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Symmetry breaking solutions of the Thirring Model

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Abstract. It is shown that the Thirring model admits of symmetry breaking operator solutions in addition to the normal solution linked with perturbation theory. The normal solution is stable only if the absolute value of the coupling constant λ is smaller than π. At $\lambda = \pm \pi$ two types of symmetry breaking solutions appear, characterized by a finite vacuum current density while for $|\lambda| > \pi$ the ground state is associated with infinite current density. All solutions except the normal one break the homogeneous Lorentz group and the group of scale transformations, while the charge group, the Touschek group and the translation group persist as symmetries of the solutions.

1. Introduction

The success of the idea of an approximate symmetry in elementary particle physics raises a fundamental question. Why does nature seem to prefer a symmetry group and nevertheless break it? One of the most fascinating attempts to answer this question is often referred to as spontaneous breakdown of symmetry. The idea is the following: Suppose a system is characterized by a set of field equations and commutation rules which are invariant under a given symmetry group. It is well known that the perturbation theory solution of these equations displays the same symmetry. However, it may happen that this solution is dynamically unstable; a small external perturbation may suffice to change the structure of the ground state completely. The most familiar example of such a system is the ferromagnet which breaks the rotation group. A similar situation arises in the theory of superconductivity. Y. NAMBU and G. JONA-LASINIO[1]) were able to transcribe the methods used in superconductivity to field theory and recently I. BIALYNICKI-BIRULA[2]) suggested the existence of symmetry breaking solutions of the Thirring model.

The purpose of the present paper is to show that the Thirring model does indeed admit symmetry breaking solutions. The symmetry groups that are broken by these solutions are the two-dimensional Lorentz group and the group of scale transformations characteristic of a system which does not contain a length.

The existence of symmetry breaking solutions is closely related to the occurrence of zero-mass Boson excitations of the system which in this theory are described by the current operator $j_\mu(x)$. In particular the infrared part of this current plays an important role in the symmetry breakdown.

Finally let us make some technical remarks concerning the use of cut-offs in our formulation of the model. We shall introduce both an ultraviolet and an infrared cut-off in order to be able to treat singular expressions like $\bar{\psi}(x)\,\gamma_\mu\,\psi(x)$ occurring in the model. We would like to emphasize that the only purpose of the cut-offs is to lead us in a simple, well-defined way to the algebra of field operators. Once this algebra is established, the cut-offs may be dropped and it will be seen that the limiting algebra of field operator distributions is in fact independent of the particular choice of cut-offs. No cut-off function occurs in the consistency requirements for symmetry break-down.

2. Definition of the Model

The Thirring model[3]) is based on a two-component spinor field ψ in one space-dimension and one time-dimension. This field satisfies the equal time commutation relations

$$\{\psi(x), \psi(y)\}_{x^0=y^0} = 0\,; \quad \{\psi(x), \psi^+(y)\}_{x^0=y^0} = \delta\,(\boldsymbol{x}-\boldsymbol{y}) \tag{2.1}$$

and the field equations[4])

$$-\,i\,\gamma^\mu\,\partial_\mu\,\psi(x) = \frac{\lambda}{2}\,\gamma^\mu\,\{j_\mu(x), \psi(x)\}\,. \tag{2.2}$$

We assume that the field operators $\psi(x)$ furnish an irreducible representation of the commutation relations (2.1) on each surface $x^\circ = \text{const}$. The current $j_\mu(x)$ is formally defined by

$$j_\mu(x) = \bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,. \tag{2.3}$$

As is well known this product of two operator-valued distributions at the same point is not well defined. The problem of how to give a meaning to an expression like (2.3) has been studied by K. Johnson[5]) and by F. L. Scarf and J. Wess[6]) in the framework of the Thirring model. These authors proposed to define $j_\mu(x)$ by means of an averaging procedure over a small region of space-time. Instead of an explicit definition of j_μ by means of a nonlocal form of (2.3) we shall make use of an implicit definition of the current. We require the current to be local and conserved[5]). The local nature of the current is expressed through equal-time commutation rules of the form[5]) $(x^0 = y^0)$

$$\begin{aligned} [j_0(x), \psi(y)] &= -\,a\,\delta\,(\boldsymbol{x}-\boldsymbol{y})\,\psi(y) \\ [j_1(x), \psi(y)] &= -\,\bar{a}\,\delta\,(\boldsymbol{x}-\boldsymbol{y})\,\gamma\,\psi(y) \end{aligned} \tag{2.4}$$

Conservation of the current and conservation of the pseudocurrent are expressed through[5])

$$\partial^\mu\,j_\mu = 0 \qquad \varepsilon^{\mu\nu}\,\partial_\mu\,j_\nu = 0 \tag{2.5}$$

The commutation rules (2.4) determine the operator j_μ up to an additive c-number provided the parameters a and $\bar{a}$ are known and the representation of the operator ψ is irreducible. The additive c-number will be determined in section 9.

As an immediate consequence of (2.4) we note that the equal-time commutation rules of the current with itself are of the form

$$[j_\mu(x), j_\nu(y)]_{x^0=y^0} = c_{\mu\nu}(x, y) , \tag{2.6}$$

where $c_{\mu\nu}(x, y)$ is a c-number distribution. To obtain the result (2.6) consider the quantity ($x^0 = y^0 = z^0$)

$$[[j_\mu(x), j_\nu(y)], \psi(z)] = [j_\mu(x), [j_\nu(y), \psi(z)]] - [j_\nu(y), [j_\mu(x), \psi(z)]] .$$

Making use of (2.4) we obtain

$$[j_\mu(x), [j_\nu(y), \psi(z)]] = [j_\nu(y), [j_\mu(x), \psi(z)]] .$$

Therefore the commutator $[j_\mu(x), j_\nu(y)]$ commutes with ψ. If the representation of the operator ψ is irreducible this quantity must be a c-number.

More information about $c_{\mu\nu}(x, y)$ can be obtained in the following fashion. The temporal development of $\psi(x)$ is governed by the field equation (2.2) while the temporal development of the current is given by the conservation laws of charge and pseudocharge, eq. (2.5). These equations of motion conserve the commutators (2.4) only if

$$c_{00}(x, y) = c_{11}(x, y) = 0 ,$$
$$c_{10}(x, y) = c_{01}(x, y) = i\,\lambda^{-1}\,(\bar{a} - a)\,\delta'(x - y) . \tag{2.7}$$

This shows that a and $\bar{a}$ coincide only if $c_{\mu\nu} = 0$. However, J. SCHWINGER[7]) has shown that a vanishing commutator of the current j_1 with the charge density j_0 leads to inconsistencies. We must therefore expect a to be different from $\bar{a}$.

