Zeitschrift: Helvetica Physica Acta

Band: 35 (1962)

Heft: VII-VIII

Artikel: Relativistic thermodynamics. III, Velocity of elastic waves and related problems

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DOI: https://doi.org/10.5169/seals-113288

Relativistic Thermodynamics III: Velocity of elastic waves and related problems

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Abstract: The equations of motion and the principle of equilibrium determine the signes of viscosity, heat conductivity, mass, elastic modulus and heat capacity in terms of the sign of absolute temperature. Furthermore, these thermodynamic conditions show that light velocity is the upper limit for the velocity of elastic waves. The equations of motion contain the 2nd time derivative of the velocity of substance, in perfect analogy to DIRAC's theory of the point election. The linear approximation of the equations is discussed. The equilibrium for a rotating fluid and for a fluid in a gravistatic field are given. In particular, a method is used (see annex), which shows that the *use of Lagrange Multipliers* is valid *not only for an extremum but also for a maximum* (*or minimum*), if the functionals involved are of the density type.

Introduction and Conclusion

In two previous articles[1] [2]) (to be referred to as I and II, *see also* LEAF[3])) we gave the equations of motion for a fluid in general and special relativity. The two laws of thermodynamics were stated as *covarient equations of continuity: homogeneous* for the *symmetric 4-tensor of momentum energy density* $\theta^{\alpha\beta}(x)$ and for the *4-vector of substance-density* $j_N^\alpha(x)$, *inhomogeneous*, with a positive definit source $i(x)$, *for the 4-vector of entropy-density* $j_S^\alpha(x)$. It was shown that this set of $4 + 2$ equations with a positive definit entropy source $i(x)$ was possible only in a manifold $\{x^\alpha\}$, where but one coordinate was of time like character (thermodynamic signature). Furthermore the *two viscosities* η and ξ' had to have *the sign of absolute temperature* T, and the *heat conductivity* $\varkappa$ had to be *positive definit.*

However, in the equations of motion additional state variables occur: *Mass*-or *enthalpy-density* m, heat capacity per unit volume c and the *elastic modulus* a. Their signs and the numerical value of the *velocity of elastic waves* $c_{||}$ ($c^2_{||} = a/m$) are of importance. These questions could not

*) Supported by the Swiss National Research Fund.

be discussed in I, because their answers follow only, if we add to the 4+2 continuity equations the *condition of equilibrium:* In a closed system, entropy increases, if time goes on to the very far *absolute future,* up to a *maximum.* We speak of absolute future, because the continuity equations are covariant even with respect to *time reversal* «*T*». The very far absolute future at which equilibrium is reached, introduces a *time-like 4-vector* into the theory, which we may call the *arrow of time.*

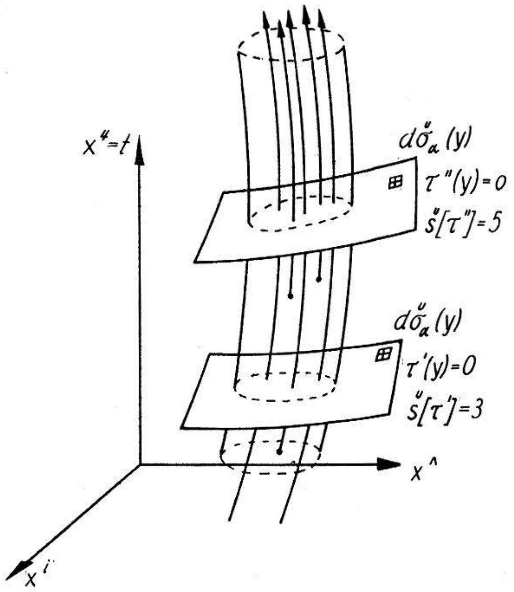

Fig.

Field lines of the entropy density 4-vector $j^\alpha{}_S(x)$, and values of the pseudochronous entropy scalar $\breve{S}[\tau]$ for 2 hypersurfaces in an orthochronous frame $\{x^\alpha\}$

This arrow of time is essentially the 4-vector of entropydensity $j^\alpha{}_S(x)$ (more exactly, its projection

$$j^\alpha{}_{S(0)}(x) = (w^\alpha\, w_\varrho\, j^\varrho{}_S)\;(x) \tag{0.1}$$

on the *4-velocity* $w^\alpha(x)$. Figure illustrates the problem: Consider an hypersurface $\tau(y) = 0$ with a *time like normal.* Its hyper-surface element at an event y, $d\breve{\sigma}_\alpha(y)$ is a *pseudochronous vector**), defined by *the scalar 4-volume element*

$$(d\delta)^4\, V(y) = d\breve{\sigma}_\alpha(y)\; \delta\breve{r}^\alpha(y) > 0 \tag{0.2}$$

where $\delta\breve{r}^\alpha(y)$ is a *time-like pseudo-chronous vector*

$$\delta\breve{r}^4(y) > 0 \tag{0.3}$$

*) An $\cup$ denotes *pseudochronous quantities.* The exact definition is given in (1. 10).

pointing to the *relative future* (relative means with respect *to the frame* $\{x^\alpha\}$). In the figure $\tau''(y) = 0$ is *later* than $\tau'(y) = 0$ in the frame $\{x^\alpha\}$.

The pseudochronous covariant surface elements $d\breve{\sigma}_\alpha(y)$ have their (+)-face oriented towards $x^4 = t$. The *entropy* $\breve{S}[\tau]$, for a given $\tau(y) = 0$, is defined by the *pseudochronous scalar*

$$\breve{S}[\tau] = \int_{\tau(y)=0} (d\breve{\sigma}_\alpha\, j^\alpha{}_S)\,(y)\,. \tag{0.4}$$

If we look at the *same surface* $'\tau('y) = \tau(y) = 0$ *) in a time-reversed frame $\{'x^\alpha\}$

$$'x = «T»\, x: \quad 'x^i = x^i\,; \quad 'x^4 = 't = -\, x^4 = -\, t \qquad (0.5)\,*)$$

we find the entropy

$$'\breve{S}['\tau] = \int_{'\tau('y)=0} ('d\breve{\sigma}_\alpha\, 'j^\alpha)\,('y) = -\, \breve{S}[\tau] \qquad (0.6)\,*)$$

because the relative orientation of the (orthochromous) 4-vector $j^\alpha{}_S(x)$ and the pseudochronous 4-vector $d\breve{\sigma}_\alpha(y)$ has changed. Thus we have in our figure

$$\breve{S}[\tau''] - \breve{S}[\tau'] = \int_{\tau'}^{\tau''} d^4V(x)\, i(x) = 5 - 3 = +\, 2 > 0 \qquad (0.7)\,**)$$

because τ'' is later than τ' in $\{x^\alpha\}$, and

$$'\breve{S}['\tau'] - '\breve{S}['\tau''] = (-\,3) - (-\,5) = +\,2 > 0 \tag{0.8}$$

in $\{'x^\alpha\}$, because now $'\tau'$ is later than $'\tau''$ in $\{'x^\alpha\}$. $\breve{S}[\tau]$ is an essentially positive quantity in an *orthochronous frame* ($j^4{}_{S(0)}(x) \overset{*}{\geqq} 0$***)).

In an *orthochronous local rest frame* the state function of *rest energy-density* $\theta^{44} \overset{*}{=} u$***) is given by

$$u = u\,[s\; n] \tag{0.9}$$

*) A prime to the left $'\breve{S}$, $'d\breve{\sigma}_\alpha$, $'j^\alpha{}_S$ denotes quantities in the transformed frame $\{'x^\alpha\}$ (see § 1).

**) The integral is extended over the region of the world tube between the 2 surfaces τ'' and τ'.

***) Equalities $\overset{*}{=}$ and inequalities $\overset{*}{>}$ hold only in a *specified frame* (*orthochromous frame, local geodesic rest frame*, ...).

as a function of two state variables: rest entropy-density

$$j^4{}_S \overset{*}{=} s \tag{0.10}$$

and rest substance-density

$$j^4{}_N \overset{*}{=} n \,. \tag{0.11}$$

Absolute temperature T and *chemical potential* μ are defined by

$$T\,[s\,n] = u_s\,[s\,n]\,; \quad \mu\,[s\,n] = u_n\,[s\,n]\,. \qquad (0.12)\,*)$$

Heat capacity per unit volume c and elastic modulus a are

$$c\,[s\,n] = (T\,T_s^{-1})\,[s\,n] = (u_s\,u_{ss}^{-1})\,[s\,n]\,, \tag{0.13}$$

$$a\,[s\,n] = (s^2\,u_{s\,s} + 2\,s\,n\,u_{s\,n} + n^2\,u_{n\,n})\,[s\,n]\,. \tag{0.14}$$

Thus the signs of the state functions

$$T^{-1}\,\eta \geqq 0\,; \quad T^{-1}\,\xi' \geqq 0\,; \quad \varkappa \geqq 0 \tag{0.15}$$

(viscosities η, ξ' and heat conductivity $\varkappa$) follow from the *continuity equations*, while those of *rest mass-or enthalpy density m*, *elastic modulus a* and *heat capacity per unit volume c*

$$T^{-1}\,m \geqq 0\,; \quad T^{-1}\,a \geqq 0\,; \quad c \geqq 0\,; \quad T^{-1}\,\mu_n \geqq 0 \tag{0.16}$$

are a consequence of the *equilibrium condition*.

