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Sum Rules for $\pi-\pi$-Scattering[1])

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Summary. Scattering amplitudes $T(s, t, u)$ having simple symmetry properties in s, t and u can be expressed as functions of suitably choosen homogeneous variables. The assumption of the validity of the Mandelstam representation determines the analyticity properties of the amplitudes as functions of these homogeneous variables. These properties lead to a new type of dispersion relations involving integrals along curves of the (s, t, u)-space. In the case of the $\pi-\pi$-system, these dispersion relations imply a set of sum rules relating S-, P- and D-wave scattering lengthes, S-wave effective ranges and integrals over physical quantities. The present experimental information does not allow a detailed test of these sum rules.

1. Introduction

Our purpose is to establish some sum rules for $\pi-\pi$-scattering which are physically meaningful consequences of analyticity, crossing symmetry and unitarity.

The three conditions of analyticity, crossing symmetry and unitarity are of quite a different nature and this makes it hard to exploit them exhaustively and simultanously. Fixed transfer dispersion relations relate two channels; their connection to the third one is completely ignored. The s-channel partial wave dispersion relations involve contributions from the t- and u-channel through the left hand cut. However, these contributions cannot be evaluated in a closed form.

In this paper, we develop an approach which takes into account the full crossing symmetry. The scattering amplitudes $T(s, t, u)$ we consider have simple symmetry properties in the variables s, t and u. Therefore, these amplitudes may be written as functions of two suitably choosen homogeneous and symmetric combinations of s, t and u. These new variables are called x and y: $T(s, t, u) = F(x, y)$. The difficulty mentioned at the beginning appears here too; the analyticity properties of $F(x, y)$ are intricate and it is not easy to express unitarity in terms of this function. Nevertheless, it is possible to extract from the analyticity properties of $F(x, y)$ some new type of dispersion relations in one variable. These relations involve integrals which extend over curves of the (s, t, u)-space.

Our sum rules are deduced from these dispersion relations. They relate scattering lengthes and effective ranges to integrals over total cross-sections and derivatives of forward absorptive parts with respect to momentum transfer.

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In Part 2 we expose our method in the case of a totally symmetric scattering amplitude. The amplitude of the process $\pi^0 + \pi^0 \to \pi^0 + \pi^0$ is an example of such a function. The results of this Part have been published previously [1][2]. In Part 3, our method is extended to the case of a partially symmetric amplitude, such as the amplitude of the process $\pi^+ + \pi^+ \to \pi^+ + \pi^+$. Some results concerning antisymmetric amplitudes, like the isospin $T = 1$ state amplitude, are described briefly in Part 4. Part 5 is devoted to a systematic application of our sum rules to the $T = 0$, 1 and 2 amplitudes of the $\pi - \pi$ system.

2. Totally Symmetric Scattering Amplitude

In this Part, we consider the scattering amplitude $T(s, t, u)$ of identical, spin zero mesons. The variables s, t and u are the usual Mandelstam variables:

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2 = 4 - s - t\,,$$

where p_1 and p_2 $(-p_3, -p_4)$ are the energy-momentum vectors of the initial (final) mesons. The meson mass has been set equal to one.

We assume that crossing symmetry implies the symmetry of $T(s, t, u)$ under any permutation of s, t and u:

$$T(s, t, u) = T(s, u, t)\,, \quad T(s, t, u) = T(u, s, t)\,. \tag{2.1}$$

2.1. *Homogeneous Variables*

It follows from (2.1) that $T(s, t, u)$ can be written as a function of two independent, homogeneous and symmetric combinations of s, t and u without introducing new, kinematical, singularities. A convenient choice is:

$$x = -\frac{1}{4}\,(s\,t + t\,u + u\,s)$$

$$y = \frac{1}{16}\,s\,t\,u\,. \tag{2.2}$$

We shall write:

$$T(s, t, u) = F(x, y)\,.$$

We shall start with a study of the change of variables (2.2). The real (s, t, u)-plane is mapped onto a domain R of the real (x, y)-plane. In order to find this domain, we take the straight line d, $t =$ const., as a search line and we look for its image d' in the (x, y)-plane. With $v = u\,s \leqslant (4 - t)^2/4$ we have:

$$d'; \quad \begin{array}{l} 4\,x = t\,(t - 4) - v \\ 16\,y = t\,v \end{array} \qquad v \leqslant \frac{1}{4}\,(4 - t)^2\,. \tag{2.3}$$

d' is a half straight line. As t varies, the extremity A' of d' describes the boundary C of R (Fig. 1):

$$C; \qquad y = y_\pm(x) = \frac{1}{27}\,[4 - 3\,(3\,x + 4) \pm (3\,x + 4)^{3/2}]\,, \quad x > -4/3\,. \tag{2.4}$$

[2]) Numbers in brackets refer to References, page 246.

This curve is composed of two branches, C_+ and C_-, corresponding to the $+$ or $-$ sign in (2.4). A' is on C_+ if $t < 4/3$, it lies on C_- if $t > 4/3$. A line d' whose extremety is on C_+ (C_-) is tangent to C_- (C_+) at a point B'.

The points of R are in one-to-one correspondence with the points of one of six sectors of the real (s, t, u)-plane; the sector $u \leqslant t \leqslant s$ for example. In this correspondence, the image of d' is the broken line ABC (B is the image of B').

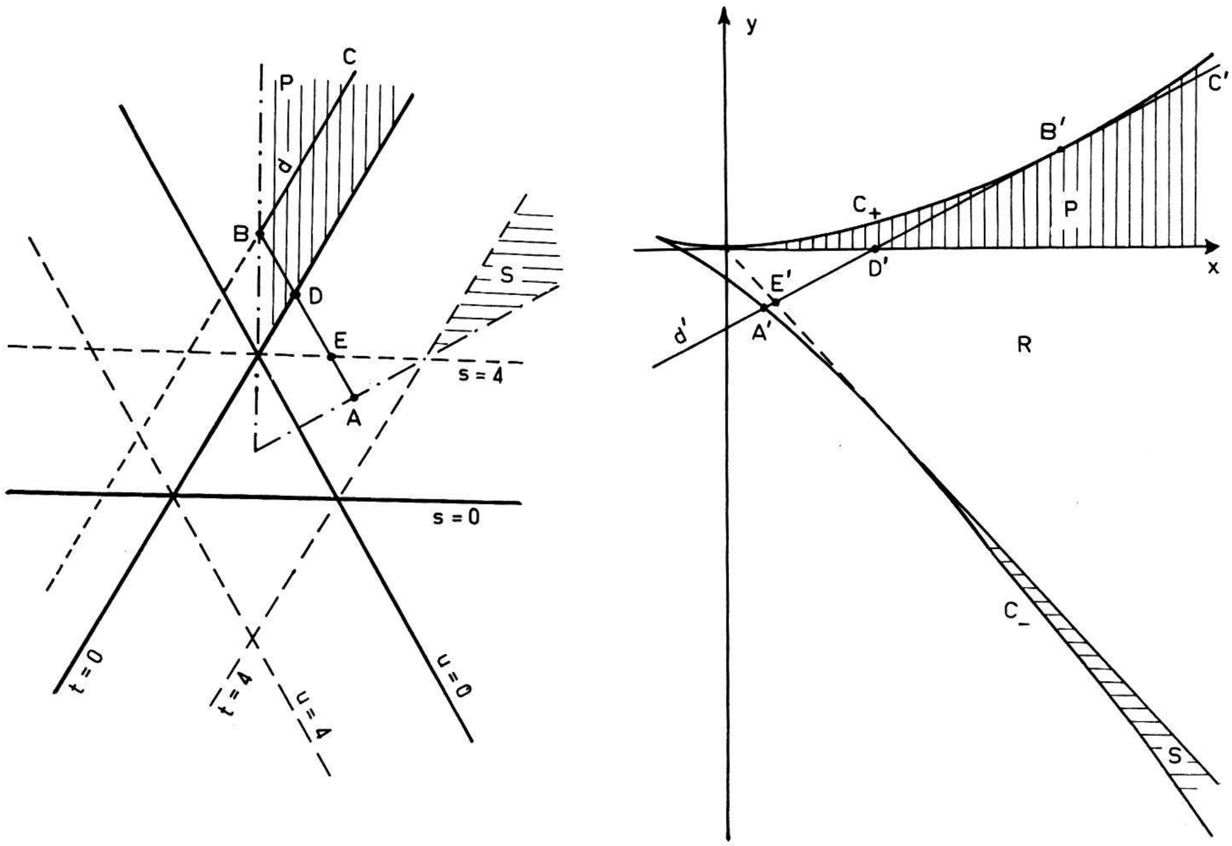

Figure 1

Totally symmetric amplitude

The real (s, t, u)-plane and its image R in the real (x, y)-plane. $[x = -(1/4)\,(s\,t + t\,u + u\,s),\ y = (1/16)\,s\,t\,u]$. P is the physical region and S is the double-spectral domain. The straight line d' is the image of d, or of the broken line $A\ B\ C$.