3. Representation of the Current in Hilbert Space

As an immediate consequence of (2.5) the current satisfies a massless free field equation

$$\Box\, j_\mu = 0 . \tag{3.1}$$

We write the Fourier representation of j_μ in the form

$$j_\mu(x) = \int d\varkappa \frac{k_\mu}{\varkappa} \{e^{-ikx} c(\varkappa) + e^{ikx} c^+(\varkappa)\} ; \quad k_\mu = (|\varkappa|, \varkappa) . \tag{3.2}$$

To find the representation of the operators $c(\varkappa)$ in Hilbert space we make use of the fact that translation invariance is not broken by the solutions to be constructed. This implies that a family of unitary operators $U(a)$ exists with the property

$$U(a)\,\psi(x)\,U^+(a) = \psi(x + a) ; \quad U(a)\,j_\mu(x)\,U^+(a) = j_\mu(x + a) .$$

Denoting by P_μ the generators of $U(a)$ we obtain

$$[P_\mu, c(\varkappa)] = -\,k_\mu\,c(\varkappa) . \tag{3.3}$$

The ground state is defined as lowest eigenstate of the Hamiltonian

$$H = P_0$$

This state is characterized by

$$c(\varkappa) \mid 0 \rangle = 0 \quad (\varkappa \neq 0) . \tag{3.5}$$

Therefore the representation of the operators $c(\varkappa)$ coincides with the ordinary Fock representation.

The conclusion (3.5) does not apply to the infrared part of the spectrum, $\varkappa = 0$. Consider e.g. the vacuum expectation value of the current. Translation invariance asserts only that $\langle 0 \mid j_\mu(x) \mid 0 \rangle$ is independent of x. A constant, nonvanishing vacuum expectation value is perfectly compatible with translation invariance and such a constant contribution arises precisely from $\varkappa = 0$.

4. Infrared Part of the Current

Let us define the infrared part of the current by

$$q_\mu = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} \frac{d\varkappa}{\varkappa} k_\mu \{c(\varkappa) + c^+(\varkappa)\} . \tag{4.1}$$

Furthermore we introduce the potentials $j(x)$ and $\tilde{j}(x)$ by

$$\begin{aligned} j(x) &= i P \int \frac{d\varkappa}{\varkappa} \{c(\varkappa)\, e^{-i k x} - c^+(\varkappa)\, e^{i k x}\} \\ \tilde{j}(x) &= i P \int \frac{d\varkappa}{\varkappa} \{\tilde{c}(\varkappa)\, e^{-i k x} - \tilde{c}^+(\varkappa)\, e^{i k x}\} \\ \tilde{c}(\varkappa) &= \varepsilon(\varkappa)\, c(\varkappa) ; \; \varepsilon(\varkappa) = \frac{\varkappa}{|\varkappa|} . \end{aligned} \tag{4.2}$$

This leads to the decompositions

$$j_\mu(x) = \partial_\mu j(x) + q_\mu \qquad j_\mu(x) = \varepsilon_{\mu\nu}\, \partial^\nu \tilde{j}(x) + q_\mu . \tag{4.3}$$

By virtue of the equal time commutation rules (2.4) the operators $j(x)$ and $\tilde{j}(x)$ satisfy the following commutation rules for arbitrary times x^0 and y^0

$$\begin{aligned} [j(x), \psi(y)] &= - \{a\, D(x-y) + \bar{a}\, \gamma\, \tilde{D}(x-y)\}\, \psi(y) \\ [\tilde{j}(x), \psi(y)] &= - \{a\, \tilde{D}(x-y) + \bar{a}\, \gamma\, D(x-y)\}\, \psi(y) , \end{aligned} \tag{4.4}$$

where the functions D and $\tilde{D}$ are defined by

$$D(x) = \frac{1}{4} \{\varepsilon(x^1 + x^0) - \varepsilon(x^1 - x^0)\} \qquad \tilde{D}(x) = \frac{1}{4} \{\varepsilon(x^1 + x^0) - \varepsilon(x^0 - x^1)\} . \tag{4.5}$$

Furthermore (2.6) and (2.7) imply

$$\begin{aligned} [j(x), j(y)] = [\tilde{j}(x), \tilde{j}(y)] &= - i \lambda^{-1} (\bar{a} - a)\, D(x-y) \\ [\tilde{j}(x), j(y)] &= - i \lambda^{-1} (\bar{a} - a)\, \tilde{D}(x-y) . \end{aligned} \tag{4.6}$$

Finally we note that q_μ behaves like a c-number.

$$[q_\mu, \psi(x)] = [q_\mu, j(x)] = [q_\mu, \tilde{j}(x)] = [q_\mu, q_\nu] = 0 . \tag{4.7}$$

The representation of the potentials is characterized by

$$j^{(+)}(x) \mid 0 \rangle = \tilde{j}^{(+)}(x) \mid 0 \rangle = 0 . \tag{4.8}$$

5. Free Field

The solutions of the model will be constructed in three steps. We first show that a field $\varphi(x)$ can be constructed out of the interacting field $\psi(x)$ such that $\varphi(x)$ satisfies the field equations and commutation rules of a free field. Then we shall determine the representation of this free field in Hilbert space and the third step is to find the representation of the interacting field in Hilbert space, given the representation of the free field.

To be on safe grounds let us introduce an ultraviolet cut-off $g_R(\varkappa)$ in the Fourier representation of the current

$$j^R_\mu(x) = \int d\varkappa \frac{k_\mu}{\varkappa} g_R(\varkappa) \{c(\varkappa) e^{-ikx} + c^+(\varkappa) e^{ikx}\} . \tag{5.1}$$

We do not regularize the field ψ. If the cut-off function $g_R(x)$ is of the form

$$g_R(\varkappa) = \begin{cases} 1 & |\varkappa| < k_R \\ 0 & |\varkappa| > k_R \end{cases} , \tag{5.2}$$

the commutation rules (4.4) to (4.6) are valid for the regularized quantities, if the functions D and $\tilde{D}$ are replaced by D_R and $\tilde{D}_R$ with

$$\varepsilon_R(\xi) = \frac{2}{\pi} \int_0^{k_R} \sin \varkappa\, \xi \frac{d\varkappa}{\varkappa} . \tag{5.3}$$

Now let us define the field $\varphi(x)$ by

$$\varphi(x) = \exp(-i\, T(x))\, \psi(x) , \tag{5.4}$$

$$T(x) = t_1\, j^R(x) + t_2 \gamma\, \tilde{j}^R(x) + Q_\mu x^\mu , \tag{5.5}$$

$$Q_\mu = l_\mu + \tilde{l}_\mu \gamma . \tag{5.6}$$

We want to show that the parameters t_1, t_2, l_μ and $\tilde{l}_\mu$ can be chosen in such a way that $\varphi(x)$ is a free field.