This condition leads also to the interesting result

$$0 \leqq c^2{}_{\|} = m^{-1}\,a \lesseqgtr 1 \tag{0.17}$$

showing *that the velocity of elastic waves* $c_{\|}$ *is smaller or at most equal to the velocity of light* ($\equiv 1$). To our knowledge, this result is new: PAULI[4]) criticizes the procedure of HERGLOTZ[5]) and LAMLA[6]) (who impose, from the condition 'maximum signal velocity = 1', an upper limit upon the *elastic modulus a*; see also LICHNEROWITZ[7]) with the words.

'... the principle of relativity can not make any statements on the magnitude of the cohesive forces.' He expects, that, at this upper limit for a, '... the phenomenological equations become incorrect'. We were therefore rather surprised, that phenomenological thermodynamics leads, for stability reasons, to this upper limit (0.17) for a.

The new mathematical problem we were faced with, was the *maximum conditions for the functional*

$$\breve{S}[\ldots] = \breve{S}' = \text{maximum} \tag{0.18}$$

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*) Symbols like u_n are *partial derivatives* $u_n\,[s\,n] = \partial u\,[s\,n]/\partial n$ of *state functions*.

when the functions to be varied were submitted to *n functional constraints*

$$G^a[\ldots] = G'^a; \quad a\, b \ldots = 12 \ldots n \tag{0.19}$$

where G'^a are *n constants of integration* of the equations of motion.

It is well known, that the *extremum conditions* for $\breve{S}[\ldots]$ can be found by the *method of n Lagrange multipliers* $\breve{\lambda}_a$: one defines a functional

$$\breve{\Psi}[\ldots] = \breve{S}[\ldots] + \breve{\lambda}_a\, G^a[\ldots] \tag{0.20 *)}$$

and looks for its extremum. We can show, that the *maximum* of $\breve{\Psi}[\ldots]$ is also a *sufficient condition for the maximum* of $\breve{S}[\ldots]$. *In the particular case, where all functionals are simple volume integrals over densities the maximum of* $\breve{\Psi}[\ldots]$ *is a necessary and sufficient condition.* The proof of this theorem is given in an annex for the case, where but one function is varied and only one constraint holds. Its generalisation to several functions and several constraints is easy, but involves a very complicated notation.

We recall in § 1 the, slightly changed, notations of I. In § 2 we restate the 4+2 continuity equations of I: only 3+2 of them are inpendent. They correspond to *the equations of motion for the 3+2 state variables:* 3-velocity $\vec{v}\,(\vec{x}\,t)$ entropy density $j^4{}_S\,(\vec{x}\,t)$ and substance density $j^4{}_N\,(\vec{x}\,t)$. They are essentially different from the n. r. (= non relativistic) case, because *2nd time derivatives of state variables occur.*

In § 3 the equilibrium condition is treated for the Lorentz case (see [8]) for a note on this subject): *In equilibrium, the fluid rotates with constant angular velocity.* Due to the 'inertia of heat', the temperature distribution is

$$T(\vec{x}) = T_0\,\big(1 - v^2(\vec{x})\big)^{-1/2}. \tag{0.21}$$

Thus, '*heat is centrifugated*', if $T > 0$.

In § 4, the linear approximation of the equations of movement is given and partially discussed. We add 2 remarks to these equations:

1. Not only the *equation of heat flow*, but also the *damped elastic waves* and the *flow of transverse momentum* allow solutions only for the (*absolute*) *future*. (This particularily holds naturally also in n. r. theory.) *Thus, if the present is known, the* (*absolute*) *future can always be predicted. However no inference can be drawn from the present to the history in the* (*absolute*) *past.* We believe that this particularity of the equations is closely related with

*) The Lagrange multipliers $\breve{\lambda}_a$ are pseudochronous constants, if the constraints refer to orthochronous quantities G^a and orthochronous constants $\breve{\lambda}_a$ if $\breve{G}^a$ is pseudochronous.

the *functional character of the transportation phenomena**). It is interesting to note the analogy of this *arrow of time in phenomenological thermodynamics* to the question of *time reversal in statistical mecanics* (see [10–13])).

2. In the linear equations of motion, the 2nd time derivative appears in the form:

$$m_0\, \partial_t \vec{v}\,(\vec{x}\, t) \overset{*}{\rightarrow} \left(-\,(\varkappa\, T)_0\, \partial_t^2 + m_0\, \partial_t\right) \vec{v}\,(\vec{x}\, t)\,. \qquad (0.22)\,^{**})$$

Thus, if no forces act, an exponential increase (*towards the absolute future*) of acceleration may occur, because, on account of (0.15) and (0.16) the coefficients of $\partial_t \vec{v}$ and $\partial_t^2 \vec{v}$ have opposit sign. This is a striking analogy to DIRAC's[14]) theory of the point electron, if the retarded self-force is chosen.

Finally, § 5 gives the correct answer***) for the *static equilibrium in a gravitastatic field.*

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(\vec{x})\,. \qquad (0.23)$$

The relation

$$T(\vec{x})\; T^{-1}(\vec{y}) = \mu(\vec{x})\; \mu^{-1}(\vec{y}) = \left(-\, g_{44}(\vec{y}) / -\, g_{44}(\vec{x})\right)^{1/2} \qquad (0.24)$$

holds in this case: In perfect analogy to the 'centrifugation of heat' (0.21), we may say that 'heat accumulates down-wards' in a gravistatic field. It is remarkable, that the ratios of absolute temperatures T in (0.21) and (0.24) are identical with the ratios of periods of an atomic clocks: $T(\vec{x})$ in (0.21) is the period of an atomic clock moving with velocity $\vec{v}(\vec{x})$; The ratio (0.24) corresponds to the red shift, if $T(\vec{x})$ is the period of an atomic clock at $\vec{x}$ measured in coordinate time $t = x^4$.

We add a word, concerning the *sign of absolute temperature* T (see [8])). T has the sign of rest-mass-or enthalpy-density m. Thus, *if we chose rest mass to be positive,* T is positive. If two systems, in our case two fluids, are in contact, their T's must be equal. Thus, all possible systems must have $T > 0$.

§ 1. Notations

If $\{x^\alpha\}$, $\alpha\beta \ldots = 12 \ldots n$ is an n-dimensional differentiable manifold, general relativity assumes a *non singular metric*

$$g_{\alpha\beta}(x) = g_{(\alpha\beta)}(x)\,; \quad \det\,(g_{..}) \neq 0\,. \qquad (1.1)\,^{****})$$

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*) i.e. We believe that this one-sidedness of the solution is also true for the exact eqs. of motion.

**) * means in an *orthochronous rest frame.* $(\ldots)_0$ are the values of the state functions $\varkappa$, T and m in equilibrium at rest.

***) In *I*, a mistake has occurred.

****) $a_{(\alpha\beta\gamma\ldots)}$ is a totaly symmetric and $b_{[\alpha\beta\gamma\ldots]}$ a totally antisymmetric tensor.

We have shown in II that only the dimensions $n = 2$ and 4 can occur and that one of the two *thermodynamic signatures*

$$\text{signat}\ (g_{\alpha\beta}) = \pm\ (111 - 1) \tag{1.2}$$

must hold: one signature may change into the other, if we cross a surface*), where two spatial coordinates change simultaneously their sign. If we do not discuss such transitions, we may restrict ourselves to the particular thermodynamic signature (111 − 1). Covariant differentiation. D_α is defined in the usual way

$$\left.\begin{aligned} D_\alpha a^{\beta\cdots}{}_{\lambda\ldots}(x) = (\partial_\alpha a^{\beta\cdots}{}_{\lambda\ldots} + G_\alpha{}^\beta{}_{\beta'}\, a^{\beta'\cdots}{}_{\lambda\ldots} + \cdots \\ - a^{\beta\cdots}{}_{\lambda'\ldots}\, G_\alpha{}^{\lambda'}{}_\lambda - \cdots), \end{aligned}\right\} \tag{1.3}$$

$$G_\alpha{}^\beta{}_\gamma(x) = (g^{\beta\beta'}\, G_{\alpha\beta'\gamma})\,(x)\,, \tag{1.4}$$

$$2\, G_{\alpha\beta\gamma}(x) = (\partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\alpha\beta})\,(x)\,. \tag{1.5}$$

As long as we are not including GÖDEL[15]) manifolds, we may assume frames in which 3 coordinates are space-like $\vec{x} = \{x^i\}$, $i\,k\ldots = 123$ and one coordinate time like $x^4 = t$. In this case, we may introduce at any *specified event* $x = x'$, *local geodesic Lorentz frames*

$$g_{i\,i}(x') \overset{*}{=} -\, g_{44}(x') \overset{*}{=} 1\,; \quad g_{\alpha \neq \beta}(x') \overset{*}{=} 0\,, \tag{1.6}$$

$$G_{\alpha\beta\gamma}(x') \overset{*}{=} 0\,; \quad (D_\alpha a^{\beta\cdots}{}_{\lambda\ldots})\,(x') \overset{*}{=} (\partial_\alpha a^{\beta\cdots}{}_{\lambda\ldots})\,(x')\,. \tag{1.7}$$

Frame transformations are written in the form

$$'x'^\alpha = \psi'^\alpha(x)\,; \quad \partial_\alpha \psi'^\alpha(x) \equiv A'^\alpha{}_\alpha(x) \tag{1.8}$$

and tensors transform according to

$$'a'^{\alpha\cdots}{}_{'\lambda\ldots}('x) = (A'^\alpha{}_{\alpha\ldots}\, a^\alpha{}_\lambda\, A^{-1\lambda}{}_{'\lambda\ldots})\,(x)\,. \tag{1.9}$$

If our transformations do not permute space- and time coordinates we may define *pseudochronous tensors*[16])

$$'\breve{a}'^{\alpha\cdots}{}_{\ldots}('x) = \text{sig}\,(A^4{}_4)\,(A'^\alpha{}_\alpha \cdots \breve{a}^{\alpha\cdots}{}_{\ldots} \cdots)\,(x)\,, \tag{1.10}$$

where sig $(t) = \pm\, 1$ (or $= 0$) for $t \gtrless 0$ (or $t = 0$).