The preceding results imply that a point (x, y) corresponds to physical values of s, t and u if:

$$x \geqslant 0\,, \quad 0 \leqslant y \leqslant y_+(x)\,. \tag{2.5}$$

This defines the physical domain P in the homogeneous variables x and y. The axis $y = 0$ corresponds to forward scattering ($t = 0$), the part of C_+ which limits P corresponds to scattering under 90° in the c.m. system ($u = t$).

The double spectral functions of the Mandelstam representation are non vanishing in parts of sectors like $s > 4$, $t > 4$. The image S of these sectors is:

$$S; \qquad x > 4\,, \quad y_-(x) < y < -x\,. \tag{2.6}$$

The image of the extended Symanzik domain ($s < 4$, $t < 4$, $u < 4$) is:

$$-\frac{4}{3} < x < 4\,, \quad y_-(x) < y < \text{Min}\,(y_+(x), -x) \tag{2.7}$$

Real (x, y) points which are outside of R correspond to complex values of s, t and u.

2.2. *Dispersion Relations in the Homogeneous Variables*

We assume that a Mandelstam representation without pole terms holds for $T(s, t, u)$:

$$T(s, t, u) = \int_4^\infty d\alpha\, d\beta\; \varrho(\alpha, \beta) \left[\frac{s^N t^N}{(\alpha - s)(\beta - t)} + \frac{t^N u^N}{(\alpha - t)(\beta - u)} + \frac{u^N s^N}{(\alpha - u)(\beta - s)}\right]$$
$$+ \sum_{p=0}^{N-1} \int_4^\infty d\alpha\; \eta_p(\alpha) \left[\frac{s^N}{\alpha - s}(t^p + u^p) + \frac{t^N}{\alpha - t}(u^p + s^p) + \frac{u^N}{\alpha - u}(s^p + t^p)\right]$$
$$+ \sum_{p,q=0}^{N-1} c_{p\,q}\,(s^p\, t^q + t^p\, u^q + u^p\, s^q) \tag{2.8}$$

$\varrho(\alpha, \beta) = \varrho(\beta, \alpha)$, $c_{p\,q} = c_{q\,p}$. This representation may be written in terms of the homogeneous variables x and y:

$$F(x, y) = \int_4^\infty d\alpha\, d\beta\, \varrho(\alpha, \beta)\; P(\alpha, \beta, x, y)\, [16\, y + 4\, x\, \alpha - \alpha^2\, (\alpha - 4)]^{-1}$$
$$\times [16\, y + 4\, x\, \beta - \beta^2\, (\beta - 4)]^{-1} + \sum_{p=0}^{N-1} \int_4^\infty d\alpha\; \eta_p(\alpha)\; Q_p(\alpha, x, y)$$
$$\times [16\, y + 4\, x\, \alpha - \alpha^2\, (\alpha - 4)]^{-1} + R(x, y)\,. \tag{2.9}$$

P, Q_p and R are polynomials in their arguments. Inspection of (2.9) shows that the analytic properties of $F(x, y)$ are not simple; this function has complex singularities which are not located on a topological product of planes. Nevertheless, we get simple properties if we consider the values taken by $F(x, y)$ on the complex plane $y = a\, x + b$ (a and b real). It follows from (2.9) that $F(x, a\, x + b)$, as a function of x, has only real singularities. These are due to the vanishing of the denominators appearing in (2.9) at real (x, y) points. For a given, real, x, this happens for:

$$y \geqslant y_1(x) = \text{Min}_\alpha\, \frac{1}{16}\, [\alpha^2\, (\alpha - 4) - 4\, x\, \alpha]$$

$$y_1(x) = \begin{cases} -x & \text{for } x < 4 \\ y_-(x) & \text{for } x > 4\,. \end{cases} \tag{2.10}$$

(This result becomes intuitively clear if we remember that the line $y = -x$ is the image of the line $s = 4$ and that S is the image of the double spectral region.) (2.10) defines a domain Σ of the real (x, y)-plane (Fig. 2). $F\,(x, a\, x + b)$ is holomorphic in the complex x-plane provided with cuts along the real axis. Let $D(a, b)$ be the real section of the plane $y = a\,x + b$. The cuts of $F(x, a\,x + b)$ correspond to those parts of $D(a, b)$ which belong to Σ. We shall confine ourselves to planes with $a \geqslant -1$. For such a plane, there is only a right-hand cut $x_1(a, b) \leqslant x < \infty$. $x_1(a, b)$ is the abscissa of the point where $D(a, b)$ intersects the boundary of Σ.

The part of $D(a, b)$ lying in R corresponds to a curve $D'(a, b)$ in the sector $-u \leqslant t \leqslant s$ of the real (s, t, u)-plane:

$$D'(a, b); \qquad t = t(x, a, b), \qquad s = s(x, a, b)\,. \tag{2.11}$$

For $x \to +\infty$,

$$t(x, a, b) \simeq -4\,a\,, \qquad s(x, a, b) \simeq (4\,x)^{1/2}\,. \tag{2.12}$$

This shows that $D'(a, b)$ has the asymptote $t = -4\,a$. We know that unitarity and analyticity imply the following, fixed t, asymptotic bound [2]:

$$|\,\mathrm{Im}\, T(s, t, 4 - s - t)\,| < C\, s^{1+\varepsilon}, \quad \varepsilon < 1, \quad \text{for } s \to \infty, \quad t < 4\,. \tag{2.13}$$

Therefore, we have from (2.12) and (2.13):

$$|\,\mathrm{Im}\, F\,(x, a\,x + b)\,| < C'\, x^{(1+\varepsilon)/2} \text{ for } x \to \infty,\ a > -1\,. \tag{2.14}$$

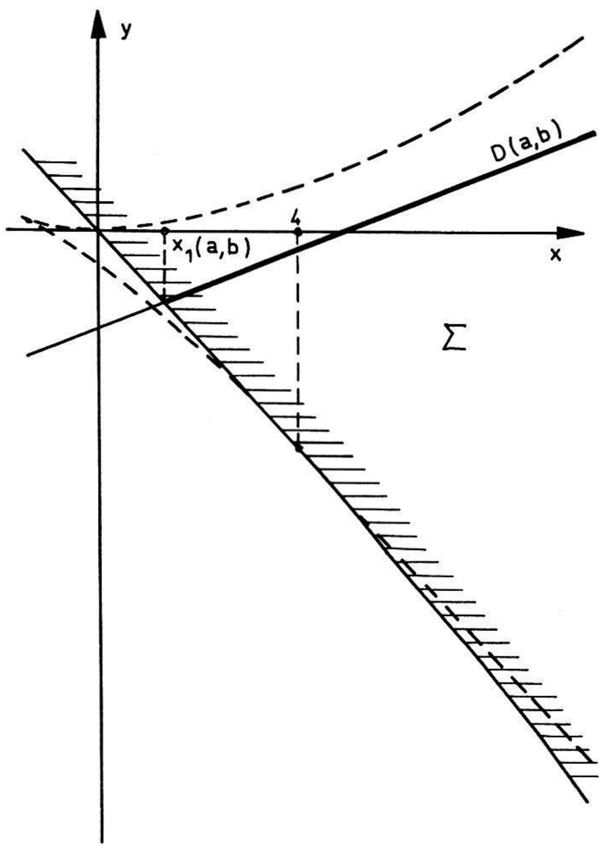

Figure 2

Totally symmetric amplitude

$D(a, b)$ is the real section of the plane $y = a\,x + b$. The singularities of $F(x, a\,x + b)$ are located on the part of D which lies in Σ.

