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A Modification of Piron's Axioms

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Abstract. We show that the lattice L_0 of yes-no observables in classical statistical mechanics fails to satisfy Piron's lattice-theoretical axioms for quantum mechanics. We weaken one of Piron's axioms, replacing completeness by σ-completeness; L_0 satisfies the modified set of axioms. Using C^*-algebra techniques, we exhibit a large class of atomic lattices which satisfy the modified set of axioms, and which do not satisfy Piron's original axioms.

1. Introduction

PIRON [5] has introduced a set of axioms for the propositional structure of a physical theory and has shown that quantum mechanics (with superselection rules) and classical Newtonian mechanics are models satisfying these axioms. It can be argued that a satisfactory axiom system would have classical statistical mechanics as a model. We show that Piron's axioms *exclude* such a model, but that a technical modification of one axiom allows its inclusion. We also discuss the relation between the modified axiom system and the C^*-algebra approach to quantum mechanics.

The papers by DAVIES [1] and PLYMEN [6] have drawn attention to an important class of operator algebras: the sequentially weakly closed C^*-algebras of operators. Such algebras may be characterized abstractly and are called *Σ^*-algebras*. The theory of Σ^*-algebras may be regarded as providing a basis for a non-commutative version of probability theory. The author establishes in [6] that the theory of Σ^*-algebras makes it possible to relate the C^*-algebra approach to quantum mechanics with the axiomatic formulation of quantum mechanics due to MACKEY [4]. Given an abstract C^*-algebra A, DAVIES [1] constructs a canonical Σ^*-algebra $A^{\sim}$ containing A, called the *σ-envelope* of A; and the relevance of the σ-envelope in quantum mechanics is discussed in [6].

Piron's second axiom requires that the partially ordered set L of questions (yes-no observables) should be a *complete* lattice (each subset of L has a least upper bound and a greatest lower bound in L). We weaken this axiom by requiring that L should be a *σ-complete* lattice (each countable subset of L has a least upper bound and a greatest lower bound in L). The resulting set of five axioms we call the essential axioms of PIRON. The main result of this paper is

Theorem 1. Let A be a type I separable C^*-algebra, and let $A^{\sim}$ be the σ-envelope of A. Then the partially ordered set $L(A^{\sim})$ of all projections in $A^{\sim}$ satisfies the essential axioms of PIRON.

Theorem 1 provides us with many atomic lattices excluded by Piron's original axioms. For example, let A be the C^*-algebra of complex-valued continuous functions on the unit interval [0,1]. Then $L(A^{\sim})$ may be identified with the lattice L_0 of Borel subsets of [0, 1]. L_0 satisfies the essential axioms of PIRON, but fails to satisfy the

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original axioms of PIRON. Theorem 1 may be regarded as relating the C^*-algebra approach to quantum mechanics with Piron's axioms.

In Section 2 we discuss classical statistical mechanics and show that it is excluded by Piron's original axioms. We argue for its inclusion as a model of a modified set of axioms. In Section 3 we discuss Σ^*-algebras and the σ-envelope. We remark upon the relevance of Σ^*-algebras in classical statistical mechanics (Remark 3.6). In Section 4 we state the essential axioms of PIRON and prove Theorem 1. We show that the partially ordered set $L(A)$ of projections in an arbitrary Σ^*-algebra A is a σ-complete orthocomplemented lattice.

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2. Classical Statistical Mechanics