Field Equations

To obtain the equation of motion for $\varphi(x)$ we need the derivative of $\exp(-iT)$ which may be evaluated by means of the well-known formula

$$\exp P \exp Q = \exp(P + Q) \exp \frac{1}{2} [P, Q] \tag{5.7}$$

valid if $[P, Q]$ commutes with P and Q. One finds

$$\partial_\mu \exp(-i\, T) = \exp(-i\, T) \{-i\, \partial_\mu T - \frac{1}{2} [\partial_\mu T, T]\} . \tag{5.8}$$

From (2.2) we conclude that $\varphi(x)$ satisfies a free field equation

$$-i\, \gamma^\mu \partial_\mu \varphi = 0 \tag{5.9}$$

provided

$$t_1 - t_2 = \lambda;\; l_\mu - \varepsilon_{\mu\nu} \tilde{l}^\nu = \lambda\, q_\mu . \tag{5.10}$$

Commutation Rules

Next we verify that the parameters included in the definition of $\varphi(x)$ can be chosen in such a fashion that $\varphi(x)$ satisfies the usual canonical commutation rules. Commuting the exponential $\exp(-iT)$ through $\psi(y)$ one obtains a phase factor

$$\exp\left(-i\,T(x)\right)\psi(y)=\psi(y)\exp\left(-i\,T(x)\right)\exp\left(-i\,\tau\,(x-y)\right)$$
$$\tau\,(x-y)=-\{(t_1\,a+t_2\,\bar{a}\,\gamma_x\,\gamma_y)\,D_R\,(x-y)+(t_1\,\bar{a}\,\gamma_y+t_2\,a\,\gamma_x)\,\tilde{D}_R\,(x-y)\}\,, \tag{5.11}$$

where the index x in γ_x indicates that the matrix γ acts on $\psi(x)$. Making again use of (5.7) we find for $x^0=y^0$

$$\varphi(x)\,\varphi(y)=-\exp C\,(x-y)\,\varphi(y)\,\varphi(x)$$
$$C\,(x-y)=i\,\tau\,(y-x)-i\,\tau\,(x-y)-[T(x),\,T(y)]\,. \tag{5.12}$$

If $x^0=y^0$ the function $D_R(x-y)$ vanishes and we have

$$C\,(x-y)=i\,\{(t_1\,\bar{a}+t_2\,a)+t_1\,t_2\,(\bar{a}-a)\,\lambda^{-1}\}\,(\gamma_x+\gamma_y)\,\tilde{D}^R\,(x-y) \tag{5.13}$$

and therefore the anticommutator of the field φ vanishes provided

$$\lambda\,(t_1\,\bar{a}+t_2\,a)+t_1\,t_2\,(\bar{a}-a)=0\,. \tag{5.14}$$

Similarly one finds

$$\varphi(x)\,\varphi^+(y)=\delta\,(\mathbf{x}-\mathbf{y})-\exp\{-C\,(x-y)\}\,\varphi^+(y)\,\varphi(x)\,, \tag{5.15}$$

where $C(x-y)$ is again given by (5.12). Therefore the condition (5.14) guarantees that $\varphi(x)$ satisfies the proper commutation rules.

Before we proceed to determine the representation of the field $\varphi(x)$ in Hilbert space we note that the definition of the free field is not unique in the sense that

$$\varphi'(x)=\exp i\,(l'\,x+\tilde{l}'\,x\,\gamma)\,\varphi(x) \tag{5.16}$$

is also a free field provided

$$l_\mu'-\varepsilon_{\mu\nu}\,l^{\nu\prime}=0\,. \tag{5.17}$$

Let us get rid of this ambiguity in the definition of $\varphi(x)$ by choosing

$$l_1=\tilde{l}_1=0\,. \tag{5.18}$$

6. Representation of the Free Field

To find the representation of the field $\varphi(x)$ in Hilbert space we make again use of the unitary translation operators introduced in Section 3. As the chosen regularization is translation invariant we have

$$[P_\mu,\,j^R(x)]=-i\,\partial_\mu\,j^R(x) \tag{6.1}$$

and similarly for $\tilde{j}^R\,(x)$ and $\psi(x)$. Therefore

$$[P_\mu,\,\varphi(x)]=-i\,\partial_\mu\,\varphi(x)+(l_\mu+\tilde{l}_\mu\,\gamma)\,\varphi(x)\,, \tag{6.2}$$

the additional term on the right hand side arising from the explicit coordinate dependence introduced in the definition of $\varphi(x)$.

Define the two number operators N and $\tilde{N}$ by

$$[N, \varphi(x)] = -\varphi(x)\,;\ [\tilde{N}, \varphi(x)] = -\gamma\,\varphi(x)\,. \tag{6.3}$$

By means of these two quantities the operator P_μ can be expressed in terms of P^f_μ, the generator of translations for the free field $\varphi(x)$

$$P_\mu = P^f_\mu - l_\mu N - \tilde{l}_\mu \tilde{N}\,. \tag{6.4}$$

In terms of the Fourier representation of $\varphi(x)$

$$\varphi(x) = (2\pi)^{-1/2} \int d\varkappa\, u(\varkappa)\, \{a(\varkappa)\, e^{-ikx} + b^+(\varkappa)\, e^{ikx}\} \tag{6.5}$$

$$u(\varkappa) = \begin{pmatrix} \theta(\varkappa) \\ \theta(-\varkappa) \end{pmatrix},$$

the state of lowest energy $H = P_0$ is characterized by

$$\begin{aligned}
a(\varkappa)\ |0\rangle &= 0 \qquad |\varkappa| - l_0 - \tilde{l}_0\, \varepsilon(\varkappa) > 0 \\
a^+(\varkappa)\ |0\rangle &= 0 \qquad |\varkappa| - l_0 - \tilde{l}_0\, \varepsilon(\varkappa) < 0 \\
b(\varkappa)\ |0\rangle &= 0 \qquad |\varkappa| + l_0 + \tilde{l}_0\, \varepsilon(\varkappa) > 0 \\
b^+(\varkappa)\ |0\rangle &= 0 \qquad |\varkappa| + l_0 + \tilde{l}_0\, \varepsilon(\varkappa) < 0\,.
\end{aligned} \tag{6.6}$$

If l_0 or $\tilde{l}_0$ are different from zero the state of lowest energy does not coincide with the usual vacuum of the free field and we are dealing with a representation of the free field commutation rules which is not equivalent to the usual Fock space representation.