This, and the fact that $F^* (x^*, a x^* + b) = F (x, a x + b)$, allow us to write a once substracted dispersion relation for each straight line $D(a, b)$ whose slope is greater than -1:

$$F (x, a x + b) = F (x_0, a x_0 + b) + \frac{1}{\pi} (x - x_0) \int_{x_1(a,b)}^{\infty} dx' \frac{\operatorname{Im} F (x' + i \varepsilon, a (x' + i \varepsilon) + b)}{(x' - x_0) (x' - x)} , \tag{2.15}$$

where $a > -1$, $x_0 < x_1(a, b)$.

We now ask what is the relation between $\operatorname{Im} F (x + i \varepsilon, a (x + i \varepsilon) + b)$ and the absorptive part of $T(s, t, u)$. To this end we consider a line $D(a, b)$ which does not cut the double spectral region S: $a > -1$, $b > -4 (1 + a)$. The s-coordinate of a point on $D'(a, b)$ and the x-coordinate of the corresponding point on $D(a, b)$ are related by:

$$x = \frac{s^2 (s - 4) - 16 b}{4 (s + 4 a)} , \tag{2.16}$$

and $x > x_1(a, b) = - b/(1 + a)$ corresponds to $s > 4$. If s has a small imaginary part, x gets an imaginary part too:

$$\operatorname{Im} x = G(s, a, b) \operatorname{Im} s , \qquad G(s, a, b) > 0 \text{ for } s > 4 . \tag{2.17}$$

Therefore, in Σ, but outside S:

$$\begin{aligned} \operatorname{Im} F \big(x + i \varepsilon, a (x + i \varepsilon) + b\big) &= \operatorname{Im} T\big(s(x, a, b) + i \varepsilon, t(x, a, b), u(x, a, b)\big) \\ &= A\big(s(x,a, b), t(x, a, b)\big) \end{aligned} \tag{2.18}$$

where $A(s, t)$ is the absorptive part of $T(s, t, u)$ in the s-channel.

The dispersion integral in (2.15) corresponds, in general, to an integral along a part of a curve $D'(a, b)$ in the (s, t, u)-space. The usual, fixed t, dispersion relations are particular cases of (2.15) obtained by taking for $D(a, b)$ a tangent to C. The generalized dispersion relations (2.15) are a specific consequence of the symmetry of $T(s, t, u)$.

As a by-product, (2.15) shows that $F(x, y)$ is real at all real (x, y)-points on the left of Σ. In other words, the extended Symanzik region is only the real portion of the domain $\mathfrak{R}$ where $T(s, t, u)$ is real. $\mathfrak{R}$ contains in addition at least all complex (s, t, u)-points which have real images in the (x, y) space, on the left of Σ. The domain $\mathfrak{R}$ extends to infinity and the asymptotic growth of $T(s, t, u)$ in $\mathfrak{R}$ is strongly restricted by (2.15).

2.3. *Sum Rules*

In this Section, we want to derive physically meaningful sum rules from (2.15). To this end, we choose a straight line $D(a, 0)$, passing through the origin (zero kinetic energy point):

$$y = a x , \qquad a > -1 ,$$

and perform the subtraction at this point: $x_0 = 0$. (2.15) gives:

$$D(x, a x) = D(0, 0) + \frac{x}{\pi} P \int_0^{\infty} dx' \frac{\operatorname{Im} F (x' + i \varepsilon, a (x' + i \varepsilon))}{x' (x' - x)} \tag{2.19}$$

where $D(x, y) = \mathrm{Re}\, F(x, y)$. If we assume a normal threshold behavior for the absorptive part $A(s, t)$ we may write:

$$A(s, t) = (s - 4)^{1/2}\, B(s, t)\ ,$$

the function $B(s, t)$ being regular in a neighbourhood of $s = 4$, $t = 0$. Thus we have:

$$\mathrm{Im}\, F(x, y) = (x + y)^{1/2}\, G(x, y)\ , \tag{2.20}$$

where $G(x, y)$ is regular at $x = y = 0$. If we insert (2.20) into (2.19) and use the fact that:

$$P \int_0^\infty dx' \frac{1}{(x')^{1/2}} \frac{1}{x' - x} = 0 \text{ for } x > 0\ ,$$

we get:

$$D(x, a\, x) - D(0, 0) = (1 + a)^{1/2} \frac{x}{\pi}\, P \int_0^\infty dx' \frac{G(x', a\, x') - G(0, 0)}{(x')^{1/2}\, (x' - x)} \text{ for } x > 0\,. \tag{2.21}$$

We divide both sides of (2.21) by x and take the limit $x \to 0+$. The integral on the left hand side of (2.21) is such that its limit is equal to the integral over the limit of the integrand. Therefore we have:

$$X + a\, Y = (1 + a)^{1/2} \frac{1}{\pi} \int_0^\infty dx \frac{1}{x^{3/2}} [G(x, a\, x) - G(0, 0)] \tag{2.22}$$

with:

$$X = \lim_{\substack{x+y \to 0+ \\ x \to 0}} \frac{\partial}{\partial x} D(x, y),\ \ Y = \lim_{\substack{x+y \to 0+ \\ x \to 0}} \frac{\partial}{\partial y} D(x, y)\,. \tag{2.23}$$

The integral on the right of (2.22) extends exclusively over physical values of x and y for $a = 0$. Therefore, we get a sum rule involving only physical quantities if we set $a = 0$ in (2.22). This sum rule is not really new, because it can as well be deduced from the ordinary fixed t ($t = 0$) dispersion relation. However, (2.22) is valid for a varying continuously in an interval around $a = 0$. We take advantage of this fact in deriving (2.22) with respect to the slope a[3]). We obtain a second sum rule by setting $a = 0$ in this new relation. Thus we have two equations involving X and Y and integrals over quantities related to forward scattering:

$$X = \frac{1}{\pi} \int_0^\infty dx \frac{1}{x^{3/2}} [G(x, 0) - G(0, 0)] \tag{2.24}$$

$$Y = \frac{1}{2} X + \frac{1}{\pi} \int_0^\infty dx \frac{1}{x^{1/2}} \frac{\partial}{\partial y} G(x, 0)\,. \tag{2.25}$$

[3]) Here, and in the following, we assume that the derivation with respect to a and the integration over x can be interchanged. A discussion of the legitimacy of such interchanges can be found in [6].

These are our sum rules. Equation (2.24) could be obtained from the forward dispersion relation; Equation (2.25) is new, it is a specific consequence of the complete symmetry of $T(s, t, u)$.

In order to clarify the physical meaning of (2.24) and (2.25), we have to identify the constants X and Y and to go back to the variables s, t and u.