We shall introduce a 4-velocity field $w^\alpha(x)$ (time-like vector), normalised to

$$(w_\alpha\, w^\alpha)\,(x) = -\,1\,. \tag{1.11}$$

*) This surface is rather to be called a line, on account of the space metric.

In this case, we may introduce at *any specified* x', where $w^\alpha(x)$ is defined, a *local geodesic rest frame*

$$\vec{w}(x') = \{w^i(x')\} \stackrel{*}{=} 0; \quad w^4(x') \stackrel{*}{=} \pm 1. \tag{1.12}$$

The two possibilities $+1$ (-1) are called *orthochronous* (*pseudochronous*) *rest frames*.

The general frames, whose space-like and time-like coordinates can be separated, divide into two classes:

$$w^4(x) \gtrless 0 \text{ in a } \begin{Bmatrix}\text{ortho-}\\ \text{pseudo-}\end{Bmatrix} \text{chronous frame}. \tag{1.13}$$

In a Lorentz frame, 4-velocity and 3-velocity are related by

$$\vec{w}(x) = (w^4 \vec{v})(\vec{x}\,t); \quad w^4(x) = \pm\left(1 - v^2(\vec{x}\,t)\right)^{-1/2}. \tag{1.14}$$

The $w^\alpha(x)$ – fields defines a *family of world-lines* $x^\alpha = z^\alpha(\lambda)$

$$\frac{d}{d\lambda} z^\alpha(\lambda) = \dot{z}^\alpha(\lambda) = w^\alpha(z(\lambda)), \tag{1.15}$$

where λ is the *proper time parameter*.

Of any scalar or tensor, we may form the *proper time derivative*

$$\dot{a}^{\alpha\cdots}(x) \equiv \left(\frac{d}{d\lambda} a^{\alpha\cdots}(z(\lambda))\right)_{z(\lambda)=x} = (w^\varrho D_\varrho a^{\alpha\cdots})(x). \tag{1.16}$$

We shall be concerned with the *symmetric 4-gradient of the 4-velocity*.

$$2\,w_{\alpha\beta}(x) = (D_\alpha w_\beta + D_\beta w_\alpha)(x) = 2\,w_{(\alpha\beta)}(x) \tag{1.17}$$

and its *spatial projection* (*normal* ($\perp$) *on* $w^\alpha(x)$)

$$2\,w_{\alpha\beta\perp}(x) = (2\,w_{\alpha\beta} + w_\alpha\,\dot{w}_\beta + w_\beta\,\dot{w}_\alpha)(x) = 2\,w_{(\alpha\beta)\perp}(x). \tag{1.18}$$

Furthermore we need the *spatial projection of the metric*

$$g_{\alpha\beta\perp}(x) = (g_{\alpha\beta} + w_\alpha\,w_\beta)(x) = g_{(\alpha\beta)\perp}(x). \tag{1.19}$$

The irreducible parts of $w_{\alpha\beta\perp}$ are the *zero trace tensor*

$$2\,w^{(0)}_{\alpha\beta\perp}(x) = \left(2\,w_{\alpha\beta\perp} - \frac{2}{3}\,g_{\alpha\beta\perp}\,w_\varrho{}^\varrho\right)(x) \tag{1.20}$$

and *the scalar*

$$w_\varrho{}^\varrho{}_\perp(x) = w_\varrho{}^\varrho(x) = (D_\varrho w^\varrho)(x). \tag{1.21}$$

The symmetric 3-gradient of 3-velocity is defined by

$$2\, v_{i\,k}\,(\vec{x}\; t) = (\partial_i v_k + \partial_k v_i)\,(\vec{x}\; t)\,. \tag{1.22}$$

In a *local geodesic rest frame* (1.12) we have the identities

$$\dot{w}_i(x) \overset{*}{=} \partial_t v_i\,(\vec{x}\; t)\,; \qquad \dot{w}_4(x) \overset{*}{=} 0\,, \tag{1.23}$$

$$2\, w_{i\,k\,\perp}(x) \overset{*}{=} (w^4\, 2\, v_{i\,k})\,(\vec{x}\; t)\,; \qquad w^4 \overset{*}{=} \pm 1\,, \tag{1.24}$$

$$2\, w_{\alpha\,4\,\perp}(x) \overset{*}{=} 0\,. \tag{1.25}$$

Further we need the *spatial projection* $T_{\alpha\,\perp}$ *of the 4-gradient of temperature* T_α:

$$D_\alpha T(x) = \partial_\alpha T(x) \equiv T_\alpha(x)\,; \qquad T_{\alpha\,\perp}(x) = (T_\alpha + w_\alpha\, \dot{T})\,(x)\,. \tag{1.26}$$

§ 2. The Equations of Motion (Continuity Equations)

From the field equations of gravitation follows the continuity equation and symmetry

$$(D_\alpha \Theta^{\alpha\,\beta})\,(x) = 0\,; \qquad \Theta^{\alpha\,\beta}(x) = \Theta^{(\alpha\,\beta)}(x)\,. \tag{2.1}$$

In n. r. theory, the *conservation of inert mass* follows from the continuity equation for momentum as an additional continuity equation, if *Galilei covariance* is required.

In *Lorenz covariant theory* (and therefore also in general relativity), no such continuity equation follows. In order to have the same number of state variables, an additional continuity equation

$$(D_\alpha j^\alpha{}_N)\,(x) = 0 \tag{2.2}$$

has to be *postulated*. (2.2) expresses a *conservation law for substance*

$$\breve{N}[\tau] = \int_\tau (d\breve{\sigma}_\alpha\, j^\alpha{}_N)\,(y) = \breve{N}'\,, \tag{2.3}$$

where $\breve{N}'$ is a constant of motion.

In special relativity, where $D_\alpha = \partial_\alpha$, (2.1) leads to 10 conservation laws for momentum-ebergy $\{\breve{\Pi}^\mu\} = \{\vec{\breve{\Pi}}, \breve{H}\}$ and *angular momentum-center of energy* $\{\breve{M}^{\mu\,\nu}\} = \{\overset{\curvearrowright}{\breve{M}}, \vec{\breve{M}}\}$

$$\breve{\Pi}^\mu[\tau] = \int_\tau (d\breve{\sigma}_\alpha\, \Theta^{\alpha\,\mu})\,(y) = \breve{\Pi}'^\mu\,, \tag{2.4Π}$$

$$\breve{M}^{\mu\,\nu}[\tau] = \breve{M}^{[\mu\,\nu]}[\tau] = \int_\tau \left(d\breve{\sigma}_\alpha\,(y^\mu\, \Theta^{\alpha\,\nu} - y^\nu\, \Theta^{\alpha\,\mu})\right)(y) = \breve{M}'^{\mu\,\nu} \tag{2.4M}$$

(2.1) and (2.2) state the 1st law in *differential form.* (2.3) and (2.4) state the 1st law in *integral form.*

The *2nd law* is the differential relation

$$(D_\alpha j^\alpha{}_S - i)\,(x) = 0\,; \quad i(x) \geqq 0 \tag{2.5}$$

(*to which we must add the condition of equilibrium,* see § 3).

Having $4+2$ equations of motion (2.1), (2.2) and (2.5), but only $3+2$ independent state variables $\vec{v}$, $j^4{}_S$ and j^4_N, an identity of the type

$$\big(w_\beta (D_\alpha \Theta^{\alpha\,\beta}) + T\,(D_\alpha j^\alpha{}_S - i) + \mu (D_\alpha j^\alpha{}_N)\big)\,(x) = 0 \tag{2.6}$$

must exist.

$w_\alpha(x)$ is a 4-vector field, $T(x)$ and $\mu(x)$ are scalar fields. (2.6) defines these 6 space-time functions only up to a common, x-dependent factor. This factor is determined, up to a sign, if we identify $w^\alpha(x)$ with the 4-velocity field, normalised by (1.11).

T and μ turn out to be absolute temperature and chemical potential in an orthochronous rest frame.