Consider the classical mechanics of a system of a very large number n of particles, for example a macroscopic physical system composed of n very small 'atoms' moving according to classical mechanical laws. The phase space of such a system may be identified with $6n$-dimensional Euclidean space R^{6n}. The physical states of such a statistical mechanical system are represented by the probability measures on R^{6n}; the observables are represented by the real-valued Borel functions on R^{6n}. To each pair consisting of an observable u and a state f we have associated the probability Borel measure on the real line given by $M \to f(u^{-1}(M))$. The number $p(u, f, M) \equiv f(u^{-1}(M))$ is the probability that a measurement of u will be in M when the system is in the state f [4, pp. 47, 61]. Let us call an observable a a *question*, if in every state f the measure $M \to p(a, f, M)$ is concentrated in the points 0 and 1, that is, if $p(a, f, \{0, 1\}) = 1$ for all f. It is easy to verify that the questions are precisely the characteristic functions of Borel subsets of R^{6n}. There is a natural partial ordering on the set L of questions as follows: $a \leqslant b$ if and only if $p(a, f, \{1\}) \leqslant p(b, f, \{1\})$ for all states f. The partially ordered set L of questions may thus be identified with the set of Borel subsets of R^{6n}, partially ordered by inclusion. Now the Borel structure underlying the metric topology of R^{6n} is standard, by definition. This standard Borel space is isomorphic, as a Borel space, with the Borel structure underlying the metric topology of the unit interval $[0, 1]$. This means that there exist $1-1$ Borel mappings $f: R^{6n} \to [0, 1]$ and $g: [0, 1] \to R^{6n}$ such that $fg = 1$ and $gf = 1$ [2, p. 357]. The Borel isomorphism preserves the partial order structure. Hence the partially ordered set L of questions of the classical statistical mechanical system may be identified with the set L_0 of Borel subsets of $[0, 1]$, partially ordered by inclusion.

Now L_0 is a σ-complete lattice (each countable subset of L_0 has a least upper bound and a greatest lower bound in L_0). Of course L_0 is not a complete lattice (each subset of L_0 has a least upper bound and a greatest lower bound in L_0) because L_0 contains each point of $[0, 1]$. There exists an orthocomplementation in L_0, $a \to a'$, which sends each element of L_0 to its complement in $[0, 1]$. The atoms of the lattice L_0 are precisely the points of the unit interval. Moreover L_0 is actually a Boolean algebra.

One of the cardinal achievements of MACKEY's axiomatization of quantum mechanics [4, pp. 61–85] is this: The formal mathematical structure of quantum mechanics differs from the formal mathematical structure of classical statistical mechanics in one and only one respect, namely the structure of the partially ordered set L of questions. On the one hand, L is isomorphic with the partially ordered set of all closed subspaces of a separable infinite-dimensional Hilbert space; on the other hand, L is isomorphic with the partially ordered set of all Borel subsets of phase space. It may be argued that any axiomatic formulation of quantum mechanics should, in a definite sense, include the case of classical statistical mechanics. In his lattice-theoretical approach to quantum mechanics, PIRON [5] requires that the partially ordered set L of questions should be a complete lattice. Thus Piron excludes the lattice L_0, hence excludes the case of classical statistical mechanics. In order to include this important case, we propose to weaken Axiom II of PIRON, replacing completeness by σ-completeness. We leave the other four axioms unchanged. The resulting set of five axioms we call the essential axioms of PIRON.

Doubts concerning Piron's second axiom were raised by GUENIN [3, p. 282], who suggested replacing completeness by σ-completeness. Note that the concept of σ-state is invariant under σ-isomorphism of Σ^*-algebras [cf. 3, p. 275].

3. On Σ^*-Algebras

For the general theory and notation concerning C^*-algebras we shall make systematic use of DIXMIER's book [2]. Since it is no restriction to assume that the C^*-algebra A has an identity, we shall always assume our C^*-algebras have identities denoted by 1. A state f of a C^*-algebra A is a linear functional on A such that $f(1) = 1$ and $f(x) \geqslant 0$ when $x \geqslant 0$. We denote by $\mathcal{B}(H)$ the C^*-algebra of all bounded operators on the Hilbert space H. We shall be concerned with the weak operator topology on $\mathcal{B}(H)$, the weakest topology on $\mathcal{B}(H)$ such that the mappings $x \to (x\,\xi, \xi)$ are continuous for each ξ in H. If $x_n \to x$ in the weak operator topology, we say $x_n \to x$ weakly. Now let A be a C^*-algebra and denote by F the set of all ordered pairs $\{x_n, x\}$ consisting of a sequence $x_n \in A$ and a point $x \in A$. If $G \subseteq F$ we denote by G^σ the set of all states f on A such that for all $\{x_n, x\} \in G$ we have $f(x_n) \to f(x)$.