The partial-wave expansion for $T(s, t, u)$ reads:

$$T(s, t, u) = \frac{4}{\pi} \left[\frac{\nu+1}{\nu}\right]^{1/2} \sum_{\substack{l=0 \\ l \text{ even}}}^{\infty} (2\,l + 1)\; T_l(\nu)\; P_l\left(1 + \frac{t}{2\,\nu}\right), \tag{2.26}$$

with $\nu = (s - 4)/4$. We know that:

$$\text{Re}\; T_l(\nu) = \frac{\nu^{(2l+1)/2}}{(\nu+1)^{1/2}}\; a_l(\nu)\;, \tag{2.27}$$

where $a_l(\nu)$ is regular at $\nu = 0$. Some algebra shows that, according to the definitions (2.23) and (2.2):

$$X = \frac{1}{\pi}\; b_0\;, \quad Y = \frac{1}{\pi}\; (b_0 + 30\; a_2)\;, \tag{2.28}$$

with the notation:

$$a_l = a_l(0)\;, \quad b_l = \frac{d}{d\nu}\; a_l(\nu) \bigg|_{\nu=0}. \tag{2.29}$$

After transforming the integrals on the right-hand sides of (2.24) and (2.25) into integrals over ν, we get finally:

$$b_0 = \frac{1}{4\,\pi^2} \int\limits_0^{\infty} d\nu\; \frac{2\,\nu+1}{\nu^{3/2}\,(\nu+1)^{3/2}}\; [\sigma(\nu) - \sigma(0)] \tag{2.30}$$

$$30\; a_2 = \int\limits_0^{\infty} d\nu \left[\frac{1}{2\,\pi^2}\; \frac{\nu^{1/2}}{(\nu+1)^{5/2}}\; \sigma(\nu) + \frac{2\,\nu+1}{\nu^2\,(\nu+1)^2}\; \frac{\partial A(\nu, 0)}{\partial t}\right] \tag{2.31}$$

$\sigma(\nu) = [\pi^2\,(\nu\,(\nu + 1))^{-1/2}]\; A(\nu, 0)$ is the total cross-section. MARTIN [3] derived an other sum rule for the D-wave scattering length. Our rule (2.31) involves only physical quantities whereas MARTINS rule uses the absorptive part $A(\nu, t)$ at the unphysical values $\nu > 0$, $t = 4$. As $\partial A(\nu, 0)/\partial t > 0$, we derive from (2.31) the non-trivial inequality:

$$a_2 > 0\,. \tag{2.31}$$

This inequality is a joint consequence of the analyticity and the symmetry of $T(s, t, u)$ and of unitarity. It is a restriction these conditions impose on the low energy interaction.

3. Partially Symmetric Scattering Amplitude

In this Part we show how the use of homogeneous variables can be extended to the case of a scattering amplitude which is not totally symmetric. We consider again the scattering of identical spin zero mesons described by the s-channel of an amplitude $T(s, t, u)$. This amplitude is symmetric under the exchange $t \leftrightarrow u$. We assume now that the (identical) t- and u-channels are distinct from the s-channel. Therefore, crossing symmetry implies no new symmetry for $T(s, t, u)$ and we have only:

$$T(s, t, u) = T(s, u, t) \tag{3.1}$$

3.1. *Homogeneous Variables*

As a consequence of (3.1), $T(s, t, u)$ can be written as a function $F(x, y)$ of the variables:

$$x = \frac{1}{4} (t + u) = \frac{1}{4} (4 - s) \qquad y = - \frac{1}{16} t\, u \tag{3.2}$$

without introducing kinematical singularities.

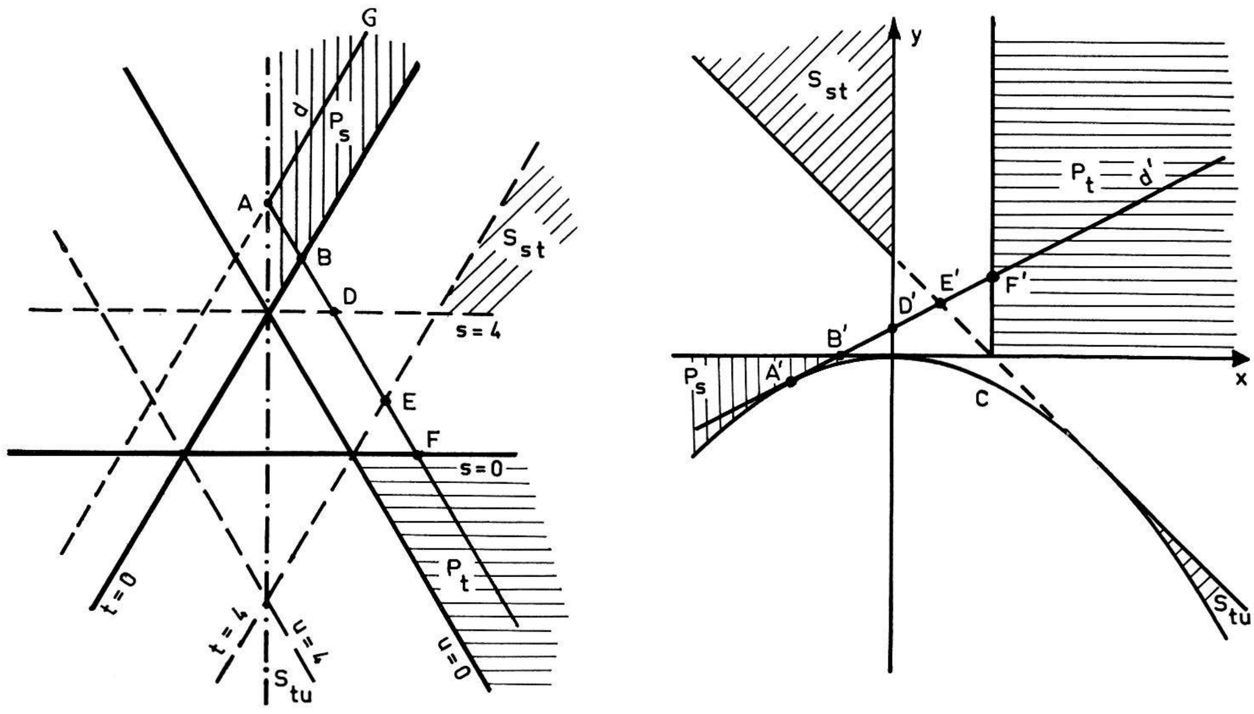

Figure 3

Partially symmetric amplitude

The real (s, t, u)-plane is mapped onto the domain above the curve C, in the real (x, y)-plane. $[x = (1/4)\ (t+u),\ y = -(1/16)\ t\, u]$. P_s and P_t are the physical regions of the s- and t-channels. $S_{s\,t}$ and $S_{t\,u}$ are the double-spectral regions. The line d' is the image of d, or of the broken line $F\,A\,G$.

The change of variables (3.2) maps the real (s, t, u)-plane onto a domain R of the real (x, y)-plane (Fig. 3):

$$R; \qquad y > - \frac{1}{4} x^2 . \tag{3.3}$$

The image of a straight line $t =$ real const. (or $u =$ real const.) is a tangent to the boundary C of R. (3.2) defines a one-to-one correspondence between the half-plane $t > u$ and R.

The image P_s of the physical region in the s-channel is defined by:

$$P_s; \qquad x < 0, \quad 0 > y > -\frac{1}{4} x^2, \tag{3.4}$$

whereas the physical domains of the identical t- and u-channels have the common image P_t:

$$P_t; \qquad x > 1, \quad y > 0. \tag{3.5}$$

The double spectral region ($u > 4$, $t > 4$) is mapped onto:

$$S_{tu}; \qquad x > 2, \quad 1 - x > y > -\frac{1}{4} x^2. \tag{3.6}$$

and the image of the double spectral regions ($s > 4$, $t > 4$) and ($s > 4$, $u > 4$) is:

$$S_{st}; \qquad x < 0, \quad y > 1 - x. \tag{3.7}$$

3.2. *Dispersion Relations in the Homogeneous Variables*

Arguments similar to those used in Section 2.2 show that $F(x, a x + b)$ (a and b real) is holomorphic in the cut x-plane. We shall always take $a > -1$. Then, there are two cuts (Fig. 4): a left-hand cut $-\infty < x \leqslant 0$ and a right-hand cut $x_1(a, b) \leqslant x < \infty$. The lower limit $x_1(a, b)$ is the solution of the equation:

$$a x + b = y_1(x)$$

$$y_1(x) = \begin{cases} 1 - x & \text{for } x < 2 \\ -\frac{1}{4} x^2 & \text{for } x > 2. \end{cases} \tag{3.8}$$

