_2) = \left(\int_{\Delta} dE(k)\, \Psi, [A(x_1), A(x_2)]\, \Omega \right)$$

for any arbitrary neighbourhood of the origin. In the limit

$$\left(\Psi, [A(x_1), A(x_2)]\, \Omega\right) = \left(E_\Omega\, \Psi, [A(x_1), A(x_2)]\, \Omega\right) \tag{10}$$

where E_Ω is the projection onto the vacuum.

Therefore

$$[A(x_1), A(x_2)]\, \Omega = W\,(x_1 - x_2)\, \Omega \tag{11}$$

where

$$W\,(x_1 - x_2) = \left(\Omega, [A(x_1)\;\; A(x_2)]\, \Omega\right) .$$

This is the conclusion of the first part of the proof.

The second part of the proof follows an argument given by R. JOST[8]) which we briefly paraphrase. Using the above result and also locality

$$\left(\Omega, A(x_0) \ldots A(x_n)\, [A(x), A(y)]\, A(y_0) \ldots A(y_m)\, \Omega\right) =$$

$$= W\,(x - y)\, \left(\Omega, A(x_0) \ldots A(x_n)\, A(y_0) \ldots A(y_m)\, \Omega\right) \tag{12}$$

if $(x - y_k)^2 < 0$ and $(y - y_k)^2 < 0$ for $k = 0, 1 \ldots m$. However, both sides of equation (12) may be continued analytically to points $(z_0 \ldots z_n, z, w, w_0 \ldots w_m)$ in the tube S

$$S: \quad \begin{array}{ll} Im\,(w_k - w_{k-1}) \in V_+ , & Im\,(w_0 - w) \in V_+ , \\ Im\,(z_k - z_{k-1}) \in V_+ , & Im\,(z - z_n) \in V_+ , \\ Im\, z = Im\, w & \text{real parts arbitrary} . \end{array}$$

Equation (12) holds for all points in S and the corresponding boundary values. Thus (12) holds for all real values of $(x_0 \ldots x_n, x, y, y_0 \ldots y_m)$ and we have, using the cyclicity of the vacuum

$$[A(x_1), A(x_2)] = W\,(x_1 - x_2) .$$

As $W(x_1 - x_2)$ is the vacuum expectation value of the commutator $W(x_1 - x_2)$ has a KÄLLEN-LEHMANN spectral representation [9][10]. Therefore

$$[A(x_1), A(x_2)] = \int_0^\infty dm^2\, \varrho(m^2)\, \Delta(x_1 - x_2, m^2)$$

which is the definition of a generalized free field.

4. Summary and Conclusions

There are a number of interesting conclusions which may be drawn from the theorem which has been proved in the preceeding section. A generalized free field is known to have support in momentum space only in or on the light cone [1][2]. Thus we may conclude that the support properties of the fields may be used to divide all local fields into one of two classes. The first class of fields have support only in or on the light cone and are generalized free fields. The second class of fields have support everywhere outside the light cone. Interacting fields are fields of the second class. However, not all fields of the second class have a scattering matrix which differs from unity. If we construct a field from the Wick product of free fields we certainly obtain a field of the second class, but this field has scattering matrix equal to unity.

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