7. Free Current

To go back from the known representation of φ to a representation of ψ we first have to express the current j_μ in terms of φ. As a preliminary step we introduce the current associated with the free field. To be on safe grounds we introduce, for the purpose of the definition of the free current only, an ultraviolet cut-off $f_r(\varkappa)$ into the Fourier representation of the free field

$$\varphi_r(x) \doteq (2\pi)^{-1/2} \int d\varkappa\, f_r(\varkappa)\, u(\varkappa)\, \{a(\varkappa)\, e^{-ikx} + b^+(\varkappa)\, e^{ikx}\}\,. \tag{7.1}$$

The free current is defined by

$$J^r_\mu(x) = :\bar{\varphi}_r(x)\, \gamma_\mu\, \varphi_r(x): \tag{7.2}$$

where the dots indicate that the product is to be ordered with respect to the annihilation and creation operators a, b, a^+, b^+. The operator J^r_μ satisfies

$$\partial^\mu J^r_\mu = 0 \qquad \varepsilon^{\mu\nu}\, \partial_\mu J^r_\nu = 0 \tag{7.3}$$

by virtue of the field equations for $\varphi_r(x)$. Furthermore the commutation rules for φ_r imply $(x^0 = y^0)$

$$[J^r_\mu(x), \varphi_r(y)] = -\gamma_0\, \gamma_\mu\, \delta_r\, (\boldsymbol{x} - \boldsymbol{y})\, \varphi_r(x)\,. \tag{7.4}$$

A more subtle quantity is the commutator of the current with itself. In a straightforward fashion one obtains ($x^0 = y^0$)

$$[J^r_\mu(x), J^r_\nu(y)] = \delta_r(\boldsymbol{x}-\boldsymbol{y}) \{\bar{\varphi}_r(x)\, \gamma_\mu \gamma_0 \gamma_\nu\, \varphi_r(y) - \bar{\varphi}_r(y)\, \gamma_\mu \gamma_0 \gamma_\nu\, \varphi_r(x)\} . \tag{7.5}$$

If no regularization had been introduced we would conclude that the δ-function instructs us to put $\boldsymbol{x} = \boldsymbol{y}$ in the curly bracket and that therefore the above commutator vanishes. To see that this argument is not correct let us examine the right hand side of (7.5) carefully. Consider a representation of the field $\varphi(x)$ of the type (6.6). Denote the space of state vectors which contain only momenta small compared to the ultraviolet cut-off by H_r and consider the quantity ($x^0 = y^0$)

$$\bar{\varphi}_r(x)\, \gamma_\mu \gamma_0 \gamma_\nu\, \varphi_r(y) = :\bar{\varphi}_r(x)\, \gamma_\mu \gamma_0 \gamma_\nu\, \varphi_r(y): + \{\bar{\varphi}^{(+)}_r(x), \gamma_\mu \gamma_0 \gamma_\nu\, \varphi^{(-)}_r(y)\} . \tag{7.6}$$

If the states $| \psi_1 \rangle$ and $| \psi_2 \rangle$ belong to H_r and if the numbers l_0 and $\tilde{l}_0$ are small compared to the ultraviolet cut-off k_r the matrix element

$$M_{\mu\nu}(x, y) = \langle \psi_1 | :\bar{\varphi}_r(x)\, \gamma_\mu \gamma_0 \gamma_\nu\, \varphi_r(y): | \psi_2 \rangle \tag{7.7}$$

is practically constant if x varies over a range of the order of k_r^{-1}. Therefore

$$\delta_r(\boldsymbol{x}-\boldsymbol{y})\, M_{\mu\nu}(x, y) = \delta_r(\boldsymbol{x}-\boldsymbol{y})\, M_{\mu\nu}(y, y) \tag{7.8}$$

and

$$\langle \psi_1 | : [J^r_\mu(x), J^r_\nu(y)] : | \psi_2 \rangle = 0 . \tag{7.9}$$

If we restrict our attention to state vectors in H_r we may write ($x^0 = y^0$)

$$[J^r_\mu(x), J^r_\nu(y)] = \mathrm{tr}\,(\gamma_1 \gamma_\mu \gamma_0 \gamma_\nu)\, \delta_r(\boldsymbol{x}-\boldsymbol{y}) \{\delta^-_r(\boldsymbol{x}-\boldsymbol{y}) - \delta^-_r(\boldsymbol{y}-\boldsymbol{x})\} , \tag{7.10}$$

where $\delta^-_r(z)$ is defined by

$$\delta^-_r(z) = \frac{1}{2\pi} \int_0^\infty d\varkappa\, | f_r(\varkappa) |^2\, e^{i\varkappa z} . \tag{7.11}$$

In the limit as the ultra-violet cut-off tends to infinity the curly bracket is singular and therefore the reasoning which was used to infer that the commutator vanishes is clearly incorrect.

In order to study the right hand side of (7.10) we consider an integral of the type

$$I = \int_{-\infty}^{+\infty} dz\, \delta_r(z) \{\delta^-_r(z) - \delta^-_r(-z)\}\, F(z) , \tag{7.12}$$

where $F(z)$ is a test funtion which contains only Fourier components small compared to k_r. This implies that $F(z)$ does not oscillate rapidly over the region where $\delta_r(z)$ does not vanish. Therefore

$$I = \int_{-\infty}^{+\infty} dz\, \delta_r(z)\, (\delta^-_r(z) - \delta^-_r(-z))\, \{F(0) + z\, F'(0) + \dots\} . \tag{7.13}$$

As the first term in the curly bracket gives no contribution by virtue of the antisymmetry of the integrand we have

$$I = I_1 F'(0) + O(k_r^{-1})$$

$$I_1 = \int_{-\infty}^{+\infty} dz\, z\, \delta_r(z) \left(\delta_r^-(z) - \delta_r^-(-z)\right) . \tag{7.14}$$

The remaining integral has the value

$$I_1 = \frac{i}{2\pi} \tag{7.15}$$

independent of the choice of the cut-off function $f_r(\varkappa)$. If we now go to the limit $k_r \to \infty$ we have ($x^0 = y^0$)

$$[J_\mu(x), J_\nu(y)] = \mathrm{tr}\,(\gamma_1 \gamma_\mu \gamma_0 \gamma_\nu)\,(2\pi i)^{-1}\,\delta'(\boldsymbol{x} - \boldsymbol{y}) . \tag{7.16}$$

This bracket relation completes the algebra of the operators $\varphi(x)$ and $J_\mu(x)$. The purpose of the ultra-violet cut-off was to obtain this commutation relation in an unambiguous fashion. Once the algebra is complete we may remove the cut-off. In the limit $k_r \to \infty$ both the restrictions on the state vectors and on the test functions are empty. Making use of the equations of motion for the field $J_\mu(x)$ we find

$$[J_\mu(x), J_\nu(y)] = \frac{i}{\pi}\,\partial_{\mu\nu} D\,(x - y)$$

$$[J_\mu(x), \varphi(y)] = -\,\{g_{\mu\nu} + \varepsilon_{\mu\nu}\gamma\}\,\partial^\nu D\,(x - y)\,\varphi(y) . \tag{7.17}$$

The important conclusion we draw from our derivation of (7.17) is that these commutation rules depend on the representation of the field $\varphi(x)$. If e.g. we start with the Fock space representation with interchanged roles of creation and destruction operators, i.e. a representation of the type

$$a^+(\varkappa)\,|\,0\rangle = 0; \quad b^+(\varkappa)\,|\,0\rangle = 0 , \tag{7.18}$$

we end up with commutation rules of the form

$$[J_\mu(x), J_\nu(y)] = -\frac{i}{\pi}\,\partial_{\mu\nu} D\,(x - y) \tag{7.19}$$

instead of (7.17). Commutation rules of this type were obtained by Scarf and Wess[6]) by means of an averaging procedure in space and time.