We construct $\Theta^{\alpha\,\beta}(x)$ as a functional of $w^\alpha(x)$, $T(x)$, $\mu(x)$. The *local term* represents a *perfect fluid*

$$\Theta^{\alpha\,\beta}_{(0)}(x) = (m\, w^\alpha\, w^\beta + g^{\alpha\,\beta}\, p)\,(x)\,. \tag{2.7}$$

It depends only on the local values of the 5 state variables. i.e. on w^α and on 2 scalars m and p. Additional terms

$$\Theta^{\alpha\,\beta}(x) = (\Theta^{\alpha\,\beta}_{(0)} + \Theta^{\alpha\,\beta}_{(\eta)} + \Theta^{\alpha\,\beta}_{(\xi')} + \Theta^{\alpha\,\beta}_{(\varkappa)})\,(x) \tag{2.8}$$

depend on *the 1st derivatives,* more exactly on the *spatial projections of the 4-gradients* defined in (1.20), (1.21) and (1.26) and imply *transportation phenomena.*

Let us consider the perfect fluid term in more detail: We remark that it is *invariant under a reversal of proper time* (1.15) ($w^\alpha \rightarrow -w^\alpha$). This is due to the fact that, if $\Theta^{\alpha\,\beta} = \Theta^{\alpha\,\beta}_{(0)}$, the entropy source vanishes identically because all phenomena are reversible. The equation of motion for w^α is

$$(D_\alpha \Theta^\alpha_{(0)\,\beta})\,(x) = \big(m\,\dot{w}_\beta + \partial_\beta p + w_\beta\, D_\alpha\,(m\, w^\alpha)\big)\,(x) \equiv 0\,. \tag{2.9}$$

In a local geodesic rest frame (1.12) (1.23), the space part takes the form

$$(D_\alpha \Theta^\alpha{}_{(0)\,i})\,(x) \stackrel{*}{=} (m\,\partial_t v_i + \partial_i p)\,(\vec{x}\,t) \equiv 0\,. \tag{2.10}$$

Comparison with n.r. hydrodynamics induces us to consider m as *rest mass-density* and p as *pression.* On the other hand we have in such a frame the expression

$$\Theta^{44}_{(0)} \stackrel{*}{=} m - p \equiv u\,[s\,n] \tag{2.11}$$

for the rest energy-density. We consider u as the *density of interior energy*, function of *rest entropy density* s and *rest substance density* n. From this consideration follows that

$$m = u + p = s\,T + n\,\mu \qquad (2.12)$$

is also the *rest enthalpy density*. Now we consider (2.6) for $\Theta^{\alpha\beta}_{(0)}$ and $j^{\alpha}{}_{S(0)}$ with $i(x) = 0$. Using (2.11) we find

$$\left.\begin{aligned} (w_\beta\, D_\alpha \Theta^{\alpha\beta}{}_{(0)})\,(x) &= -\,(\dot{u} + m\, D_\alpha w^\alpha)\,(x) \\ &\equiv -\,(T\, D_\alpha j^\alpha{}_{S(0)} + \mu\, D_\alpha j^\alpha{}_N)\,(x)\,. \end{aligned}\right\} \qquad (2.13)$$

This identity is satisfied on account of (0.10) and the 2nd equation (2.12), if we pose

$$j^\alpha{}_{S(0)} = (s\, w^\alpha)\,(x)\,; \quad j^\alpha{}_N(x) = (n\, w^\alpha)\,(x)\,. \qquad (2.14)\,^*)$$

Thus s and n are equal to the rest densities *only in an orthochronous (1.13) local rest frame. Therefore T and μ have their true signification only in orthochronous frames.* This is not surprising, because energy $H[\tau] = \breve{\Pi}^4[\tau]$ is *an (orthochronous) scalar with respect* to time reversal, while entropy $\breve{S}[\tau]$ is a *pseudochronous scalar* (see [16]); Under time reversal, we have

$$'t = -\,t\,; \quad 'H = H\,; \quad '\breve{S} = -\,\breve{S}\,. \qquad (2.15)$$

Thus T and μ should be defined as pseudochronous scalars. We have not used this way, because for general space-time continua (for ex. of the GÖDEL[15]) type) the term pseudo-chronous has no meaning: In the purely differential relations like (2.6) only orthochronous scalar and tensor fields can occur.

We shall now give the transportation terms, depending on the two viscosities ξ' and η and on the heat conductivity $\varkappa$ in terms of $w_{\alpha\beta\perp}$ (1.18) and $T_{\alpha\perp}$ (1.26):

$$\left.\begin{aligned} \Theta^{\alpha\beta}_{(\eta)}(x) &= -\,(\eta\, 2\, w_\perp^{\alpha\beta(0)})\,(x)\,, \\ \Theta^{\alpha\beta}_{(\xi')}(x) &= -\,(\xi'\, g_\perp^{\alpha\beta}\, w_\varrho{}^\varrho)\,(x)\,, \\ \Theta^{\alpha\beta}_{(\varkappa)}(x) &= (w^\alpha\, q^\beta + w^\beta\, q^\alpha)\,(x)\,. \end{aligned}\right\} \qquad (2.16)$$

---

*) $j^\alpha{}_N$ is the definitive form. $j^\alpha{}_{S(0)}$ however is only the, time-like, projection of $j^\alpha{}_S$ (see (2. 19)) on w^α (see (0. 1)).

q^α is the *4-vector of heat flow*

$$q_\alpha(x) = -\left(\varkappa\,(T_{\alpha\perp} + \dot{w}_\alpha\,T)\right)(x) = q_{\alpha\perp}(x)\,. \tag{2.17}$$

In a local geodesic rest frame, q^4 vanishes and its space part is

$$\vec{q}\,(\vec{x}\,t) \overset{*}{=} -\left(\varkappa\,(\overrightarrow{\text{grad}}\,T + T\,\partial_t\vec{v})\right)(\vec{x}\,t)\,. \tag{2.18}$$

We remark the term proportional to $\partial_t\vec{v}$: It may be interpreted as an 'inertia of heat'.

The *4-vector of entropy density* is

$$j^\alpha{}_S(x) = (j^\alpha{}_{S(0)} + j^\alpha{}_{S(\varkappa)})\,(x) = (s\,w^\alpha + T^{-1}\,q^\alpha)\,(x) \tag{2.19}$$

and its source

$$\left.\begin{aligned} i(x) = \big(T^{-1}\,\eta\,w^{(0)}_{\alpha\beta\perp}\,w^{\alpha\beta(0)}_{\perp} + T^{-1}\,\xi'(w_\varrho{}^\varrho)^2 \\ + T^{-2}\,\varkappa\,(T_{\alpha\perp} + \dot{w}_\alpha\,T)\,(T^\alpha{}_\perp + \dot{w}_\alpha\,T)\big)\,(x) \geqq 0\,. \end{aligned}\right\} \tag{2.20}$$

The scalar products are positive definit, if only one coordinate is of time like charakter. Positive definitness of $i(x)$ requires therefore (0.15). We shall not give the explicit form of the equations of motion, but only their linear approximations in § 4.

§ 3. The Equilibrium Condition

As it has been stated in the introduction (0.20) (proof see annex), the *maximum of* $\breve{S}[\ldots]$, submitted to the 7 constraints*)

$$\left.\begin{aligned} H[\ldots] - H' &= 0\,, \\ \breve{\vec{\Pi}}[\ldots] - \breve{\vec{\Pi}}' &= 0\,, \\ \breve{\overleftrightarrow{M}}[\ldots] - \breve{\overleftrightarrow{M}}' &= 0\,, \\ \breve{N}[\ldots] - \breve{N}' &= 0 \end{aligned}\right\} \tag{3.1}$$

is given by *the maximum of*

$$\breve{\Psi}[\ldots] = \breve{S}[\ldots] + \breve{\vartheta}\,H[\ldots] - (\vec{\zeta}, \breve{\vec{\Pi}}[\ldots]) - (\overleftrightarrow{\omega}, \breve{\overleftrightarrow{M}}[\ldots]) - \nu\,\breve{N}[\ldots] \tag{3.2}$$

---

*) $\vec{M} = \{M^{i4}\} = \vec{M}'$ is not a constraint, because it depends explicitly on t.

where $\breve{\vartheta}$, $\vec{\zeta}$, $\overset{\curvearrowright}{\omega}$ and ν are Lagrange multipliers. As $i(x) = 0$, we may use the perfect fluid terms:

$$\breve{S}[\ldots] = \int (d^3V\, j^4{}_S)\,(\vec{x})\,; \quad j^4{}_S = s\, w^4 = j^4_{s(0)}\,. \tag{3.3$\breve{S}$}$$

$$H[\ldots] = \int \left(d^3V \left(m\,(w^4)^2 - p\right)\right)(\vec{x})\,, \tag{3.3H}$$

$$\breve{\vec{\Pi}}[\ldots] = \int (d^3V\, m\, w^4\, \vec{w})\,(\vec{x})\,, \tag{3.3$\breve{\vec{\Pi}}$}$$

$$\breve{\overset{\curvearrowright}{M}}[\ldots] = \int (d^3V\, m\, w^4\, [\vec{x} \wedge \vec{w}])\,(\vec{x})\,, \tag{3.3$\breve{\overset{\curvearrowright}{M}}$}$$

$$\breve{N}[\ldots] = \int (d^3V\, j^4_N)\,(\vec{x})\,. \tag{3.3$\breve{N}$}$$

The *extremum condition* is $\delta^{(1)}\breve{\Psi}[\ldots] = 0$, where $\delta^{(1)}$ is the 1st variation, linear in the variation of the functions to be varied. Calculation is easiest, if we chose $\vec{w}$, $j^4_S = j^4_{S(0)}$ and j^4_N as state variables.

