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## Representation of Group Generators by Boson or Fermion Operators Application to Spin Perturbation Theory

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(27. V. 66)

A representation of spin by fermion operators was known for a long time for  $S = 1/2$  [1]<sup>1)</sup>, whereas the generalization to  $S > 1/2$  is more recent [2]. The interest of such a representation is to avoid the complications of a generalized Wick's theorem valid for spin [3]. Here we remark first that such representations have a much larger generality, valid for fermions as well as for bosons [4]. We then discuss the elements of perturbation theory in the representation of spin by fermion operators.

1. Let  $G_i$  ( $i = 1, \dots, r$ ) be the generators of a simple Lie group numbered such that the first  $l$  commute with each other. A basis of an irreducible representation  $D^N$  may then be chosen such that

$$G_i |\nu\rangle = m_i(\nu) |\nu\rangle; \quad i = 1, \dots, l; \quad \nu = 0, 1, \dots, N - 1. \quad (1)$$

One calls  $r$  the order and  $l$  the rank of the group,  $N$  the dimension and  $m_i(\nu)$  the weights of the representation [5].

Let  $a_\nu$  and  $a_\nu^*$  be boson or fermion operators such that

$$[a_\nu a_{\nu'}^*]_{\mp} = \delta_{\nu\nu'}. \quad (2)$$

It is then easy to prove that the expressions  $\sum_{\nu\nu'} a_\nu^* g_i^{\nu\nu'} a_{\nu'}$  ( $i = 1, \dots, r$ ) satisfy the commutation rules of the  $G_i$ ,  $[G_i G_j]_- = \sum_k \gamma_{ij}^k G_k$ , if the matrices  $g_i$  do so.

If  $|\rangle$  is the vacuum state of these fictitious particles such that

$$a_\nu |\rangle = 0 \quad (3)$$

then the identification

$$|\nu\rangle = a_\nu^* |\rangle \quad (4)$$

$$G_i = \sum_{\nu\nu'} a_\nu^* g_i^{\nu\nu'} a_{\nu'} \quad (5)$$

reproduces all the properties of the representation (1), of which the matrices  $g_i$  are a realization.

The inconvenience of this representation by *fictitious* particles  $\nu$  is that neither the vacuum  $|\rangle$  nor the states with more than one particle,  $a_{\nu_1}^* \dots a_{\nu_n}^* |\rangle$ ,  $n > 1$ , have

<sup>1)</sup> Numbers in brackets refer to References, page 465.

physical meaning. For fermions  $n \leq N$  and hence the dimension of the Hilbert space of all particles is  $2^N$ .

The case of spin is obtained with the group  $SU(2)$  for which  $r = 3, l = 1$ , and  $G_1 = S_3, G_2 = S_+, G_3 = S_-, N = 2S + 1, m(\nu) = S - \nu$ .

For spins localized on different atoms labeled by  $n$  one has [2]

$$S_n = \sum_{\nu\nu'} a_{\nu n}^* s^{\nu\nu'} a_{\nu' n}. \tag{6}$$

It is interesting to note that in elementary particle theory, localized  $G_i$  have an interpretation as generalized charge densities and (5) expresses their bilinear form in the fields of the particles  $\nu$  which, in this case, are *physical*.

2. Consider now perturbation theory for the coupling of one single spin, Equation (6) with indices  $n$  omitted. The unperturbed hamiltonian is, for an external magnetic field  $\mathfrak{H}$ ,

$$H_0 = -\gamma S_z = -\gamma \sum_{\nu=0}^{2S} (S - \nu) a_{\nu}^* a_{\nu}; \quad \gamma = 2\mu_B \mathfrak{H} > 0. \tag{7}$$

$|0\rangle = a_0^* | \rangle$  is the ground state. The expressions to be calculated are of the form  $\langle \prod_{\nu=0}^{2S} \tau_{\nu} \rangle$ . Here  $\tau_{\nu}$  is a product ordered according to imaginary times of operators  $a_{\nu}^*(-i\tau), a_{\nu}(-i\tau)$  where  $O(t) = e^{iH_0 t} O e^{-iH_0 t}$ .  $\langle \rangle$  is an unperturbed canonical average taken over the physical states  $| \nu \rangle$ , i.e.

$$\langle O \rangle \equiv \sum_{\nu=0}^{2S} \langle \nu | e^{-\beta H_0} O | \nu \rangle / \sum_{\nu=0}^{2S} \langle \nu | e^{-\beta H_0} | \nu \rangle. \tag{8}$$

One calculates

$$\langle \nu | e^{-\beta H_0} \prod_{\nu'} \tau_{\nu'} | \nu \rangle = e^{\beta\gamma(S-\nu)} \langle \nu | \tau_{\nu} | \nu \rangle \prod_{\nu' \neq \nu} \langle | \tau_{\nu'} | \rangle. \tag{9}$$

