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Electromagnetic Formfactors with SL(2,C)[1]

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Abstract. The Budini-Fronsdal symmetry group $G = P \boxed{\sim} \mathrm{SL}(2, C)$ is used to construct the production vertex of two spin 1/2 particles by a spin 1 particle and to calculate the appropriate form factors. The representations of the group are described by means of homogeneous variables as introduced by Leutwyler and Gorgé. The ratio of the electromagnetic form factors in the production channel is numerically computed in its full energy dependence. The result is in agreement with a formula for this ratio given by Bebié.

1. Introduction

In connection with an elaborate application of the Budini-Fronsdal symmetry group $G = P \boxed{\sim} \mathrm{SL}(6, C)$ to the problem of particle interactions, Leutwyler and Gorge [1] have given a reformulation of the old representation theory of Gelfand and Naimark [4] in terms of homogeneous functions. This reformulation proved to be especially useful in the task of reducing the complex mathematical apparatus involved in calculations of the coupling of three representations of G [1, 2].

In this paper we want to investigate a physical model based on the group $G = P \boxed{\sim} \mathrm{SL}(2, C)$ (P is the Poincaré group, SL(2, C) the complex unimodular group in 2 dimensions) with the technique developed by Leutwyler and Gorgé. We shall calculate the ratio of the electromagnetic form factors of a vertex involving two spin 1/2 particles and a spin 1 particle and find for it a twofold integral expression which will be computed numerically. The results obtained are of special interest not from the point of view of physical applicability, but from the point of view of methodology and consistency: Bebié [3] has given an explicit formula for the same ratio, using the same symmetry group, with which we can compare our results. Full agreement is found.

In Section 2 and 3 we review briefly the method of construction of particle states involving representations of the model symmetry group $G = P \boxed{\sim} \mathrm{SL}(2, C)$. This represents in principle an application of the theory outlined in Ref. [1]. The particle states are constructed by reducing out of the full representations of G the representations of a 'little group' which is here taken to be the physical Poincaré group in the rest system of the particle. Moving states are then generated by the application of appropriate boosts. This leads to real particles with timelike momenta.

In Section 4, we discuss the vertex of two spin 1/2 particles and one spin 1 particle (briefly called baryon-meson-vertex). Since the states describe particles on the mass shell, we are forced to calculate a vertex involving only timelike momenta, i.e. we are forced to restrict ourselves upon the production channel.

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In Section 5 and 6 the 12 real integrals over the group parameters appearing in the vertex are reduced to two and the form factors are expressed by their means. In order to have a control in hand we calculate, beside the form factors of the vector current, also those of the axial current. The latter should vanish whenever the parity of the vertex is uniquely determined by the group.

Section 7 contains the results of a numerical evaluation of the integrals: first the axial form factors vanish and the coupling of the three representations is unique, and secondly the ratio of the vector form factors shows very accurately the same energy dependence as the mentionned formula given by BEBIÉ [3].

In order not to repeat considerations contained in Ref. [1] and [2] and in other literature quoted there, we do not discuss in this paper the fundamentals of the symmetry group G, the technique of homogeneous variables and the problems connected with field theory and analyticity. And since there exists Bebié's closed form expression for the form factors, we do not either discuss here points like asymptotic behaviour in energy and analytic continuation below the production threshold. Instead, we have found it more instructive to show details of the effective calculations.

2. The State Vectors of the Group G

a) *Notation, Group Parameters*

The symmetry group G to be considered in this model can be described in two equivalent ways [5, 2], either as

$$G = P \boxed{\sim} \mathrm{SL}(2, \mathrm{C}) \tag{2.1}$$

or, more conveniently for practical purposes

$$G = P^* \otimes \mathrm{SL}(2, \mathrm{C})\,. \tag{2.2}$$

Here P stands for the physical Poincaré group generated by the energy momentum operator P_μ and the total angular momentum operators $M_{\mu\nu}$, whereas P^* is the (unphysical) Poincaré group generated by P_μ and the orbital angular momentum operators $L_{\mu\nu}$. The generators of SL(2, C) are the spin operators $S_{\mu\nu}$, and $L_{\mu\nu} = M_{\mu\nu} - S_{\mu\nu}$. We do not further explain this group here nor try to justify it, but refer the reader to Ref. [2] and the original literature quoted there.