The result is:

$$\left.\begin{aligned} &\delta^{(1)}\breve{\Psi}[\vec{w}(\,),\, j^4{}_S(\,),\, j^4{}_N(\,)] \\ &\quad = \int \{d^3V\, [(\breve{\vartheta}\,(m - a\,v^2)\, w^4\, \vec{v} - m\, w^4\, \vec{\zeta}' + a(\vec{\zeta}', \vec{w})\, \vec{v},\, \delta\vec{w}) \\ &\quad + \left(1 + \breve{\vartheta}\left(m_s\, w^4 - (w^4)^{-1}\, p_s\right) - m_s(\vec{\zeta}', \vec{w})\right) \delta j^4{}_S \\ &\quad + \left(\breve{\vartheta}\left(m_n\, w^4 - (w^4)^{-1}\, p_n\right) - m_n(\vec{\zeta}', \vec{w}) - \nu\right) \delta j^4{}_N]\}\,(\vec{x}) \end{aligned}\right\} \tag{3.4}$$

where

$$\vec{\zeta}'(\vec{x}) = \vec{\zeta} + [\overset{\curvearrowright}{\omega} \wedge \vec{x}]\,. \tag{3.5}$$

The coefficient of $\delta\vec{w}(\vec{x})$ vanishes, if

$$\breve{\vartheta}\, \vec{v}(\vec{x}) = \vec{\zeta}'(\vec{x}) = \vec{\zeta} + [\overset{\curvearrowright}{\omega} \wedge \vec{x}]\,. \tag{3.6}$$

Thus the motion at equilibrium is a translation at *constant velocity* $\breve{\vartheta}^{-1}\,\zeta$, on which a rotation at *constant angular velocity* $\breve{\vartheta}^{-1}\,\overset{\curvearrowright}{\omega}$ is superposed. Evidently, the fluid is supposed to be limited by a *rigid container* $\vec{x} \in C(\vec{y}) = 0$, whose velocity corresponds to the *boundary velocity* $(v^2(\vec{y}) < 1)$, which may, however, be as close to the velocity of light as we like

$$1 - v^2(\vec{y}) = \varepsilon^2\,; \quad \varepsilon \to +0\,. \tag{3.7}$$

The coefficient of $\delta j^4{}_S(x)$ vanishes (we have eliminated $\vec{\zeta}'$ by (3.6)), if

$$T(\vec{x}) = u_s(\vec{x}) = -\overset{\cup}{\vartheta}{}^{-1}\, w^4(\vec{x}) = T_0\left(1 - v^2(\vec{x})\right)^{-1/2} \left(\leqq T_0\, \varepsilon^{-1}{}^{*)}\right), \tag{3.8}$$

$$T_0 = -\overset{\cup}{\vartheta}{}^{-1} \operatorname{sig}(w^4)\,. \tag{3.9}$$

The formula (3.8) corresponds to the equation (368) of PAULI[4]): PAULI interprets his formula as a Lorentz transformation of temperature. According to (3.8), $T(\vec{x})$ takes (for $T_0 > 0$) arbitrarily high values at the bounary. This is due to the 'inertia of heat': *'heat is centrifugated in a rotating fluid'*.

The coefficient of δj^4_N vanishes, if

$$\mu(\vec{x}) = u_n(\vec{x}) = \overset{\cup}{\vartheta}{}^{-1}\, \nu\, w^4(\vec{x}) = \mu_0\left(1 - v^2(\vec{x})\right)^{-1/2} \left(\leqq \mu_0\, \varepsilon^{-1}{}^{**)}\right), \tag{3.10}$$

$$\mu_0 = \overset{\cup}{\vartheta}{}^{-1}\, \nu \operatorname{sig}(w^4)\,. \tag{3.11}$$

At this stage, one has to verify, whether the equations of motion are satisfied by (3.6), (3.8) and (3.10).

Calculation shows, that $w_{\alpha\,\beta\,\perp}(\vec{x})$ and $q_\alpha(\vec{x}) = q_{\alpha\,\perp}(\vec{x})$ vanish. Thus one is left, as we have postulated, with the equations of motion of the perfect fluid. These are satisfied by (3.6), (3.8), and (3.10).

Next, we calculate the 2nd variation $\delta^{(2)}\overset{\cup}{\Psi}[\ldots]$, bilinear in the functions to be varied. The condition for a maximum is $\delta^{(2)}\overset{\cup}{\Psi}[\ldots] \leqq 0$.

A somewhat lengthy calculation gives the following result. (We have substituted, *after* the variation was performed (3.6), (3.8) and (3.10) in order to eliminate the multipliers):

$$\left.\begin{aligned} \delta^{(2)}\overset{\cup}{\Psi}[\ldots] &= \ldots = -\operatorname{sig}(w^4) \int \left(d^3V\, T^{-1}\,(1 - v^2)^{-1/2}\right)(\vec{x})\big) \\ &\times \Big\{\big(m\,(g^{ik} - v^i\, v^k) - a\,(1 - v^2)\, v^i\, v^k\big)\, \delta w_i\, \delta w_k \\ &+ (1 - v^2)\left(u_{s\,s}(\delta j^4_S)^2 + 2\, u_{s\,n}\, \delta j^4_S\, \delta j^4_N + u_{n\,n}(\delta j^4_N)^2\right) \\ &+ (\ldots)^i\, \delta w_i\, \delta j^4_S + (\ldots)^i\, \delta w_i\, \delta j^4_N\Big\}\,(\vec{x}) \leqq 0\,. \end{aligned}\right\} \tag{3.12}$$

The variation changes its sign under time reversal, because $\overset{\cup}{\Psi}$ is pseudochronous. We have now to decide on the frame (ortho- or pseudo chronous, $w^4 > 0$ or $w^4 < 0$) in which this maximum is reached. We decide, arbitrarily, that

$$\overset{\cup}{S}[\ldots] \overset{*}{=} \text{maximum, if } \operatorname{sig}(w^4) \overset{*}{>} 0\,. \tag{3.13}$$

This defines the orthochronous frames. The contribution due to $\delta j^4{}_S(\vec{x})$ and $\delta j^4{}_N(\vec{x})$ is negative definit if a has the sign of T and if $c = u_s\, u_{s\,s}{}^{-1} \geqq 0$

---

*) if $T_0 > 0$.

**) if $\mu_0 > 0$.

(0.16). In order to evaluate the contribution due to $\delta\vec{w}(\vec{x})$, we introduce, at $\vec{x}$, a local frame $\vec{v} = (v, o, o)$. The condition takes the form

$$\{T^{-1}\,[(m - a\,v^2)\,(1 - v^2)\,(\delta w_1)^2 + m\,\big((\delta w_2)^2 + (\delta w_3)^2\big)]\}\,(\vec{x}) \geqq 0\,. \qquad (3.14)$$

Thus, for $(\delta w_2)^2\,(\vec{x})$, the condition $T^{-1}\,m \geqq 0$ in (0.16) follows. For $(\delta w_1)^2\,(\vec{x})$ we find

$$T^{-1}\,(m - a\,v^2) \geqq 0\,. \qquad (3.15)$$

In the limit (3.7) we obtain (multiply (3.15) with the positive definit factor $T\,m^{-1}\,v^{-2} \geqq 0$:

$$c_{\|}^2 = m^{-1}\,a \leqq v^{-2} \leqq 1 + \varepsilon^2\,; \qquad \varepsilon \to 0 \qquad (3.16)$$

i.e. condition (0.17).

From (0.16) and (2.12) follows:

$$T^{-1}\,m = s + T^{-1}\,n\,\mu \geqq 0\,. \qquad (3.17)$$

As s and n are independent variables, the condition follows:

$$s \geqq 0\,; \qquad j^4{}_{S(0)}(x) = (s\,w^4)\,(x) \overset{*}{\geqq} 0\,, \qquad (3.18)$$

the 2nd inequality being valid in an orthochronous frame. *Thus the 4-vector $j^{\alpha}_{S(0)}(x)$ is the arrow of time.* (3.18) may be considered as a rudimentary form of *Nernst's 3rd law: Entropy is (in an orthochronous frame) a positive definit quantity.*

§ 4. The linear Approximation

We consider, in zeroth approximation, the fluid at rest

i.e.: $\vec{v}(\vec{x}, t) = \vec{v}_0 = 0\,;\ w^4\,(\vec{x}\,t) = w^4{}_0 = \pm 1\,;\ s\,(\vec{x}\,t) = s_0\,;\ n\,(\vec{x}\,t) = n_0\,.$

All state functions in this static equilibrium will be marked by $(\ldots)_0$. Then, we allow an infinitesimal departure from equilibrium:

$$\vec{v}\,(\vec{x}\,t) = \vec{v}\,(\vec{x}\,t) - v_0\,; \qquad s\,(\vec{x}\,t) - s_0\,; \qquad n\,(\vec{x}\,t) - n_0$$

are infinitesimal quantities. Taking but the first order of this approximation, the equation of motion for $\vec{v}(\vec{x}\,t)$ is:

$$\left.\begin{aligned} \partial_\alpha \Theta^\alpha{}_i(x) \to \Big(&- (w^4\,\varkappa\,T)_0\,\partial^2{}_t\vec{v} + m_0\,\partial_t\vec{v} - (w^4\,\varkappa)_0\,\overrightarrow{\mathrm{grad}}\,\partial_t T \\ &+ \overrightarrow{\mathrm{grad}}\,p - (w^4\,\eta)_0\,(-\,\overset{\frown}{\mathrm{rot}}\,\overset{\frown}{\mathrm{rot}}\,\vec{v}) \\ &- \Big(w^4\Big(\xi' + \frac{4}{3}\,\eta\Big)\Big)_0\,\overrightarrow{\mathrm{grad}}\,\mathrm{div}\,\vec{v}\Big)\,(\vec{x}\,t) = 0\,. \end{aligned}\right\} \qquad (4.1)$$