Here  $\langle | \tau_{\nu} | \rangle$  is obtained by applying the usual diagram technique for *zero temperature* and is expressed in terms of free propagators which, however, do not have the usual form because of the imaginary time order. They are

$$G_{\nu}(\tau) \equiv \langle | T_{\tau} (c_{\nu}(-i\tau) a_{\nu}^*(0)) | \rangle \\ = \frac{1}{2\pi} \int_{-i\gamma(S-\nu+0)-\infty}^{-i\gamma(S-\nu+0)+\infty} d\omega \frac{e^{i\omega\tau}}{i\omega - \gamma(S-\nu)} = \begin{cases} e^{\tau\gamma(S-\nu)}; & \tau > 0 \\ 0 & ; \tau < 0 \end{cases}. \tag{10}$$

The expression  $\langle \nu | \tau_{\nu} | \nu \rangle$  is not directly accessible to the usual technique because the normal products obtained by applying the ordinary Wick's theorem to  $\tau_{\nu}$ , do not all vanish when taken between the states  $| \nu \rangle$ . To handle it, we introduce the unperturbed canonical average taken over the Hilbert space of the fictitious particles. For fermions this is

$$\langle\langle O \rangle\rangle \equiv \sum_{\{n_{\nu}=0,1\}} \langle | \prod_{\nu} (a_{\nu})^{n_{\nu}} e^{-\beta H_0} O \prod_{\nu} (a_{\nu}^*)^{n_{\nu}} | \rangle \\ \times \left\{ \sum_{\{n_{\nu}=0,1\}} \langle | \prod_{\nu} (a_{\nu})^{n_{\nu}} e^{-\beta H_0} \prod_{\nu} (a_{\nu}^*)^{n_{\nu}} | \rangle \right\}^{-1}. \tag{11}$$

One then finds that  $\langle \nu | \tau_\nu | \nu \rangle$  can be expressed in terms of  $\langle | \tau_\nu | \rangle$  and  $\ll \tau_\nu \gg$ ,

$$e^{\beta\gamma(S-\nu)} \langle \nu | \tau_\nu | \nu \rangle = (1 + e^{\beta\gamma(S-\nu)}) \ll \tau_\nu \gg - \langle | \tau_\nu | \rangle. \quad (12)$$

$\ll \tau_\nu \gg$  can be calculated by the usual diagram technique for *finite temperature* [6] and is expressed in terms of free propagators,

$$g_\nu(\tau) \equiv \ll T_\tau (a_\nu(-i\tau) a_\nu^*(0)) \gg = \frac{1}{\beta} \sum_{r=-\infty}^{+\infty} \frac{e^{i\omega_r \tau}}{i\omega_r - \gamma(S-\nu)}; \quad \omega_r = \frac{\pi}{\beta} (2r+1). \quad (13)$$

The general result now is

$$\begin{aligned} \langle \prod_\nu \tau_\nu \rangle &= e^{-\beta\gamma S} \frac{1 - e^{-\beta\gamma}}{1 - e^{-\beta\gamma(2S+1)}} \left\{ \sum_{\nu=0}^{2S} (1 + e^{\beta\gamma(S-\nu)}) \right. \\ &\quad \left. \times \ll \tau_\nu \gg \prod_{\nu' \neq \nu} \langle | \tau_{\nu'} | \rangle - (2S+1) \langle | \prod_\nu \tau_\nu | \rangle \right\}. \quad (14) \end{aligned}$$

Since  $\prod_\nu \tau_\nu$  usually is a product of operators (6) one has  $\langle | \prod_\nu \tau_\nu | \rangle = 0$ . For low temperatures,  $\beta\gamma \gg 1$ , Equation (14) simplifies considerably:

$$\langle \prod_\nu \tau_\nu \rangle \cong \ll \tau_0 \gg \prod_{\nu \neq 0} \langle | \tau_\nu | \rangle. \quad (15)$$

The procedure described here is to be compared with that of Abrikosov's second paper [7] while in his first paper [2], ABRIKOSOV uses an artificial hamiltonian which is obtained from (7) by the substitution  $\gamma(S-\nu) \rightarrow \lambda$ . In this case, Equation (14) goes over into

$$\langle \prod_\nu \tau_\nu \rangle = \frac{e^{\beta\lambda+1}}{2S+1} \sum_\nu \ll \tau_\nu \gg \prod_{\nu' \neq \nu} \langle | \tau_{\nu'} | \rangle. \quad (16)$$

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$$| \nu \rangle = [(2S - \nu)! \nu!]^{-1/2} (a_0^*)^{2S-\nu} (a_1^*)^\nu | \rangle$$
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