Definition 3.1. A Σ^*-algebra A is a C^*-algebra together with a subset $G \subseteq F$, called the set of *σ-convergent sequences* in A and denoted $x_n \to x$, such that the following properties hold:

(i) if $x_n \to x$ then there is a constant K such that for all n we have $\| x_n \| \leqslant K < \infty$;
(ii) if $x_n \to x$ and $y \in A$ then $x_n\, y \to x\, y$;
(iii) if $x_n \in A$ is a sequence such that $f(x_n)$ converges for all $f \in G^\sigma$ then there is some $x \in A$ such that $x_n \to x$;
(iv) if $0 \neq x \in A$ then there is some $f \in G^\sigma$ such that $f(x) \neq 0$.

G^σ is called the set of *σ-states* of the Σ^*-algebra A.

Example 3.2. A set A of bounded operators on the Hilbert space H shall be called *σ-closed* if given any sequence $x_n \in A$ which converges weakly to $x \in \mathcal{B}(H)$, we then have that $x \in A$. Given any set A there is a smallest σ-closed set containing it, which we call its *σ-closure* and denote by $\sigma(A)$. If A is a sub-C^*-algebra of $\mathcal{B}(H)$ then $\sigma(A)$ is a C^*-algebra. Let A be a sub-C^*-algebra of $\mathcal{B}(H)$ such that $A = \sigma(A)$. A becomes a Σ^*-algebra if we define the σ-convergent sequences to be the weakly convergent sequences. We call such algebras *Σ^*-algebras of operators*; clearly $\mathcal{B}(H)$ is itself a Σ^*-algebra of operators. By a σ-representation π of the Σ^*-algebra A on the Hilbert space H we shall mean a representation such that if $x_n \to x$ then $\pi(x_n) \to \pi(x)$. By a faithful σ-representation we shall mean a faithful representation such that $\pi(A)$ is σ-closed and $x_n \to x$ if and only if $\pi(x_n) \to \pi(x)$.

Lemma 3.3. Every Σ^*-algebra A has a faithful σ-representation as a Σ^*-algebra of operators on a Hilbert space.

The proof of this Lemma is in [1].

Example 3.4. Let X be a set with a given σ-ring of subsets. The algebra $B\{X\}$ of all bounded measurable functions on X is a commutative C^*-algebra in an obvious sense. We say that a sequence $u_n \in B\{X\}$ is σ-convergent to $u \in B\{X\}$ if and only if $\| u_n \| \leqslant K$ for some K and all n, and u_n also converges pointwise to u. Then $B\{X\}$ is a Σ^*-algebra; and the family of σ-states is exactly the set of probability measures on X. There is a discussion of this example in [1]. Note that $B\{X\}$ is in general not the dual of a Banach space, hence not a W^*-algebra [7].

Let Q denote the set of all positive linear functionals on A, and let $\phi = \oplus_{f \in Q} \pi_f$, where π_f is the canonical cyclic representation defined by f. Then ϕ is called the *universal representation* of A [2, p. 43].

Definition 3.5. The σ-envelope $A^{\sim}$ of the C^*-algebra A is the σ-closure of $\phi(A)$, where ϕ is the universal representation of A.

Now $A^{\sim}$ is a C^*-algebra, hence a Σ^*-algebra. We regard $A^{\sim}$ as a canonical Σ^*-algebra containing A. There is a close analysis of the structure of the algebra $A^{\sim}$ in [1]. When A is a separable commutative C^*-algebra, $A^{\sim}$ may be identified with the Σ^*-algebra $B\{\hat{A}\}$ of all complex-valued bounded Borel functions on the spectrum $\hat{A}$ of A [1].

Remark 3.6. Let R^{6n} be phase space of a classical statistical mechanical system. Consider the Σ^*-algebra $B\{R^{6n}\}$ of all complex-valued bounded Borel functions on R^{6n}. Each real function in $B\{R^{6n}\}$ represents a bounded observable, and each σ-state of $B\{R^{6n}\}$ represents a physical state. The partially ordered set of projections in the C^*-algebra $B\{R^{6n}\}$ may be identified with the partially ordered set of Borel subsets of R^{6n}, hence with the lattice L_0.