For $x \to -\infty$, $y = a x + b$, relation (3.2) gives:

$$s \simeq -4 x, \quad t \to -4 a$$

and, for $x \to +\infty$, we have:

$$t \simeq 4 x, \quad u \to -4 a.$$

Therefore, the bound (2.13) leads to:

$$|\operatorname{Im} F(x, a x + b)| < C'' |x|^{1+\varepsilon} \text{ for } |x| \to \infty, \quad a > -1. \tag{3.9}$$

Thus, we have a twice subtracted dispersion relation for each straight line $D(a, b)$ ($y = a x + b$) whose slope is greater than -1. For instance, we may write:

$$F(x, a x + b) = \frac{1}{x_1} [F(x_1, a x_1 + b) x - F(0, b)(x - x_1)] + \frac{1}{\pi} x (x - x_1)$$
$$\times \left[\int_{-\infty}^{0} dx' \frac{\operatorname{Im} F(x' + i\varepsilon, a(x' + i\varepsilon) + b)}{x'(x' - x_1)(x' - x)} + \int_{x_1}^{\infty} dx' \frac{\operatorname{Im} F(x' + i\varepsilon, a(x' + i\varepsilon) + b)}{x'(x' - x_1)(x' - x)} \right], \tag{3.10}$$

if $D(a, b)$ does not cross the double spectral region S_{st}.

Let $A_s(s, t)$ and $A_t(t, u)$ be the absorptive parts of $T(s, t, u)$ in the s- and t-channels:

$$A_s(s, t) = \operatorname{Im} T\,(s + i\,\varepsilon, t, 4 - s - t) \text{ for } s > 4, - s < t < 4$$
$$A_t(t, u) = \operatorname{Im} T\,(4 - t - u, t + i\,\varepsilon, u) \text{ for } t > 4, - t < u < 4\,. \tag{3.11}$$

The change of variables (3.2) transforms $A_s(s, t)$ into a function $U(x, y)$ defined in the domain $x < 0$, $y < 1 - x$:

$$U\left(\frac{1}{4}\,(4 - s)\,,\ -\frac{1}{16}\,(4 - s - t)\right) = A_s(s, t)\,, \tag{3.12}$$

and it may be shown that:

$$\operatorname{Im} F\left(x + i\,\varepsilon, a\,(x + i\,\varepsilon) + b\right) = -\,U\,(x, a\,x + b) \tag{3.13}$$

on the left-hand cut $x < 0$, as far as $(1 + a)\,x < 1 - b$.

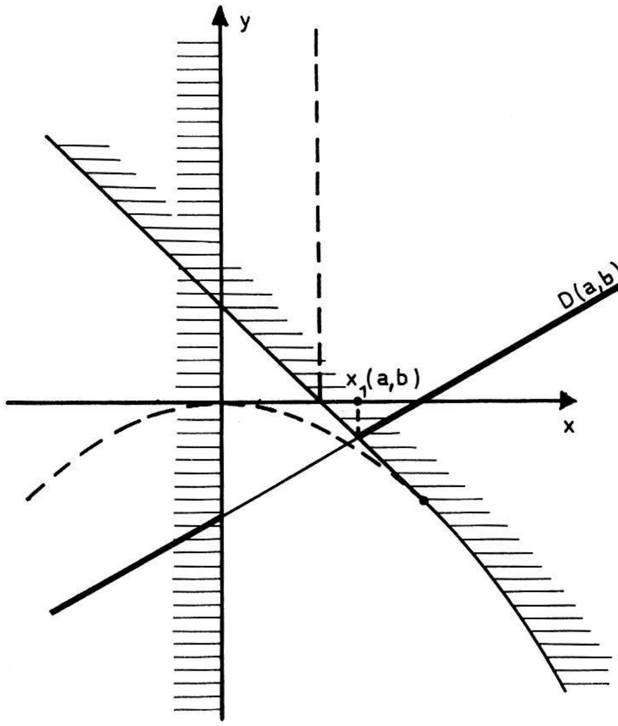

Figure 4

Partially symmetric amplitude

The plane $y = a\,x + b$ cuts the real (x, y)-plane along $D(a, b)$. $F(x, a\,x + b)$ is singular on those parts of $D(a, b)$ which are in the shaded areas.

For the right-hand cut it is convenient to replace the variable x by a new variable z defined by:

$$z = x + y\,. \tag{3.14}$$

The straight lines $z =$ const. are parallel to the line $z = 1$ corresponding to the elastic threshold in the t-channel. Equations (3.2) and (3.14) transform $A_t(t, u)$ into a function $V(z, y)$ defined for $z > 1$, $z > y$:

$$V\left(\frac{1}{4}\,(t + u) - \frac{1}{16}\,t\,u,\ -\frac{1}{16}\,t\,u\right) = A_t(t, u)\,. \tag{3.15}$$

One has:

$$\operatorname{Im} F\left(x + i\,\varepsilon,\, a\,(x + i\,\varepsilon) + b\right) = V\left(z,\, \frac{1}{1+a}\,(a\,z + b)\right) \tag{3.16}$$

if $z > 1$, $(1 + a)\,z > b$, and with $z = x\,(1 + a) + b$.

Therefore, if $D(a, b)$ does not cross the double spectral regions S_{tu} and S_{st}, (3.10) becomes:

$$F\,(x,\, a\,x + b) = \left[F\left(\frac{1-b}{1+a},\, \frac{a+b}{1+a}\right) \frac{1+a}{1-b}\, x - F(0, b)\left(\frac{1+a}{1-b}\, x - 1\right)\right]$$
$$+ \frac{1}{\pi}\, x\left(x\,(1+a) - 1 + b\right)\left[-\int_{-\infty}^{0} dx' \,\frac{U\,(x',\, a\,x' + b)}{x'\,(x'\,(1+a) - 1 + b)\,(x' - x)}\right.$$
$$\left. + (1 + a)\int_{1}^{\infty} dz' \,\frac{V\,(z',\, (1/1+a)\,(a\,z' + b))}{(z' - b)\,(z' - 1)\,(z' - b - x\,(1+a))}\right]. \tag{3.17}$$

3.3. *Sum Rules*

In Section 2.3 we started from the dispersion relations corresponding to the family of straight lines we may draw through the zero kinetic energy point $x = y = 0$. Among this family, there is one line which leads to a dispersion integral over physical quantities only. This is the line $y = 0$. The sum rules we established involve integrals over the physical part of that line.

In the present case, we have two straight lines leading to integrals extended over physical values of x and y. These are the lines $y = 0$ and $x = 1$. Furthermore, as we have now two distinct channels, there are two zero kinetic energy points through which families of straight lines may be drawn: the origin $x = y = 0$ and the point $x = 1$, $y = 0$. Therefore, several ways of constructing sum rules have to be explored.