8. Relation Between Interacting and Free Currents

In order to express j_μ in terms of quantities derived from φ we compute the commutator $[j_\mu, \varphi]$. This quantity may be obtained from the definition of φ, eq. (5.4) and the commutation rules (4.4) and (4.6).

$$[j_\mu(x), \varphi(y)] = -\,[\{a + \lambda^{-1} t_1 (\bar{a} - a)\}\, g_{\mu\nu} +$$
$$+ \{\bar{a} + \lambda^{-1} t_2 (\bar{a} - a)\}\,\varepsilon_{\mu\nu}\gamma]\,\partial^\nu D\,(x - y)\,\varphi(y) . \tag{8.1}$$

Due to (5.10) the curly brackets on the right hand side coincide. The normalisation of the current was left open up to now. A change in this normalisation amounts simply to a redefinition of the coupling constant. It is convenient to fix the norm of j_μ by

$$a + \lambda^{-1} t_1 (\bar{a} - a) = \bar{a} + \lambda^{-1} t_2 (\bar{a} - a) = 1 , \tag{8.2}$$

$$[j_\mu(x), \varphi(y)] = - \{g_{\mu\nu} + \varepsilon_{\mu\nu} \gamma\} \partial^\nu D (x - y) \varphi(y) . \tag{8.3}$$

Comparison with (7.17) shows that $j_\mu(x) - J_\mu(x)$ commutes with $\varphi(y)$. If the representation of φ is irreducible we conclude that

$$j_\mu = J_\mu + c_\mu , \tag{8.4}$$

where $c_\mu(x)$ is a real c-number function. Actually c_μ must be independent of x by virtue of (4.8).

Finally, comparison of the commutation rules (4.6 and (7.17) leads to

$$\pi (\bar{a} - a) = \lambda . \tag{8.5}$$

The relations (5.10), (5.14), (8.2) and (8.5) are equivalent to

$$a = 1 - \frac{t_1}{\pi} ; \quad \bar{a} = 1 - \frac{t_2}{\pi} , \tag{8.6}$$

$$t_1 - t_2 = \lambda ; \qquad t_1 + t_2 - \frac{t_1 t_2}{\pi} = 0 . \tag{8.7}$$

If we introduce instead of λ the parameter Λ by

$$\lambda = 2 \pi \sinh\Lambda , \tag{8.8}$$

we find the following expressions for a, $\bar{a}$, t_1 and t_2

$$a = e^{-\Lambda} ; \quad \bar{a} = e^{\Lambda}$$

$$t_1 = \pi (1 - e^{-\Lambda}) ; \quad t_2 = \pi (1 - e^{\Lambda}) . \tag{8.9}$$

9. Vacuum Expectation Value of the Current

What remains to be done is to determine the parameters l_μ, $\bar{l}_\mu$ and c_μ related to the vacuum expectation value of the current. In Section 2 we defined the current j_μ through the commutation rules between j_μ and ψ. This definition is incomplete, as an arbitrary c-number can be added to the current without changing the commutation rules. We now supplement this definition by relating the vacuum expectation value of $j_\mu(x)$ to the two-point-function of the interacting field. Let us introduce the Fourier components of the interacting field by

$$\psi(x) = (2 \pi)^{-1/2} \int d\varkappa \, u(\varkappa) \{A (\varkappa, x^0) e^{-i k x} + B^+ (\varkappa, x^0) e^{i k x}\} . \tag{9.1}$$

In analogy with (7.2) we define

$$\langle 0 | j_\mu(x) | 0 \rangle = \langle 0 |; \; \bar{\psi}(x) \gamma_\mu \psi(x); \; | 0 \rangle , \tag{9.2}$$

where the semicolon indicates ordering with respect to the annihilation and creation operators A, B, A^+, B^+. Equivalenty

$$\langle 0 \mid j_\mu(x) \mid 0 \rangle = \langle 0 \mid \bar{\psi}(x)\, \gamma_\mu\, \psi_+(x) + \bar{\psi}_-(x)\, \gamma_\mu\, \psi_-(x) - \mathrm{tr}\, \gamma_\mu\, \psi_-(x)\, \bar{\psi}_+(x) \mid 0 \rangle$$

$$\psi_\pm(x) = \int\limits_{x^0 = y^0} \{P_+\, \delta_\pm\, (\boldsymbol{x} - \boldsymbol{y}) + P_-\, \delta_\mp\, (\boldsymbol{x} - \boldsymbol{y})\}\, \psi(y)\, d\boldsymbol{y}$$

$$P_\pm = \frac{1}{2}\,(1 \pm \gamma)\,, \tag{9.3}$$

or finally

$$\langle 0 \mid j_\mu(x) \mid 0 \rangle = \int \mathrm{tr}\, [\gamma_\mu\, (P_+\, \delta_-(\boldsymbol{z}) + P_-\, \delta_+(\boldsymbol{z}))\, H(z)]\, d\boldsymbol{z}$$

$$H_{\alpha\beta}\,(x - y) = \langle 0 \mid \bar{\psi}_\beta(y)\, \psi_\alpha(x) - \psi_\alpha(y)\, \bar{\psi}_\beta(x) \mid 0 \rangle\,. \tag{9.4}$$

The results obtained so far may be summarized in the following way. Given the coupling constant λ, the parameters t_1 and t_2 are determined by (8.9). If we in addition suppose l_μ and $\tilde{l}_\nu$ to be known, the definition of the free field (5.4) can be inverted and the interacting field expressed in terms of the free field and its current potentials. (Note that by virtue of (8.4) the free and interacting current potentials coincide.) On the other hand the representation of the free field in Hilbert space was determined in Section 6. Therefore the representation of the interacting field is known, provided l_μ and $\tilde{l}_\mu$ are given. This allows us to compute the function $H(x - y)$ and to determine $\langle 0 \mid j_\mu \mid 0 \rangle$ through (9.4). The calculation is sketched in the Appendix and leads to the result

$$\langle 0 \mid j_0(x) \mid 0 \rangle = \frac{l_0}{\pi} = \langle 0 \mid J_0(x) \mid 0 \rangle \qquad \langle 0 \mid j_1(x) \mid 0 \rangle = \frac{\tilde{l}_0}{\pi} = \langle 0 \mid J_1(x) \mid 0 \rangle\,. \tag{9.5}$$