We can separate the transversal ($\perp$) and longitudinal ($||$) parts

$$\vec{v} = \vec{v}_{\perp} + \vec{v}_{||} = \overrightarrow{\text{rot}}\, \vec{a} - \overrightarrow{\text{grad}}\, \varphi \,. \tag{4.2}$$

Then the two expression

$$\left(- (w^4 \varkappa\, T)_0\, \partial^2{}_t + m_0\, \partial_t - (w^4\, \eta)_0\, \Delta\right) \vec{v}_{\perp}\, (\vec{x}\, t) = 0 \tag{4.3}$$

and (we omit the $\overrightarrow{\text{grad}}$ sign and explicit p and T in terms of $s - s_0$ and $n - n_0$):

$$\left.\begin{aligned} &\Big\{\Big[- (w^4\, \varkappa\, T)_0\, \partial_t{}^2 + m_0\, \partial_t - \Big(w^4 \Big(\xi' + \frac{4}{3}\,\eta\Big)\Big)_0 \Delta\Big]\, (-\,\varphi) \\ &\quad - (w^4\, \varkappa\, u_{s\,s})_0\, \partial_t\, (s - s_0) - (w^4\, \varkappa\, u_{s\,n})_0\, \partial_t\, (n - n_0) \\ &\quad + (s\, u_{s\,s} + n\, u_{n\,s})_0\, (s - s_0) + (s\, u_{s\,n} + n\, u_{n\,n})\, (n - n_0)\Big\}\, (\vec{x}\, t) = 0 \end{aligned}\right\} \tag{4.4}$$

have to vanish separatly.

The equation of motion for $s - s_0$ and $n - n_0$ involve only the longitudinal part $\vec{v}_{||}$. They are, in terms of φ:

$$\left.\begin{aligned} &\{\partial_t\, (s - s_0) + (w^4\, \varkappa)_0\, \Delta \partial_t \varphi - s_0\, \Delta\varphi - (w^4\, \varkappa\, c^{-1})_0\, \Delta\, (s - s_0) \\ &\qquad\qquad - (w^4\, \varkappa\, T^{-1}\, u_{s\,n})_0\, \Delta\, (n - n_0)\}\, (\vec{x}\, t) = 0\,, \end{aligned}\right\} \tag{4.5}$$

$$\{\partial_t\, (n - n_0) - n_0\, \Delta\varphi\}\, (\vec{x}\, t) = 0\,. \tag{4.6}$$

As mentioned in the introduction (0.20), the 2nd time derivative of $\vec{v}$ apears, if $\varkappa > 0$.

We discuss the following particular cases:

1. Heat Flow:

If we put $\vec{v}_{||} = \varphi = 0$; $n = n_0$ (4.5) reduces to the n. r. equation of heat flow

$$(\partial_t - w_0^4\, b_0\, \Delta)\, (s - s_0)\, (\vec{x}\, t) = 0\,; \qquad b = \varkappa\, c^{-1} \geqq 0\,. \tag{4.7}$$

Its general solution is

$$(s - s_0)\, (\vec{x}\, t) = \int d^3V(\vec{y})\, K_{(b)}\, (\,|\vec{x} - \vec{y}\,|\,,\, w^4{}_0\, t)\, (s - s_0)\, (\vec{y}\, 0) \tag{4.8}$$

whose the kernel $K_{(b)}\, (\,|\vec{z}\,|\,,\, t)$ satisfies (4.7) (with $w_0^4 = +\,1$) and

$$K_{(b)}(\,|\vec{z}\,|\,,\, 0) = \delta(\vec{z})\,. \tag{4.9}$$

The kernel is, as is well known:

$$K_{(b)}(|\vec{z}|, t) = (2\pi)^{-3} \int d^3k \, e^{i(\vec{k}, \vec{z}) - b_0 k^2 t} = (4\pi b_0 t)^{-3/2} \exp\left(-\frac{|\vec{z}|^2}{4 b_0 t}\right). \quad (4.9)$$

As was mentioned in the introduction, *only solutions for the absolute future* $w_0{}^4 t \geqq 0$ *exist*. Equilibrium is asymptotically reached as $w_0^4 t \to +\infty$. This was to be expected on account of the condition (3.13). We remark however, that $\varphi = 0$, $n = n_0$ does not satisfy (4.4). Thus, heat flow is always connected[9]) with the

2. Elastic Waves:

We assume $\varkappa_0 = 0$. The wave equation is obtained if we operate with ∂_t on (4.4) and eliminate $\partial_t (s - s_0)$ and $\partial_t (n - n_0)$ from (4.5) and (4.6):

$$\left(m_0 \partial_t{}^2 - a_0 \Delta - \left(w^4 \left(\xi' + \frac{4}{3}\eta\right)\right)_0 \Delta \partial_t\right) \varphi (\vec{x}\, t) = 0\,. \quad (4.10)$$

This is exactly the equation of n. r. theory: Undamped waves ($\xi'_0 = \eta_0 = 0$) propagate with the velocity $0 \leqq c_{||} \leqq 1$ (0.15).

In a viscous fluid, the general solution is

$$\varphi (\vec{x}\, t) = \int d^3V(\vec{y}) \, [D_{(t)} (|\vec{x} - \vec{y}|, w^4{}_0 t) \, \varphi (\vec{y}\, 0) + D (|\vec{x} - \vec{y}|, w_0{}^4 t) \, w^4{}_0 \partial_t \varphi (\vec{y}\, 0)] \quad (4.11)$$

where the two kernels $D_{(t)}$ and D satisfy the wave equation and

$$D_{(t)}(|\vec{z}|, 0) = \delta(\vec{z})\,; \qquad \partial_t D_{(t)}(|\vec{z}|, 0) = 0\,; \quad (4.12)$$

$$D(\vec{z}, 0) = 0\,; \qquad \partial_t D(|\vec{z}|, 0) = \delta(\vec{z})\,. \quad (4.13)$$

We give the Fourrier representation of the kernels. They involve two terms $k^2 < k^2_{max}$ and $k^2 > k^2{}_{max}$.

The representation follows from the wave equation in the form

$$(\partial_t^2 - c_{||}^2 \Delta - w_0{}^4 c_{||}^2 \tau \Delta \partial_t) \, \varphi (\vec{x}\, t)\,, \quad (4.14)$$

with

$$c_{||}^2 \tau = \left(m^{-1} \left(\xi' + \frac{4}{3}\eta\right)\right) \geqq 0\,, \quad (4.15)$$

$$k^2{}_{max} = 4\, (c^2{}_{||} \tau^2)^{-1} \quad (4.16)$$

in terms of the following functions of k^2

$$\left.\begin{aligned} k^2 < k^2_{max}: \qquad & \gamma(k^2) = \frac{1}{2} c_{\|}^2 \tau k^2 \geqq 0 , \\ & \omega(k^2) = \left(c^2_{\|} k^2 - \frac{1}{4} c_{\|}{}^4 \tau^2 k^4\right)^{1/2} \geqq 0 , \end{aligned}\right\} \quad (4.17)$$

$$\left.\begin{aligned} & k^2 > k^2_{max}: \\ & \gamma_{\substack{1\\2}}(k^2) = \gamma(k^2) \left(1 \mp (1 - k^2_{max} k^{-2})\right) \to \begin{cases} +0 \\ c_{\|}^2 \tau k^2 \to +\infty , \end{cases} \end{aligned}\right\} \quad (4.18)$$

$$\left.\begin{aligned} D_{(t)}(|\vec{z}|, t) = (2\pi)^{-3} \Bigg[& \int_0^{k_{max}} d^3k \, 2^{-1} \left(e^{i((\vec{k}, \vec{z}) - \omega t) - \gamma t} (1 + i \gamma \omega^{-1}) + c.\,c.\right) \\ & + \int_{k_{max}}^{\infty} d^3k \, (\gamma_2 - \gamma_1)^{-1} e^{i(\vec{k}, \vec{z})} \left(\gamma_2 e^{-\gamma_1 t} - \gamma_1 e^{-\gamma_2 t}\right) \Bigg] , \end{aligned}\right\} \quad (4.19)$$

$$\left.\begin{aligned} D(|\vec{z}|, t) = (2\pi)^{-3} \Bigg[& \int_0^{k_{max}} d^3k \, (2 i \omega)^{-1} \left(e^{i((\vec{k}, \vec{z}) - \omega t) - \gamma t} - c.\,c.\right) \\ & + \int_{k_{max}}^{\infty} d^3k \, (\gamma_2 - \gamma_1)^{-1} e^{i(\vec{k}, \vec{z})} \left(e^{-\gamma_1 t} - e^{-\gamma_2 t}\right) \Bigg] . \end{aligned}\right\} \quad (4.20)$$

On account of (4.18), both kernels exist but for $t \geqq 0$. Therefore, in perfect analogy to the general solution of heat flow, *damped waves can only be predicted for the absolute future* $w_0^4 \, t \geqq 0$. (If there is no damping, $D(|\vec{z}|, t)$ is the 'D^0-function' of electrodynamics with $c_{\|}$ instead of 1, and the relation $D_{(t)}(|\vec{z}|, t) = \partial_t D(|\vec{z}|, t)$ holds).