4. Piron's Axioms

In this section we consider the lattice-theoretical approach to quantum mechanics, as formulated by Piron [5]. Among all possible observables of a physical system, we shall consider those for which the result of measurement can be expressed by yes or no and we shall call them *questions*. For a given system, a question is said to be *true* if the answer is yes with certainty. If this definition is to be meaningful, if a is true, it should be possible to measure a without perturbing the system. We shall admit this. If a and b are two questions, it may happen that one implies the other, i.e., that every time a is true then b is also true. We shall write this $a \leqslant b$, and $a = b$ means $a \leqslant b$ and $b \leqslant a$. On the set L of questions, we shall impose the following axioms.

Axiom I (i) $a \leqslant a$ for all a in L,
(ii) $a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$.

Remark 4.1. Thus L is a partially ordered set. Let A be a C^*-algebra, and let $L(A)$ be the set of all projections in A. Then $L(A)$ has a natural partial ordering by positivity, $a \leqslant b$ if and only if $b-a \geqslant 0$.

Axiom II (weakened) (i) There exists an element, denoted 0, in L such that $0 \leqslant a$ for all a in L.

(ii) Each sequence a_n in L has a greatest lower bound, denoted $\Lambda\, a_n$.

Lemma 4.2. If A is a Σ^*-algebra then $L(A)$ satisfies Axiom II.

Proof. Pass to a faithful σ-representation of A as a Σ^*-algebra of operators on the Hilbert space H. Let a_n be a sequence in $L(A)$. The least upper bound of $a_1, \ldots, a_n$ is the range projection a_n' of $a_1 + \cdots + a_n$, and a_n' lies in A [1]. The increasing sequence a_n' of projections converges weakly to a projection a in $\mathcal{B}(H)$, $a_n' \to a$. Since A is sequentially weakly closed, a lies in A. Now a is the least upper bound of the a_n'; hence a is the least upper bound $V\, a_n$ of the a_n. Then $1 - V\,(1 - a_n)$ is the greatest lower bound $\Lambda\, a_n$ of the a_n.

Axiom III. There exists an orthocomplementation in L, i.e. there is a mapping $a \to a'$ of L into L such that

(i) $(a')' = a$,

(ii) $a' \,\Lambda\, a = 0$,

(iii) $a' \leqslant b' \Rightarrow b \leqslant a$.

Remark 4.3. The mapping $a \to 1 - a$ is an orthocomplementation in $L(A)$ where A is a Σ^*-algebra. It follows from Axioms II and III that there exists an element, denoted 1, in L such that $a \leqslant 1$ for all a in L. It also follows from Axioms II and III that each sequence a_n in L has a least upper bound, denoted $V\, a_n$, namely $(\Lambda\, a_n')'$. Thus L is a σ-complete orthocomplemented lattice.

An atom is, by definition, an element $p \neq 0$ in L such that $0 \leqslant x \leqslant p \Rightarrow x = 0$ or $x = p$.

Axiom IV (i) If $a \in L$ and $a \neq 0$ then there exists an atom p such that $p \leqslant a$.

(ii) If p is an atom, then $a \leqslant x \leqslant a\, V\, p \Rightarrow x = a$ or $x = a\, V\, p$.

The representation π of the C^*-algebra A is type I if the von Neumann algebra generated by $\pi(A)$ is type I. The C^*-algebra A is, by definition, type I if all its representations are type I [2, p. 111]. A separable C^*-algebra is type I if and only if it is a G.C.R. algebra [2, p. 168].

Lemma 4.4. If A is a type I separable C^*-algebra then $L(A^{\sim})$ satisfies Axiom IV.

Proof. Following [2], let $\hat{A}$ denote the spectrum of A, and let $\hat{A}_n$ denote the set of unitary equivalence classes of n-dimensional irreducible representations of A. The Borel structure underlying the topology of $\hat{A}$ is a standard Borel space [2, p. 95]. Therefore each point of $\hat{A}$ is a Borel subset of $\hat{A}$. Now $\hat{A}_n$ is a Borel subspace of $\hat{A}$, hence each point of $\hat{A}_n$ is a Borel subset of $\hat{A}_n$. Let H_n denote an n-dimensional Hilbert space, separable for $n = \infty$. We say a function $u: \hat{A}_n \to \mathcal{B}(H_n)$ is a Borel function if for each $\xi, \eta \in H_n$ the function $(u(\pi)\, \xi, \eta)$ is a Borel function on $\hat{A}_n$. The space $B\{\hat{A}_n, \mathcal{B}(H_n)\}$ of all norm bounded Borel functions $u: \hat{A}_n \to \mathcal{B}(H_n)$ is a C^*-algebra in an obvious way. If K is the Hilbert space of all functions $u: \hat{A}_n \to H_n$ of countable support such that