We conclude that c_μ vanishes, i.e. free and interacting currents coincide.

$$j_\mu(x) = J_\mu(x)\,. \tag{9.6}$$

Comparing (9.5) with (5.10) we get the selfconsistency conditions

$$\left(1 - \frac{\lambda}{\pi}\right) l_0 = 0; \quad \left(1 + \frac{\lambda}{\pi}\right) \tilde{l}_0 = 0\,. \tag{9.7}$$

If $|\lambda| \neq \pi$ we have $l_0 = \tilde{l}_0 = 0$ characteristic of the normal solution. A breakdown of symmetry can occur only if $\lambda = \pm\,\pi$

$$\lambda = \pi; \quad \langle 0 \mid j_0 \mid 0 \rangle \neq 0; \quad \langle 0 \mid j_1 \mid 0 \rangle = 0$$

$$\lambda = -\,\pi; \quad \langle 0 \mid j_0 \mid 0 \rangle = 0; \quad \langle 0 \mid j_1 \mid 0 \rangle \neq 0\,. \tag{9.8}$$

The value $\lambda = +\,\pi$ admits of symmetry breaking solutions with nonvanishing charge density of the vacuum, while the symmetry breaking solutions for $\lambda = -\,\pi$ are characterized by a nonvanishing current density.

10. Spontaneous Breakdown of Symmetry

The symmetry breaking solutions, characterized by a nonvanishing vacuum expectation value of j_μ, lead to inequivalent representations of the free field, defined in (6.6). In other words the symmetry breaking solutions and the normal solution are defined on different Hilbert spaces. This makes it difficult to see how the symmetry breakdown arises. To understand why symmetry can be broken we introduce a comparison system which is enclosed in a box. In this comparison system the normal and the symmetry breaking solutions will be represented by equivalent representations, defined on the same Hilbert space. We put

$$\varkappa_0 = \frac{2\pi}{L}\,, \qquad a_n = \sqrt{\frac{L}{2\pi}} \int\limits_{\varkappa_0 (n-1/2)}^{\varkappa_0 (n+1/2)} d\varkappa\, a(\varkappa)\,; \quad b_n = \sqrt{\frac{L}{2\pi}} \int\limits_{\varkappa_0 (n-1/2)}^{\varkappa_0 (n+1/2)} d\varkappa\, b(\varkappa)$$

$$\varphi_L(x) = L^{-1/2} \sum_n u_n \{a_n\, e^{-i k_n x} + b_n^+\, e^{i k_n x}\}\,, \tag{10.1}$$

where L is the length of the Box. If the free current is defined in terms of the free field $\varphi_L(x)$ in the same way as in Section 7 we arrive at the same commutation rules (7.17) with D replaced by D_L

$$D_L(x) = \frac{1}{4} \{\varepsilon_L\,(x^1 + x^0) - \varepsilon_L\,(x^1 - x^0)\}\;;\quad \varepsilon_L(z) = \frac{4}{L} \sum_{n=1}^{\infty} \frac{\sin \varkappa_0\, n\, z}{\varkappa_0\, n} + \frac{2\, z}{L}\,. \tag{10.2}$$

By construction J_μ^L is of the form

$$J_\mu^L(x) = \sum_{n \neq 0} \frac{k_\mu^n}{\varkappa_0\, n} \{C_n\, e^{-i k_n x} + C_n^+\, e^{i k_n x}\} + K_\mu\,, \tag{10.3}$$

where the operators C_n are bilinear expressions in a_m, b_m, a_m^+, b_m^+. The infrared part K_μ reads explicitly

$$K_\mu = L^{-1} \sum_n \bar{u}_n\, \gamma_\mu\, u_n\, (a_n^+\, a_n - b_n^+\, b_n)\,. \tag{10.4}$$

Note that in the comparison system the infrared part of the current does not commute with φ_L

$$[K_\mu, \varphi_L(x)] = -\, L^{-1}\, \gamma_0\, \gamma_\mu\, \varphi_L(x)\,. \tag{10.5}$$

The potential $J^L(x)$ is defined as

$$J^L(x) = i \sum_{n \neq 0} \frac{1}{\varkappa_0\, n} \{C_n\, e^{-i k_n x} - C_n^+\, e^{i k_n x}\} \tag{10.6}$$

and similarly for $J_L(x)$.

We now introduce ψ_L and j_μ^L by

$$\psi_L(x) = \exp\,(i\, T_L(x)\, \varphi_L(x)\,, \tag{10.7}$$

$$T_L(x) = t_1\, J^L(x) + t_2\, \tilde{J}^L(x)\, \gamma + \lambda\, (K_0 - \gamma\, K_1)\, x^0\,, \tag{10.8}$$

$$j_\mu^L(x) = J_\mu^L(x)\,, \tag{10.9}$$

where t_1 and t_2 are given by (8.9). The fields ψ_L and j_μ^L satisfy the required commutation rules and field equations.

To establish the correspondence between the comparison system and the actual system we consider the normal solution first. Denote by $|\,0)$ the vacuum of φ_L. By the definition of K_μ we have

$$(\,0\,|\,K_\mu\,|\,0) = 0\,. \tag{10.10}$$

If we compute with this state vacuum expectation values of products of ψ_L, $\bar{\psi}_L$ and j^L_μ and then go to the limit $L \to \infty$ the results are the same as those obtained from the normal solution of the actual system.

We are now in a position to analyze the question: is the normal solution stable? This problem concerns the spectrum of the Hamiltonian

$$H_L = H^f_L - \frac{1}{2}\lambda\, L\, (K_0^2 - K_1^2)\,, \tag{10.11}$$

where H^f_L denotes the Hamiltonian of the field φ_L. In order for a solution to be stable the spectrum of H_L must be bounded below. We now show that this is the case only if $|\,\lambda\,| \leqslant \pi$.