3. *Flow of Transverse momentum:*

Equation (4.3), to which of course div $\vec{v}_\perp = 0$ must be added, can be discussed without approximation. We consider first the 3 limiting cases.

a) $\varkappa_0 = 0$: This leads to the equation of heat-flow (4.8), with

$$b = \eta \, m^{-1} \geqq 0 . \quad (4.21)$$

The general solution exists only for the absolute future $w_0{}^4 \, t \geqq 0$.

b) $\eta_0 = 0$: The solution for the acceleration is

$$\partial_t \vec{v}_\perp (\vec{x}\, t) = e^{w_0{}^4 \beta t} \, \partial_t \vec{v} (\vec{x}\, 0) , \quad (4.22)$$

$$\beta = (m \, \varkappa^{-1} \, T^{-1})_0 \geqq 0 . \quad (4.23)$$

Unless the initial acceleration vanishes, the accelaration increases indefinitly towards the absolute future $w_0^4\, t \geqq 0$ ($\beta^{-1} \sim 10^{-24} - 10^{-27}$ cm).

This feature is perfectly analogous to the 'run away' solution of DIRAC's[14]) theory of the point electron ($\beta^{-1} \sim 10^{-13}$ cm).

c) $m_0 = 0$: This case is of purely theoretical interest, the rest mass of a fluid being always finite.

Two kernels for

$$\vec{v}_\perp (\vec{x}\, t) = \int d^3V(\vec{y})\, K^{(\pm)} (|\vec{x} - \vec{y}|, w_0^4\, t)\, \vec{v}_\perp (\vec{y}\, 0) \tag{4.24}$$

exist. They are, in terms of

$$\alpha^2 = (\eta\, \varkappa^{-1}\, T^{-1})_0 \geqq 0\,, \quad \alpha \geqq 0\,, \tag{4.25}$$

$$\left.\begin{aligned} K^{(\pm)}(|\vec{z}|, t) &= (2\pi)^{-3} \int d^3k\, e^{i(\vec{k}, \vec{z}) \mp \alpha t} \\ &= \pi^{-2} (\alpha^2 t^2 + |\vec{z}|^2)^{-2} (\pm \alpha t)\,, \end{aligned}\right\} \tag{4.26}$$

$K^{(+)}$ exists only for $t \geqq 0$ and $K^{(-)}$ only for $t \leqq 0$.

We have of course

$$K^{(\pm)} (|\vec{z}|, \pm 0) = \delta(\vec{z})\,. \tag{4.27}$$

Thus a solution, which exists only in the absolute past $w^4{}_0\, t \leqq 0$ *is equally possible.* The apparance of this non thermodynamic solution is the generalization of the 'run away' solution (4.22).

d) *General case:* In terms of α and β the two kernels are

$$K^{(\pm)}(|\vec{z}|, t) = (2\pi)^{-3} \int d^3k\, e^{i(\vec{k}, \vec{z}) - \gamma^{\pm}(k^2)\, t}\,, \tag{4.28}$$

$$\gamma^{\pm}(k^2) = \frac{1}{2}\beta \pm \left(\frac{1}{4}\beta^2 + \alpha^2 k^2\right)^{1/2} \to \pm \alpha k \to \pm \infty\,. \tag{4.29}$$

We have again the thermodynamic solution and the, unphysical, 'run away' solution existing only in the absolute past $w^4{}_0\, t \leqq 0$. It is interesting to write down the equations for small wave vectors $|\vec{k}|$:

$$k^2 \ll k^2_{max} = \frac{1}{4} (\eta\, \varkappa\, T)^{-1}{}_0\, m^2{}_0\,. \tag{4.30}$$

The two kernels are

$$K^{(\pm)}(|\vec{z}|, t) \cong \binom{1}{e^{\beta t}} (2\pi)^{-3} \int_0^{k_{max}} d^3k\, e^{i(\vec{k}, \vec{z}) \mp b_0 k^2} \tag{4.31}$$

where b_0 is given by (4.21). *Thus, the physical solution* $K^{(+)}$, *valid only for* $t \geqq 0$ *reduces to the heat flow kernel as in the n. r. case:* The upper limit k_{max} goes to $k_{max} \to +\infty$, if $\varkappa\, T \to 0$.

The *unphysical solution* $K^{(-)}$, valid only for $t \leqq 0$, is the 'run away' factor exp $(\beta\, t)$ multiplied with the heat flow kernel, in which the substitution $t \to -t$ has been made.

§ 5. Equilibrium in a Gravistatic Field*)

We consider a field $g_{\alpha\beta}(x) = g_{\alpha\beta}(\vec{x})$ independent of t, and we look for a stationary solution $\partial_t(\ldots) = 0$ with $w^i(\vec{x}) = 0$. Thus we have

$$(w_\alpha\, w^\alpha)\,(\vec{x}) = \left(g_{44}(w^4)^2\right)(\vec{x}) = -1\,. \tag{5.1}$$

Equilibrium requires a vanishing $i(\vec{x})$. This implies $w_{\alpha\beta\perp}(\vec{x}) = 0$ and $(T_{\alpha\perp} + \dot{w}_a\, T)\,(\vec{x}) = 0$. Calculation shows, that $w^\alpha(x) = \left\{0,\, w^4(\vec{x})\right\}$ satisfies the 1st condition. The 2nd condition gives us the temperature distribution $T(\vec{x})$. The *space part* of the 2nd condition leads to $T_{i\perp}(\vec{x}) = \partial_i T(\vec{x})$, because $\dot{T}(\vec{x}) = 0$. Further we have (cf. (1.4) and (1.5))

$$\dot{w}_i(\vec{x}) = (w^4\, D_4 w_i)\,(\vec{x}) = -\,(w^4\, w_4\, G_4{}^4{}_i)\,(\vec{x}) = -\left((w^4)^2\,\frac{1}{2}\,\partial_i g_{44}\right)(\vec{x})\,. \tag{5.2}$$

Thus, on account of (5.1),

$$(T_{i\perp} + \dot{w}_i\, T)\,(\vec{x}) = \left(\partial_i T + \left(\frac{1}{2}\,(g_{44})^{-1}\,\partial_i g_{44}\right) T\right)(\vec{x}) \equiv 0\,. \tag{5.3}$$

The solution is (0.24). In order to find $\mu(\vec{x})$, we use the equation of motion (space part for $\dot{w}_i$, remembering $m = s\,T + n\,\mu$ and (from (2.12) $\partial_i p(\vec{z}) = (s\,\partial_i T + n\,\partial_i \mu)\,(\vec{x})$) $(i(x) = 0$ has again reduced the problem to the *perfect fluid case*). We find, from (2.12) and (5.2):

$$\left.\begin{aligned}(m\,\dot{w}_i + \partial_i p)\,(\vec{x}) = \Big[s\left(\frac{1}{2}\,(g_{44})^{-1}\,(\partial_i g_{44})\;T + \partial_i T\right)\\ + n\left(\frac{1}{2}\,(g_{44})^{-1}\,(\partial_i g_{44})\;\mu + \partial_i \mu\right)\Big]\,(\vec{x}) \equiv 0\,.\end{aligned}\right\} \tag{5.5}$$

From (5.3) and (5.5), (0.24) follows for μ. The equations of motion for s and n are easily verified, if we write them in the form

$$(\dot{s} + s\,D_\alpha w^\alpha) = (\dot{n} + n\,D_\alpha w^\alpha) = \dot{s} = \dot{n} = 0\,, \tag{5.7}$$

because $D_\alpha w^\alpha = w_\alpha{}^\alpha = w_{\alpha\perp}^{\;\alpha}$ vanishes and because the substantial derivatives of scalars $\dot{T}$, $\dot{\mu}$, $\dot{n}$, $\dot{s}$, ... vanish in the static case.

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*) The relations in I are not quite correct: They assume a point $\vec{y}$, where $g_{44}(\vec{y}) = 1$.