$$\sum_{\pi \in \hat{A}_n} \| u(\pi) \|^2 < \infty$$

then $B\{\hat{A}_n, \mathcal{B}(H_n)\}$ is naturally identified with a Σ^*-algebra of operators on K. If $u, u_m \in B\{\hat{A}_n, \mathcal{B}(H_n)\}$ then u_m is σ-convergent to u if and only if for some k, all m and all $\pi \in \hat{A}_n$ we have

$$\| u_m(\pi) \| \leqslant k < \infty$$

and for all $\pi \in \hat{A}_n$ the sequence $u_m(\pi)$ converges to $u(\pi)$ in the weak operator topology. By Theorem 4.5 of [1], each $n = \infty, 1, 2, \ldots$ defines a central projection e_n in the σ-envelope $A^{\sim}$ and so a σ-ideal $A_n^{\sim} = e_n A^{\sim} e_n$ such that

$$A^{\sim} = \bigoplus_{n=1}^{n=\infty} A_n^{\sim} .$$

Each Σ^*-algebra $A_n^{\sim}$ has a faithful σ-representation as $B\{\hat{A}_n, \mathcal{B}(H_n)\}$, the Σ^*-algebra of all bounded Borel functions from $\hat{A}_n$ to $\mathcal{B}(H_n)$, where H_n is an n-dimensional Hilbert space, separable for $n = \infty$.

The projections in $B\{\hat{A}_n, \mathcal{B}(H_n)\}$ are precisely the projection-valued Borel functions from $\hat{A}_n$ to $\mathcal{B}(H_n)$. We first describe the minimal projections in $A^{\sim}$, which are precisely the atoms in $L(A^{\sim})$. Let k be a natural number or ∞. Let π_k be a point in $\hat{A}_k$, a_k a 1-dimensional projection in $\mathcal{B}(H_k)$. Define u_k: $\hat{A}_k \to \mathcal{B}(H_k)$ as follows: $u_k(\pi_k) = a_k$, $u_k(\pi) = 0$ $(\pi \neq \pi_k)$. Choose $\xi, \eta \in H_k$; then the function $\pi \to (u_k(\pi)\, \xi, \eta)$ has two values, 0 and $(a_k\, \xi, \eta)$. Since each point of $\hat{A}_k$ is a Borel subset of $\hat{A}_k$, u_k is a Borel function, hence

$$u_k \in B\{\hat{A}_k, \mathcal{B}(H_k)\} .$$

The sequence v_n given by $v_k = u_k$, $v_n = 0$ $(n \neq k)$ is a minimal projection in $A^{\sim}$, and every minimal projection in $A^{\sim}$ arises in this way. Clearly each nonzero projection in $A^{\sim}$ contains a minimal projection, which proves (i). If a is a projection in $B\{\hat{A}_k, \mathcal{B}(H_k)\}$ then $a \leqslant x \leqslant a \vee u_k \Rightarrow x = a$ or $x = a \vee u_k$. Thus if $a, a_0 \in L(A^{\sim})$, a_0 minimal, then $a \leqslant x \leqslant a \vee a_0 \Rightarrow x = a$ or $x = a \vee a_0$, which proves (ii).

Axiom V. If $a \leqslant b$ then the sublattice of L generated by a and b is a Boolean algebra.

Remark 4.5. Let A be a Σ^*-algebra, and pass to a faithful σ-representation of A as a Σ^*-algebra of operators on H. Now $L(A) \subseteq L(\mathcal{B}(H))$ and $L(\mathcal{B}(H))$ satisfies Axiom V. Hence $L(A)$ satisfies Axiom V.

Theorem 1 is a consequence of the lemmas and remarks 4.1 to 4.5.

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