A. *Sum Rules Generated from the Family $y = a\,(x - 1)$ and Involving Integrals over the Physical Parts of $y = 0$*

The threshold behavior of $A_t(t, u)$ [$A_t(t, u) = (t - 4)^{1/2} \times$ function regular at $t = 4$, $u = 0$] implies:

$$V(z, y) = (z - 1)^{1/2}\, L(z, y) \tag{3.18}$$

with $L(z, y)$ regular at $z = 1$, $y = 0$. Using an argument similar to one used in Section 2.3, we transform (3.17) (with $b = -\,a$) into:

$$D\left(x,\, a\,(x - 1)\right) - D(1, 0) = D(1, 0)\,(x - 1) - D(0, -\,a)\,(x - 1) + \frac{1}{\pi}\, x\,(x - 1)$$
$$\times\left[(1 + a)^2\, P\int_{1}^{\infty} dz' \,\frac{(1/z' + a)\, L\,(z',\, (a/1+a)\,(z' - 1)) - (1/1+a)\, L(1, 0)}{(z' - 1)^{1/2}\,(z' + a - x\,(1+a))}\right.$$
$$\left. - \int_{-\infty}^{0} dx' \,\frac{U(x',\, a\,(x' - 1)}{x'\,(x' - x)\,(x' - 1)}\right] \tag{3.19}$$

if $x > 1$ and where $D(x, y) = \mathrm{Re}\, F(x, y)$. Dividing both sides of (3.19) by $(x-1)$ and taking the limit $x \to 1+$, we get:

$$X_1 + a\, Y_1 = D(1, 0) - D(0, -a) + \frac{1}{\pi} (1+a)^2 \int_1^\infty dz \frac{1}{(z-1)^{3/2}}$$
$$\times \left[\frac{1}{z+a} L\left(z, \frac{a}{1+a} (z-1)\right) - \frac{1}{1+a} L(1, 0)\right]$$
$$- \frac{1}{\pi} \int_{-\infty}^0 dx \frac{1}{x(x-1)^2} U\left(x, a(x-1)\right) \tag{3.20}$$

with the notation:

$$X_1 = \lim_{\substack{x+y \to 1+ \\ y \to 0}} \frac{\partial}{\partial x} D(x, y)\,, \quad Y_1 = \lim_{\substack{x+y \to 1+ \\ y \to 0}} \frac{\partial}{\partial y} D(x, y)\,. \tag{3.21}$$

From (3.20) and from the relation obtained by deriving (3.20) with respect to the slope a, we get two sum rules:

$$X_1 = D(1, 0) - D(0, 0) + \frac{1}{\pi} \int_1^\infty dz \frac{1}{(z-1)^{3/2}} \left[\frac{1}{z} L(z, 0) - L(1, 0)\right]$$
$$- \frac{1}{\pi} \int_{-\infty}^0 dx \frac{1}{x(x-1)^2} U(x, 0)\,, \tag{3.22}$$

$$Y_1 = Y_0 + \frac{1}{\pi} \int_1^\infty dz \left\{\frac{1}{(z-1)^{3/2}} \left[\frac{1}{z} L(z, 0) - L(1, 0)\right] + \frac{1}{z(z-1)}\right.$$
$$\left. \times \left[\frac{1}{z} V(z, 0) + \frac{\partial}{\partial y} V(z, 0)\right]\right\} - \frac{1}{\pi} \int_{-\infty}^0 dx \frac{1}{x(x-1)} \frac{\partial}{\partial y} U(x, 0) \tag{3.23}$$

where:

$$Y_0 = \frac{\partial}{\partial y} D(0, 0)\,. \tag{3.24}$$

B. *Sum Rules Generated from the Family $y = a\,x$ and Involving Integrals over the Physical Parts of $y = 0$*

The threshold behavior of $A_s(s, t)$ leads to the form:

$$U(x, y) = (-x)^{1/2} K(x, y)\,, \tag{3.25}$$

where $K(x, y)$ is regular at $x = y = 0$. Applying our standard procedure to (3.17), in the case $b = 0$, we get:

$$D(x, a\,x) = (1+a) \left[D\left(\frac{1}{1+a}, \frac{a}{1+a}\right) x - D(0, 0) \left(x - \frac{1}{1+a}\right)\right]$$
$$+ \frac{1}{\pi} x \left(x - \frac{1}{1+a}\right) \left\{(1+a)^2 \int_1^\infty dz' \frac{V(z', (a/1+a)\, z')}{z' (z'-1) (z' - x(1+a))} + P \int_{-\infty}^0 dx'\right.$$
$$\left. \times \frac{1}{(-x')^{1/2} (x'-x)} (1+a) \left[\frac{1}{x'(1+a) - 1} K(x', a\,x') + K(0, 0)\right]\right\}, \tag{3.26}$$

if $x < 0$. The limit $x \to 0-$ of (3.26), after division by x, gives:

$$X_0 + a\,Y_0 = (1+a)\left[D\left(\frac{1}{1+a}, \frac{a}{1+a}\right) - D(0,0)\right] - \frac{1}{\pi}(1+a)\int_1^\infty dz\,\frac{1}{z^2(z-1)}$$

$$\times V\left(z, \frac{a}{1+a}\,z\right) + \frac{1}{\pi}\int_{-\infty}^{0} dx\,\frac{1}{(-x)^{3/2}}\left[\frac{1}{(1+a)\,x-1}\,K(x, a\,x) - K(0,0)\right], \qquad (3.27)$$

with:

$$X_0 = \lim_{x\to 0-} \frac{\partial}{\partial x} D(x, 0)\,. \qquad (3.28)$$

Equation (3.27) leads again to two sum rules:

$$X_0 = D(1,0) - D(0,0) - \frac{1}{\pi}\int_1^\infty dz\,\frac{1}{z^2\,(z-1)}\,V(z,0)$$

$$+ \frac{1}{\pi}\int_{-\infty}^{0} dx\,\frac{1}{(-x)^{3/2}}\left[\frac{1}{x-1}\,K(x,0) + K(0,0)\right] \qquad (3.29)$$

$$Y_0 = D(1,0) - D(0,0) - X_1 + Y_1 - \frac{1}{\pi}\int_1^\infty dz\,\frac{1}{(z-1)\,z}\left[\frac{1}{z}\,V(z,0) + \frac{\partial}{\partial y}\,V(z,0)\right]$$

$$- \frac{1}{\pi}\int_{-\infty}^{0} dx\,\frac{1}{x\,(x-1)}\left[\frac{1}{x-1}\,U(x,0) + \frac{\partial}{\partial y}\,U(x,0)\right]. \qquad (3.30)$$

As is readily seen, the sum rule (3.30) is equivalent to the difference of the preceding sum rules (3.22) and (3.23). Furthermore, the rules (3.29) and (3.22) could be derived as well from an ordinary, fixed u ($u = 0$), dispersion relation. Therefore, the use of homogeneous variables provides us with one specifically new sum rule, (3.30) for example.

C. *Sum Rule Involving an Integral over the Physical Part of $x = 1$*

One could be tempted to apply our technique to the family of straight lines $x = 1 + a\,y$. $F\,(1 + a\,y, y)$ is holomorphic in the y-plane provided with the right-hand cut $0 \leqslant y < +\infty$ and the left-hand cut $-\infty < y \leqslant y_1(a)$. The contribution of the dispersion integral along the left-hand cut vanishes for $a = 0$, because $y_1(0) = -\infty$. The right-hand cut is entirely in the physical domain P_t for $a \geqslant 0$. However, if we take the derivative of the dispersion integral along the left-hand cut (which contains unphysical quantities) with respect to the slope a, we do not know if this derivative vanishes in the limit $a \to 0$. Therefore, we have only one sum rule, which could also be derived from the fixed s ($s = 0$) dispersion relation:

$$Y_1 = \frac{1}{\pi}\int_1^\infty dz\,\frac{1}{(z-1)^{3/2}}\,[L\,(z, z-1) - L(1,0)]\,. \qquad (3.31)$$