Annex: 2nd Order Variation of Functionals

If $F = F[\xi(\ )]$ is a functional of the function $\xi(x)$, $x = \{x^i\}$ *), *the functional derivative at x* $F_{,x} = F_{,x}[\xi\ (\ )]$ is defined by

$$\delta^{(1)} F[\ldots] \equiv \int_V (dV\ \delta\xi)\ (x)\ F_{,x}[\ldots]\,. \qquad \text{(A. 1)}$$

In *thermodynamics*, we deal with fonctionals of the *density type*:

$$F[\ldots] = \int_V dV(x)\ f[x, \xi(x)] \equiv \int_V dV(x)\ f(x)\,. \qquad \text{(A. 2)}$$

V is the region of integration and symbolises also the volume

$$V = \int_V dV(x)$$

of the region. $f(x) = f[x, \xi(x)]$ is *the density of F*. With $f'[\ldots] = \partial f[x, \xi]/\partial\xi$, $f'' = \ldots$, the functional derivative is

$$F_{,x}[\ldots] = f'(x)\,. \qquad \text{(A. 3)}$$

If we look for the maximum of $F[\ldots]$, submitted to a constraint

$$G[\xi(\ )] = G'\,, \qquad \text{(A. 4)}$$

we introduce the *particular variation*

$$\delta\xi(x) = \delta\xi^1 = \text{const}\,, \quad \text{if } x \in V_1\,, \qquad \text{(A. 5)}$$

where V_1 is a small region of V. According to

$$\delta^{(1)} G[\ldots] = V_1\, G_{,x_1}[\ldots]\, \delta\xi^1 + \int_{V'} (dV\ \delta\xi)\ (x)\ G_{,x}[\ldots] = 0\,, \qquad \text{(A. 6)}$$

$\delta\xi^1 = \delta\xi^1[\xi(\ )]$ is a functional of $\xi(x)$ in $V' = V - V_1$. We write

$$\left.\begin{aligned} \delta\xi^1[\ldots] &= -\,(G_{,1}[\ldots.])^{-1} \int_{V'} (dV\ \delta\xi)\ (x)\ G_{,x}[\ldots] \\ &\equiv \int_{V'} (dV\ \delta\xi)\ (x)\ \xi^1{}_{,x}[\ldots]\,, \end{aligned}\right\} \qquad \text{(A. 7)}$$

introducing the functional derivative $\xi^1{}_{,x}$ of ξ^1. $G_{,1}[\ldots] = V_1\, G_{,x_1}[\ldots]$ is the *partial derivative* $\partial G[\ldots]/\partial\xi^1$. x_1 is a point *inside* V_1 ($\in V_1$) (mean value theorem of integration).

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*) We omit the $\rightarrow$ in $\vec{x} = x = \{x^i\}$.

The 1st variation of $F[\ldots]$ is now

$$\left.\begin{aligned}\delta^{(1)}F[\ldots] &= F_{,1}\,\delta\xi^1 + \int\limits_{V'} (dV\,\delta\xi)\,(x)\,F_{,x} \\ &= \int\limits_{V'} (dV\,\delta\xi)\,(x)\,\big(F_{,x} + (-\,G^{-1}{}_{,1}\,F_{,1})\,G_{,x}\big) \equiv 0\,.\end{aligned}\right\} \quad \text{(A. 8) *)}$$

It has to vanish for an *extremum*. Thus

$$\overline{F}_{,x}[\ldots] = \big(F_{,x} + (G^{-1}{}_{,1}\,F_{,1})\,G_{,x}\big)\,[\ldots] \equiv 0\,; \quad x \in V' \qquad \text{(A. 9)}$$

is the functional derivative of F in $V' = V - V_1$.

The *method of Lagrange multipliers* introduces the functional

$$\Psi[\ldots] = (F + \lambda\,G)\,[\ldots] \qquad \text{(A. 10)}$$

and looks for the extremum of Ψ with unrestricted variations. Or, for our particular variation (A. 5), we have

$$\delta^{(1)}\Psi[\ldots] = (F_{,1} + \lambda\,G_{,1})\,\delta\xi^1 + \int\limits_{V'} (dV\,\delta\xi)\,(x)\,\Psi_{,x} \equiv 0\,. \qquad \text{(A. 11)}$$

The variation being arbitrary, the value of the multiplier is:

$$\lambda = -\,(G^{-1}{}_{,1}\,F_{,1})\,[\ldots]\,. \qquad \text{(A. 12)}$$

Thus, for the 1st variation, the multiplier method is equivalent to the elimination method.

We show now, that the condition $\delta^{(2)}\Psi \leqq 0$ is sufficient for $\delta^{(2)}F \leqq 0$, if the constraint holds: We form the 2nd functional derivative of $\overline{F}[\ldots]$. (A. 9), considered as functional of $\xi(x)$ in V',

$$\left.\begin{aligned}\overline{F}_{,xy}[\ldots] &= \Psi_{,xy} + \Psi_{,x1}\,\xi^1{}_{,y} + (G^{-2}{}_{,1}\,G_{,1y}\,F_{,1} - G^{-1}_{,1}\,F_{,1y}) \\ &\quad \times G_{,x} + (G^{-2}{}_{,1}\,G_{,11}\,F_{,1} - G^{-1}{}_{,1}\,F_{,11})\,G_{,x}\,\xi^1{}_{,y} \\ &= \Psi_{,xy} + \Psi_{,x1}\,\xi^1{}_{,y} + \Psi_{,1y}\,G^1_{,x} + \Psi_{,11}\,\xi^1{}_{,x}\,\xi^1{}_{,y}\,; \\ &\quad x\,y \in V'\,,\end{aligned}\right\} \quad \text{(A. 13)}$$

where (A. 12) and the definition of $\xi^1{}_{,x}$ in (A. 7) have been used. The 2nd variation is therefore

*) The points x_1 in $F_{,1} = V_1\,F_{,x_1}$ and in $G_{,1} = V_1\,G_{,x_1}$ are in general *different*, but are both inside V_1 ($\in V_1$).

$$\left.\begin{aligned}\delta^{(2)}F &= \int\limits_{V'} (dV\,\delta\xi)\,(x) \int\limits_{V'} (dV\,\delta\xi)\,(y)\,\bar{F}_{,xy} \\ &= \int\limits_{V'} (dV\,\delta\xi)\,(x) \int\limits_{V'} (dV\,\delta\xi)\,(y)\,\Psi_{,xy} + 2 \int\limits_{V'} (dV\,\delta\xi)\,(x) \\ &\times \Psi_{,x1}\,\delta\,\xi^1 + \Psi_{,11}(\delta\xi^1)^2 = \int\limits_{V} (dV\,\delta\xi)\,(x) \\ &\times \int\limits_{V} (dV\,\delta\xi)\,(y)\,\Psi_{,xy}[\ldots] = \delta^{(2)}\Psi[\ldots] \leqq 0\,.\end{aligned}\right\} \quad (A.\,14)$$

The third equality is only ralid for the particalar variation (A.5) (A.7). However, if $\delta^{(2)}\Psi[\ldots] \leqslant 0$ for an arbitrary variation, $\delta^{(2)}F[\ldots] \leqslant 0$ results. Thus $\delta^{(2)}\Psi[\ldots] \leqslant 0$ is a *sufficient condition* for $\delta^{(2)}F[\ldots] \leqslant 0$.

In the particular case, where $F[\ldots]$ and $G[\ldots]$ (with the density $g(x) = g[x, \xi(x)]$) are of the form (A.2), we have

$$\Psi[\ldots] = \int\limits_{V} (dV\,\psi)\,(x)\,; \quad \psi(x) = f(x) + \lambda\,g(x)\,, \qquad (A.\,14)$$

$$\left.\begin{aligned}\Psi_{,xy} &= \psi''(x)\,\delta\,(x-y) \\ \Psi_{,x1} &= 0 \\ \Psi_{,11} &= V_1\,\psi''(x_1)\end{aligned}\right\}\quad x\,y \in V'\,. \qquad (A.\,15)$$

Therefore

$$\delta^{(2)}F[\ldots] = \int\limits_{V'} (dV\,\psi''(\delta\xi)^2)\,(x) + V_1\,\psi''(x_1)\,(\delta\xi^1)^2 \leqq 0\,. \qquad (A.\,16)$$

Now, we take the particular variation

$$\delta\xi(x) = \begin{Bmatrix} 0 \\ V_0^{-1}\,\delta\eta \end{Bmatrix} \text{ for } x \begin{Bmatrix} \notin \\ \in \end{Bmatrix} V_0\,; \quad V_0 \in V' \qquad (A.\,17)$$

from which

$$\delta\xi^1 = -\,V_1^{-1}\left(\frac{g'(x_0)}{g'(x_1)}\right)\delta\eta \qquad (A.\,18)$$

follows, (x_0 is a point inside V_0).

For this variation, we have

$$\left.\begin{aligned}&\delta^{(2)}F[\ldots] = \left(V_0^{-1}\,\psi''(x_0) + V_1^{-1}\,\psi''(x_1)\left(\frac{g'(x_0)}{g'(x_1)}\right)^2\right)(\delta\eta)^2 \leqq 0\,, \\ &\qquad x_0\,x_1 \in V\,.\end{aligned}\right\} \qquad (A.\,19)$$

The volumes V_0 and V_1 are arbitrary and independent of each other. x_0 and x_1, are therefore two (arbitrary) points $\in V$. Therefore

$$\psi''(x) \leqq 0 ; \quad x \in V \tag{A. 20}$$

is a *necessary condition.* Thus $\delta^{(1)}\Psi[\ldots] = 0$, $\delta^{(2)}\Psi[\ldots] \leqq 0$ are *necessary and sufficient* conditions for $\delta^{(1)}F[\ldots] = 0$ and $\delta^{(2)}F[\ldots] \leqq 0$, if $F[\ldots]$ and $G[\ldots]$ are of the form (A. 2).

The method can easily be generalized to the case, where several (n) constaints (0.18) are imposed, and if several (ω) functions $\xi^\alpha(x)$ ($\alpha\,\beta \ldots = 12 \ldots \omega$) are to be varied. Higher order variations can also be formed in the same way. We conclude therefore:

The maximum condition for density functionals with constraints, which are also density functionals, can be computed by the method of Lagrange multipliers.

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$$v^\alpha \rightarrow w^\alpha, \quad \sigma \rightarrow s, \quad \nu \rightarrow n, \quad n^\alpha \rightarrow j^\alpha{}_N,$$

$$s^\alpha \rightarrow j^\alpha{}_S, \quad \xi = \xi' - \frac{2}{3}\eta .$$

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