For this purpose consider a special state of the form

$$|n_1, n_2, n_3, n_4) = \prod_1^{n_1} a_n^+ \prod_1^{n_2} a_{-m}^+ \prod_1^{n_3} b_p^+ \prod_1^{n_4} b_{-q}^+\, |0)\,. \tag{10.12}$$

The eigenvalue of H_L in this state is given by

$$E\,(n_1, n_2, n_3, n_4) = L^{-1}\,\{\pi \sum_{i=1}^{4} n_i\,(n_i + 1) - 2\,\lambda\,(n_1 - n_3)\,(n_2 - n_4)\}\,. \tag{10.13}$$

Suppose $\lambda > 0$ and write

$$E\,(n_1, n_2, n_3, n_4) = L^{-1}\,\{(\pi - \lambda) \sum n_i^2 + \pi \sum n_i + \lambda\,(n_1 - n_2)^2 + \\ + \lambda\,(n_3 - n_4)^2 + 2\,\lambda\,(n_1\,n_4 + n_2\,n_3)\}\,. \tag{10.14}$$

If $\pi > \lambda > 0$ all terms on the right hand side are positive. However, if $\lambda > \pi$ we have e.g.

$$E\,(n, n, o, o) = 2\,L^{-1}\,n\,\{(\pi - \lambda)\,n + \pi\}\,, \tag{10.15}$$

which is arbitrarily large and negative for large n. Similarly for $\lambda < 0$. We conclude that the normal solution is stable only if $|\,\lambda\,| < \pi$. If $|\,\lambda\,| > \pi$ four ground states outside the Hilbert space under consideration here are possible

$$\prod_{n=-\infty}^{+\infty} a_n^+\,|\,0)\,; \quad \prod_{n=-\infty}^{+\infty} b_n^+\,|\,0) \quad (\lambda > \pi)\,,$$

$$\prod_{n=0}^{\infty} a_n^+\,b_{-n}^+\,|\,0)\,; \quad \prod_{n=0}^{\infty} a_{-n}^+\,b_n^+\,|\,0) \quad (\lambda < -\pi)\,. \tag{10.16}$$

These states define representations of the field operators which are not equivalent to those considered, not even in a box of finite length. It must be expected that these representations lead to different current-current commutation rules as explained in Section 7. We shall not investigate this problem here. The above states correspond to infinite charge or current densities. As $|\lambda|$ passes through π the vacuum flips from a state of zero charge and current density to a state of infinite charge or current density.

What is the situation for $\lambda = \pm \pi$? Suppose e.g. $\lambda = \pi$ and consider the state $|n, n, o, o)$. If in the limit $L \to \infty$ we simultaneously increase n in such a way that

$$2 \pi L^{-1} n = \alpha , \tag{10.17}$$

we obtain precisely the representation (6.6) of the operators a, a^+, b, b^+ with the values $l_0 = \alpha$; $\tilde{l}_0 = 0$. Negative values of l_0 are obtained from $|0, 0, n, n)$ and the representations $l_0 = 0$, $\tilde{l}_0 \neq 0$ arise from $|n, 0, 0, n)$ and $|0, n, n, 0)$.

The energy required to add two additional particles of momenta $\pm 2 \pi L^{-1} (n+1)$ is $2 \pi L^{-1}$. This energy vanishes in the limit $L \to \infty$. In other words the ground state is not stable against excitation of particle pairs with opposite momenta. Finally we note that the total charge in the state $|n, n, 0, 0)$ is given by $2n$ while the total current vanishes. In the limit $L \to \infty$ this gives rise to a finite charge density $L^{-1} 2n = l_0/\pi$. This is precisely the relation (9.5). Similarly for the other types of inequivalent representations.

In summary we have the following result. The normal solution is stable for $|\lambda| < \pi$. As λ reaches $+\pi$ the vacuum becomes unstable against excitation of particle pairs or antiparticle pairs. A Fermi sea of particles or antiparticles can arise with vanishing total momentum and the finite total energy $2 \pi L^{-1} n = |l_0|$. The charge density of this sea is given by l_0/π, the current density vanishes. Similarly, for $\lambda = -\pi$ a Fermi sea of particle-antiparticle pairs with opposite momenta can arise, characterized by vanishing total momentum, total energy $|\tilde{l}_0|$, vanishing charge density and current density $\tilde{l}_0/\pi$. As $\lambda > \pi$ either all particle states or all antiparticle states are occupied and we get out of the Hilbert space. Likewise for $\lambda < -\pi$.

11. Symmetry Groups and Inequivalent Representations

The Thirring model admits of five basic symmetry groups:

1. The gauge group associated with charge conservation

$$\psi'(x) = e^{i\alpha} \psi(x) . \tag{11.1}$$

2. The group of Touschek[8]) transformations characteristic of zero mass spinor fields

$$\psi'(x) = e^{i\beta\gamma} \psi(x) . \tag{11.2}$$

3. The translation group

$$\psi'(x) = \psi(x + a) . \tag{11.3}$$

4. The homogeneous Lorentz group

$$\psi'(x) = S_L \psi(L x) . \tag{11.4}$$

5. The group of scale transformations characteristic of field equations which do not contain constants of the dimension of a length

$$\psi'(x) = \varrho^{1/2} \psi(\varrho x) . \tag{11.5}$$

The problem of whether or not these symmetry groups are broken is equivalent to the question: are there unitary operators U

$$\psi'(x) = U\,\psi(x)\,U^{+} \tag{11.6}$$

leaving the vacuum invariant? It is easy to see that all our solutions do admit of such operators for the first three groups in the above list, the generators being given by N, $\tilde{N}$ and P_μ respectively. On the other hand the group of homogeneous Lorentz transformations and the group of scale transformations are broken.

Group of scale transformations

Let us first consider the group of scale transformations. For the annihilation and creation operators of the free field $\varphi(x)$ this group induces the transformation

$$a'(\varkappa) = \varrho^{-1/2}\,a(\varrho^{-1}\,\varkappa) \tag{11.7}$$

and similarly for $b(\varkappa)$ and $c(\varkappa)$. This transformation leaves the commutation rules for the operators a and b in fact invariant. However, if $a(\varkappa)$ belongs to the representation characterized by l_0, $\tilde{l}_0$, defined in (6.6) the operator $a'(\varkappa)$ belongs to the representation characterized by

$$l_0' = \varrho\,l_0; \quad \tilde{l}_0' = \varrho\,\tilde{l}_0\,. \tag{11.8}$$

In other words, the transformed solution is defined on an inequivalent representation of the free field commutation rules and no unitary operator connecting the two solutions can possibly exist.

An exception occurs only for the normal solution characterized by $l_0 = \tilde{l}_0 = 0$. In this case a family of unitary operators generating scale transformations does indeed exist. The operators U coincide with the unitary representation of the scale group associated with the free field.