In a last step, we relate the constants appearing in our sum rules to the partial wave amplitudes in the s- and t-channel. We write the partial wave expansions:

$$T\left(4(\nu+1),\,-2\nu(1-\cos\theta),\,-2\nu(1+\cos\theta)\right)=\frac{4}{\pi}\left[\frac{\nu+1}{\nu}\right]^{1/2}$$
$$\times\sum_{\substack{l=0\\ l\text{ even}}}^{\infty}(2l+1)\,T_l^{(s)}(\nu)\,P_l(\cos\theta) \tag{3.32}$$

in the s-channel, and:

$$T\left(-2\nu(1+\cos\theta),\,4(\nu+1),\,-2\nu(1-\cos\theta)\right)=\frac{4}{\pi}\left[\frac{\nu+1}{\nu}\right]^{1/2}$$
$$\times\sum_{l=0}^{\infty}(2l+1)\,T_l^{(t)}(\nu)\,P_l(\cos\theta) \tag{3.33}$$

in the t-channel. We have:

$$\operatorname{Re}T_l^{(s)}(\nu)=\frac{\nu^{(2l+1)/2}}{(\nu+1)^{1/2}}\,a_l^{(s)}(\nu)\qquad \operatorname{Re}T_l^{(t)}(\nu)=\frac{\nu^{(2l+1)/2}}{(\nu+1)^{1/2}}\,a_l^{(t)}(\nu) \tag{3.34}$$

where $a^{(s)}(\nu)$ and $a^{(t)}(\nu)$ are regular at $\nu=0$. From the definitions (3.21), (3.24) and (3.28) one derives, using (3.2), (3.32), (3.33) and (3.34):

$$\begin{aligned} D(0,0)&=\frac{4}{\pi}\,a_0^{(s)}\,, & D(1,0)&=\frac{4}{\pi}\,a_0^{(t)}\\ X_0&=-\frac{4}{\pi}\,b_0^{(s)}\,, & Y_0&=\frac{120}{\pi}\,a_2^{(s)}\\ X_1&=\frac{4}{\pi}\,b_0^{(t)}+\frac{12}{\pi}\,a_1^{(t)}\,, & Y_1&=\frac{4}{\pi}\,b_0^{(t)}-\frac{12}{\pi}\,a_1^{(t)}\,, \end{aligned} \tag{3.35}$$

with the notation:

$$a_l^{(s),(t)}=a^{(s),(t)}(0)\,,\quad b_l^{(s),(t)}=\frac{d}{d\nu}\,a_l^{(s),(t)}(0)\,. \tag{3.36}$$

— a_l is equal to the $2l$-th power of the l-wave scattering length, whereas b is related to the l-wave effective range.

Equation (3.30) and the difference of (3.23) and (3.31) lead to two sum rules involving S-, P- and D-wave scattering lengthes. Forming suitable combinations of these rules, we get:

$$24\,a_1^{(t)}+4\left(a_0^{(s)}-a_0^{(t)}\right)=\frac{1}{\pi^2}\int_0^{\infty}d\nu\left\{\frac{1}{\nu^{1/2}(\nu+1)^{3/2}}\,\sigma_s(\nu)\right.$$
$$+\frac{1}{\nu^{3/2}}\left[\frac{1}{(\nu+1)^{1/2}}\,\sigma_t(\nu)-\frac{1}{2}\,\frac{1}{(\nu+1)^3}\,(\nu^3+3\nu^2+8\nu+2)\,\sigma_t(0)\right]$$
$$\left.-\frac{2\nu+1}{(\nu(\nu+1))^{3/2}}\left[\bar{\sigma}_t(\nu)-\bar{\sigma}_t(0)\right]\right\} \tag{3.37}$$

$$120\, a_2^{(s)} = \frac{1}{\pi^2} \int\limits_0^\infty d\nu \left\{ - \frac{1}{\nu^{3/2}\,(\nu+1)^3} \left[(3\,\nu+1)\,(\nu+1)^{1/2}\,\sigma_t(\nu) - \frac{1}{2}\,(\nu^3+3\,\nu^2+8\,\nu+2)\,\sigma_t(0) \right] + \frac{2\,\nu+1}{(\nu\,(\nu+1))^{3/2}}\,[\bar\sigma_t(\nu)-\bar\sigma_t(0)] + 4\,\pi^2\,\frac{1}{\nu\,(\nu+1)} \left[\frac{1}{\nu}\,\frac{\partial}{\partial t}\,A_s(\nu,0) + \frac{1}{\nu+1}\,\frac{\partial}{\partial u}\,A_t(\nu,0) \right] \right\} \tag{3.38}$$

with the notation:

$$\sigma_s(\nu) = \frac{\pi^2}{(\nu\,(\nu+1))^{1/2}}\,A_s(\nu, t=0) \qquad \sigma_t(\nu) = \frac{\pi^2}{(\nu\,(\nu+1))^{1/2}}\,A_t(\nu, u=0)$$

$$\bar\sigma_t(\nu) = \frac{\pi^2}{(\nu\,(\nu+1))^{1/2}}\,A_t\,(\nu, u=-4\,\nu)\ , \tag{3.39}$$

$\sigma_s(\nu)$ and $\sigma_t(\nu)$ are the total cross-sections in the s- and t-channels; $\bar\sigma_t(\nu)$ is proportional to the absorptive part in the t-channel for backward scattering.

Furthermore, we have two rules for S-wave effective ranges:

$$4\,b_0^{(s)} = 4\,(a_0^{(s)} - a_0^{(t)}) + \frac{1}{\pi^2} \int\limits_0^\infty d\nu \left\{ \frac{1}{\nu^{1/2}\,(\nu+1)^{3/2}}\,\sigma_t(\nu) + \frac{1}{\nu^{3/2}} \left[\frac{\sigma_s(\nu)}{(\nu+1)^{1/2}} - \sigma_s(0) \right] \right\} \tag{3.40}$$

$$4\,b_0^{(t)} = 12\,a_1^{(t)} + \frac{1}{\pi^2} \int\limits_0^\infty d\nu\,\frac{2\,\nu+1}{(\nu\,(\nu+1))^{3/2}}\,[\bar\sigma_t(\nu)-\bar\sigma_t(0)]\,. \tag{3.41}$$

It is easily seen that the sum rules established in Part 2 are equivalent to the rules established in this Part if the s-channel and the t-channel are identical. Therefore, Part 3 gives no further properties of a scattering amplitude which is completely symmetric in s, t and u.

4. Scattering Amplitude Antisymmetric under the Exchange $t \leftrightarrow u$

Consider the amplitude $T^1(s, t, u)$ which coincides in its s-channel with the scattering amplitude of the $T = 1$ isotopic spin state of a π-meson pair. This amplitude is antisymmetric under the exchange $t \leftrightarrow u$. Therefore, the function $T^1(s, t, u)/(t-u)$ is symmetric under this exchange, and the discussion of Part 3 applies. However, we have different sum rules, because we have now once subtracted dispersion relations.

One obtains a set of six independent sum rules. Some of these turn out to be equivalent to previous ones. Two rules involve an F-wave scattering length and a P-wave effective range. These parameters are not easily measurable and the integrals which expresse them are not simple. A last rule is a relation between integrals over cross-sections and derivatives of absorptive parts with respect to momentum transfer. These integrals are intricate and it would be hard to test the rule. In conclusion, the sum rules considered here do not appear as very useful and we shall not write them down.

5. Sum Rules for $\pi-\pi$ Scattering

Let $T^I(\nu, t)$ be the scattering amplitude of the $\pi-\pi$ system in its $T = I$ isotopic spin state ($I = 0, 1, 2$). These amplitudes define three functions $A(s, t, u)$, $B(s, t, u)$ and $C(s, t, u)$ which, as a consequence of crossing symmetry, have simple symmetry properties [4]:

$$A\left(4(\nu+1), t, -4\nu - t\right) = \frac{1}{3}\left[T^0(\nu, t) - T^2(\nu, t)\right]$$

$$B\left(4(\nu+1), t, -4\nu - t\right) = \frac{1}{2}\left[T^1(\nu, t) + T^2(\nu, t)\right]$$

$$C\left(4(\nu+1), t, -4\nu - t\right) = -\frac{1}{2}\left[T^1(\nu, t) - T^2(\nu, t)\right] \qquad (5.1)$$

for $\nu \geqslant 0$, $-4\nu \leqslant t \leqslant 0$. $A(s, t, u)$ is symmetric under the exchange $t \leftrightarrow u$:

$$A(s, t, u) = A(s, u, t) \qquad (5.2)$$

and the functions $B(s, t, u)$ and $C(s, t, u)$ are related to $A(s, t, u)$ through cyclic permutations of s, t and u:

$$B(s, t, u) = A(t, u, s), \quad C(s, t, u) = A(u, s, t). \qquad (5.3)$$

The combination:

$$A(s,t,u) + B(s,t,u) + C(s,t,u) = \frac{1}{3} T^0\left(\frac{1}{4}(s-4), t\right) + \frac{2}{3} T^2\left(\frac{1}{4}(s-4), t\right) \qquad (5.4)$$

is totally symmetric in s, t and u, and the sum rules (2.30) and (2.31) hold for this function.