Homogeneous Lorentz group

In the present treatment of the symmetry breaking solutions of the Thirring model we have adapted the coordinate system in such a way that the total momentum of the Fermi sea representing the ground state vanishes. In this preferred system of coordinates the vacuum expectation value of j_μ points along the time-axis for $\lambda = \pi$ and along the space-axis for $\lambda = -\pi$. We did not consider other coordinate frames up to now. It is easy to see what happens if one looks at the system from a frame moving with respect to the Fermi sea. Under Lorentz transformations

$$\varphi'(x) = S_L\,\varphi(L\,x) \tag{11.9}$$

with

$$L = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}; \quad S_L = \exp\frac{\chi}{2}\,\gamma \tag{11.10}$$

the Fourier components of $\varphi(x)$ transform as

$$a'(\varkappa) = e^{-\chi/2}\,a(e^{-\chi}\,\varkappa) \quad \varkappa > 0 \tag{11.11}$$
$$a'(\varkappa) = e^{\chi/2}\,a(e^{\chi}\,\varkappa) \quad \varkappa < 0$$

and likewise for $b(\varkappa)$. If e.g. $\lambda = \pi$, $\tilde{l}_0 = 0$, $l_0 > 0$ this transformation brings us to a representation of the free field commutation rules of the type

$$\begin{aligned} a'(\varkappa) \mid 0 \rangle &= 0 \quad \varkappa > e^{\chi} l_0;\ \varkappa < - e^{-\chi} l_0 \\ a'^{+}(\varkappa) \mid 0 \rangle &= 0 \quad - e^{-\chi} l_0 < \varkappa < e^{\chi} l_0 \\ b'(\varkappa) \mid 0 \rangle &= 0 \quad all\ \varkappa . \end{aligned} \tag{11.12}$$

However, the state $\mid 0 \rangle$ is not a state of lowest energy with respect to the Hamiltonian associated with the new coordinate system. This is intuitively clear as the kinetic energy of the Fermi sea can be made free by rearranging the particles in such a way that their momenta cancel pairwise. Formally

$$[H', a'(\varkappa)] = -(\mid \varkappa \mid - l_0 \cosh \chi)\, a'(\varkappa) \tag{11.13}$$

and therefore if e.g. $\chi > 0$ the state

$$a'(e^{\chi} l_0 - \varepsilon) \mid 0 \rangle \tag{11.14}$$

belongs to a smaller energy eigenvalue than the state $\mid 0 \rangle$. The requirement that there exists a state of lowest energy singles out the particular Lorentz frame wherein the Fermi sea is at rest.

12. Conclusion

Our results may be summarized in the following way. The Thirring model admits of three essentially different solutions:

1. The normal solution which coincides with the perturbation theory result. The vacuum expectation value of the current vanishes. This solution displays the full symmetry of the model and is stable for $\mid \lambda \mid < \pi$.

2. A symmetry breaking solution, characterized by a nonvanishing timelike vacuum expectation value of the current occours if $\lambda = \pi$. Both the group of scale transformations and the homogeneous Lorentz group are broken. The structure of this solution depends on the scale chosen to describe the system; different scales lead to inequivalent representations of the field operators. The requirement that there exists a state of lowest energy singles out a particular Lorentz frame wherein the vacuum expectation value of the current points along the time-axis.

3. A symmetry breaking solution characterized by a spacelike vacuum expectation value of the current occurs if $\lambda = -\pi$. In this case the vacuum singles out a Lorentz frame in such a way that the vacuum expectation value of the current points along the space-axis.

If $\mid \lambda \mid > \pi$ the vacuum expectation value of the current becomes infinite. This case is not treated in the present paper.

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Appendix

To compute the two-point-function we invert the definition of $\varphi(x)$

$$\langle 0 \mid \psi(x)\, \bar{\psi}(y) \mid 0 \rangle = \langle 0 \mid e^{i\,T(x)}\, \varphi(x)\, \varphi^{+}(y)\, e^{-i\,T(y)}\, \gamma^{0} \mid 0 \rangle .$$

We decompose the current potentials contained in $T(x)$ in positive and negative frequency parts. Making repeated use of (5.7) one finds

$$\langle 0 \mid \psi(x)\, \bar{\psi}(y) \mid 0 \rangle = \exp i \{F\,(x-y) + Q\,(x-y)\} \langle 0 \mid \varphi(x)\, \bar{\varphi}(y) \mid 0 \rangle$$

$$F\,(x-y) = -\frac{\lambda^2}{\pi} \{D_R^{+}\,(x-y) - D_R^{+}(0)\}$$

and similarly

$$\langle 0 \mid \bar{\psi}(x)\, \psi(y) \mid 0 \rangle = \exp i \{F\,(x-y) - Q\,(x-y)\} \langle 0 \mid \bar{\varphi}(x)\, \varphi(y) \mid 0 \rangle .$$

For equal times the function $H(x-y)$ defined in (9.4) becomes

$$H\,(x-y) = \exp i\, F\,(y-x) \langle 0 \mid \bar{\varphi}(y)\, \varphi(x) - \varphi(y)\, \bar{\varphi}(x) \mid 0 \rangle .$$

Making use of (6.6) one finds ($x^0 = y^0$)

$$\langle 0 \mid \bar{\varphi}(y)\, \varphi(x) - \varphi(y)\, \bar{\varphi}(x) \mid 0 \rangle =$$

$$\frac{1}{\pi} \{P_{+} \int_0^{l_0+\tilde{l}_0} d\varkappa \cos\varkappa\,(\boldsymbol{x}-\boldsymbol{y}) + P_{-} \int_0^{l_0-\tilde{l}_0} d\varkappa \cos\varkappa\,(\boldsymbol{x}-\boldsymbol{y})\} .$$

As this is a well-behaved function, the expression (9.4) for the vacuum expectation value of the current is well-defined. Because $H(z)$ is even, only the even parts of $\delta_{\pm}\,(z)$ contribute

$$\delta_{\pm}(z) = \frac{1}{2}\,\delta(z) \pm (2\,\pi\, i)^{-1}\, P\frac{1}{z} .$$

Therefore we have

$$\langle 0 \mid j_{\mu}(x) \mid 0 \rangle = \frac{1}{2} \operatorname{tr} \{\gamma_{\mu}\, H(0)\} ,$$

explicitly

$$\langle 0 \mid j_0(x) \mid 0 \rangle = \frac{l_0}{\pi} \qquad \langle 0 \mid j_1(x) \mid 0 \rangle = \frac{\tilde{l}_0}{\pi} .$$

Footnotes

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[4]) Notation: Metric $g_{\mu\nu}: g_{00} = -g_{11} = 1$.
Antisymmetric tensor $\varepsilon^{\mu\nu}: \varepsilon^{01} = -\varepsilon^{10} = \varepsilon_{10} = -\varepsilon_{01} = 1$.
γ-matrices: $\gamma_0 = \sigma^2$, $\gamma_1 = i\,\sigma^1$, $\gamma = \gamma_0\,\gamma_1 = \sigma_3$.
The component x^1 of the vector x^{μ} is denoted by $\boldsymbol{x}$.

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