The function:

$$T(s, t, u) = B(s, t, u) + C(s, t, u) = T^2\left(\frac{1}{4}(s-4), t\right) \qquad (5.5)$$

is symmetric in u and t. The discussion of Part 3 applies to this function; it verifies the sum rules (3.37), (3.38), (3.40) and (3.41). In order to get the physical content of these rules, we have to identify $T(s, t, u)$ in the t-channel in terms of the T^I's:

$$T\left(-4\nu - u, 4(\nu+1), u\right) = A\left(4(\nu+1), u, -4\nu - u\right) + B\left(4(\nu+1), u, -4\nu - u\right)$$

$$= \frac{1}{3} T^0(\nu, u) + \frac{1}{2} T^1(\nu, u) + \frac{1}{6} T^2(\nu, u) \qquad (5.6)$$

for $\nu \geqslant 0$, $-4\nu \leqslant u \leqslant 0$. The use of (5.4), (5.5) and (5.6) shows that the sum rule obtained from (3.41) is equivalent to a combination of the relations obtained from the other rules. Hence, we are finally left with a set of five sum rules which are listed in Table 1. According to the remark put at the end of Part 3, these rules exhaust the results of our method.

A remarkable fact is that the different terms behaving like $\nu^{-3/2}\left[\sigma^I(\nu) - \sigma^I(0)\right]$ as $\nu \to 0$ compensate each other exactly in the integrals appearing in (A), (B) and (C). Thus, the scattering lengthes are less sensitive to the behavior of the total cross sections at threshold than the effective ranges. In spite of this, all our rules are plainly

Table 1

Sum rules for $\pi - \pi$ scattering

Notation: $a_l^I = \lim_{\nu \to 0+} [(\nu+1)^{1/2} \nu^{-(2l+1)/2} \exp(i\,\delta_l^I(\nu)) \sin\delta_l^I(\nu)]$, $b_l^I = \lim_{\nu \to 0+} (d/d\nu)\, [(\nu+1)^{1/2} \nu^{-(2l+1)/2} \exp(i\,\delta_l^I(\nu)) \sin\delta_l^I(\nu)]$, $\sigma^I(\nu)$ = total cross-section of the isotopic spin $T = I$ state, $A^I(\nu, t)$ = absorptive part of the $T = I$ scattering amplitude. ν = (momentum)^2, t = (momentum transfer)^2 in the *C.M.* system, in units of (meson mass)^2.

(A)
$$18\, a_1^1 = 2\, a_0^0 - 5\, a_0^2 - \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{1}{(\nu(\nu+1))^{3/2}} \times [2\,\nu\,\sigma^0(\nu) - 3\,(3\,\nu+2)\,\sigma^1(\nu) - 5\,\nu\,\sigma^2(\nu)]$$

(B)
$$a_2^0 = \frac{1}{180\,\pi^2} \int_0^\infty d\nu \left\{ \frac{1}{\nu^{3/2}(\nu+1)^{5/2}} [\nu^2\,\sigma^0(\nu) + 3\,(\nu^2+3\,\nu+1)\,\sigma^1(\nu) + 5\,\nu^2\,\sigma^2(\nu)] + \frac{2\,\pi^2}{(\nu(\nu+1))^2} \left[(4\,\nu+3) \frac{\partial A^0(\nu,0)}{\partial t} - 3\,\nu \frac{\partial A^1(\nu,0)}{\partial t} + 5\,\nu \frac{\partial A^2(\nu,0)}{\partial t} \right] \right\}$$

(C)
$$a_2^2 = \frac{1}{360\,\pi^2} \int_0^\infty d\nu \left\{ \frac{1}{\nu^{3/2}(\nu+1)^{5/2}} [2\,\nu^2\,\sigma^0(\nu) - 3\,(\nu^2+3\,\nu+1)\,\sigma^1(\nu) + \nu^2\,\sigma^2(\nu)] + \frac{2\,\pi^2}{(\nu(\nu+1))^2} \left[(4\,\nu+3) \frac{\partial A^0(\nu,0)}{\partial t} + 3\,\nu \frac{\partial A^1(\nu,0)}{\partial t} + (7\,\nu+6) \frac{\partial A^2(\nu,0)}{\partial t} \right] \right\}$$

(D)
$$3\, b_0^0 = 2\, a_0^0 - 5\, a_0^2 + \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{1}{(\nu(\nu+1))^{3/2}} \times [(4\,\nu+3)\,\sigma^0(\nu) - 3\,(2\,\nu+1)\,\sigma^0(0) - 3\,\nu\,\sigma^1(\nu) + 5\,\nu\,\sigma^2(\nu)]$$

(E)
$$6\, b_0^2 = -2\, a_0^0 + 5\, a_0^2 + \frac{1}{4\pi^2} \int_0^\infty d\nu \frac{1}{(\nu(\nu+1))^{3/2}} \times [2\,\nu\,\sigma^0(\nu) + 3\,\nu\,\sigma^1(\nu) + (7\,\nu+6)\,\sigma^2(\nu) - 6\,(\nu+1)^{3/2}\,\sigma^2(0)]$$

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low energy relations, because of the rapid decrease of all weight functions; they behave at least like ν^{-2} at high ν values, i.e., like the inverse of the fourth power of the energy in the C.M. system. Indeed, this decrease is so strong that the energy domains where resonances occur and where the scattering amplitudes are large and relatively well known do not contribute significantly to the integrals. For these reasons our present knowledge of the $\pi - \pi$ scattering amplitudes does not allow a reliable quantitative test of our sum rules.

Recently Wolf [5] proposed a phase shift analysis of the $\pi - \pi$ scattering experimental results. This analysis is based on the hypothesis that $a_1^1 > 0$. We find that the S-wave scattering lengthes and the cross sections Wolf obtains and our rule (A) are compatible with this assumption.

As was mentioned in Part 2, the positivity of $A^I(\nu, 0)$ and $(\partial/\partial t)\, A^I(\nu, 0)$, which follows from unitarity, implies that:

$$a_2^0 + 2\, a_2^2 > 0\,. \tag{5.7}$$

One may ask if our sum rules, combined with these positivity conditions, imply that other linear combinations of the scattering lengthes have a definite sign. The answer is negative. All what one can show is that a combination is bounded by an integral over the S-wave, $T = 0$ absorptive part $A_0^0(\nu)$:

$$2\, a_0^0 - 5\, a_0^2 - 18\, a_1^1 - 30\, (a_2^0 + 2\, a_2^2) + \alpha\, 60\, (a_2^0 - a_2^2) < \frac{2}{\pi} \int_0^\infty d\nu\, \frac{A_0^0(\nu)}{\nu^{3/2}\, (\nu+1)^{5/2}} \tag{5.8}$$

for $0 < \alpha < 5/4$.

It should be noted that MARTIN [3] gives a sum rule for a_2^0 which ensures that this quantity is positive. Taking this result into account and eliminating a_2^2 from (5.8) by choosing $\alpha = 1$, we get:

$$18\, a_1^1 > 2\, a_0^0 - 5\, a_0^2 - \frac{2}{\pi} \int_0^\infty d\nu\, \frac{A_0^0(\nu)}{\nu^{3/2}\, (\nu+1)^{5/2}}\,. \tag{5.9}$$

In other words, the S-wave scattering of the $T = 1$ and $T = 2$ states determines a lower bound for the P-wave scattering length of the $T = 1$ state.

I have begun to play with homogeneous variables several years ago, with Prof. H. LEHMANN, during a stay in Hamburg. I am grateful to Mr. Y. STOLL for a careful verification of the calculations leading to the results presented in this paper. Several stimulating discussions with Prof. A. MARTIN, from CERN, favoured the progress of this work.

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