A general element $g \in G$ can, according to (2.2), be decomposed as

$$g = l\, s = s\, l \tag{2.3}$$

where $l = l(L, a) \in P^*$ and $s \in \mathrm{SL}(2, \mathrm{C})$. Here L denotes the transformation matrix of the homogeneous part of P^* and a the translation parameter of the translatory part of P^*.

In the special case of a physical Lorentz transformation characterized by the Lorentz matrix Λ and the translation vector a we have

$$g(\Lambda, a) = l(\Lambda, a)\, s_\Lambda = s_\Lambda\, l(\Lambda, a)\,, \tag{2.4}$$

where s_Λ is the corresponding Lorentz transformation in the covering group SL(2, C) linked to Λ by [6]

$$s_\Lambda\, \tilde{\sigma}_\mu\, s_\Lambda^+ = \Lambda_\mu^\nu\, \tilde{\sigma}_\nu\,. \tag{2.5}$$

(For the σ-matrices we use the same conventions as in Ref. [2]; see footnote [2]) on page 226.)

According to (2.2) the unitary irreducible representations (UIR) of the full group G are direct products of the UIR of P^* and of SL(2, C).

b) *The UIR of P^**

The UIR of P^* are characterized by an invariant mass M, and a 'spin' which here amounts to an orbital angular momentum in the rest system of the particle. For the group P^* to be physically interpretable, there should be no such orbital angular momentum [2], so we restrict ourselves to the case of 'scalar' representations for P^*. The basis vectors $\psi(p)$ are then eigenstates of the energy momentum operator P_μ,

$$P_\mu \psi(p) = p_\mu \psi(p), \quad p_\mu(\omega_p, \boldsymbol{p}), \quad \omega_p = \boldsymbol{p}^2 + M^2. \tag{2.6}$$

The unitary transformations $U(l)$ of P^* are defined on the function space spanned by them and given by [6]

$$U[l(L, a)]\, \psi(p) = e^{i L^{-1} p a}\, \psi(L^{-1} p). \tag{2.7}$$

These representations are unitary with respect to the scalar product

$$(\varphi, \psi) = \int d\Omega(p)\, \varphi^*(p)\, \psi(p); \quad d\Omega(p) = d^3p\, \frac{M}{\omega_p}.$$

c) *The UIR of* SL(2, C)

For the UIR of SL(2, C), we restrict ourselves to the socalled principal series of Gelfand and Naimark [5]. We make use of the formulation using basis vectors labelled by complex spinor variables ζ_α ($\alpha = 1, 2$) [1]. The basis vectors $F_{m,\varrho}(\zeta)$ are homogeneous functions of ζ_α with a degree of homogeneity characterized by an integer m and a real parameter ϱ:

$$F_{m,\rho}(\lambda\, \zeta) = \lambda^{-m/2 + i\varrho/2 - 1}\, \lambda^{*m/2 + i\varrho/2 - 1}\, F_{m,\rho}(\zeta). \tag{2.8}$$

The unitary representations $U(s)$ of SL(2, C) are then defined in the space of these functions by

$$U(s)\, F_{m,\rho}(\zeta) = F_{m,\rho}(\zeta\, s); \quad (\zeta\, s)_\alpha = \zeta_\beta\, s^\alpha_\beta. \tag{2.9}$$

These representations are unitary with respect to the principal series scalar product

$$(F, G) = \int d\mu(\zeta)\, F^*_{m,\varrho}(\zeta)\, G_{m,\rho}(\zeta); \quad d\mu(\zeta) = \delta^{(2)}(\zeta_2 - 1)\, d\zeta$$

$d\zeta$ stands for the product of the four differentials of the real and imaginary parts of ζ_1 and ζ_2.

d) *The UIR of G*

The UIR of the full symmetry group can then be constructed as direct products of 'scalar' representations of P^* and appropriate representations (m, ϱ) of SL(2, C),

$$U(g) = U(l)\, U(s) = U(s)\, U(l),$$

and are defined over the product space spanned by the vectors

$$\phi_{m,\rho}(\zeta, p) = F_{m,\rho}(\zeta)\, \psi(p)\,. \tag{2.10}$$

For these states the transformation formula for the special case of a physical Lorentz transformation reads

$$U(\Lambda, a)\, \phi_{m,\rho}(\zeta, p) = e^{i\Lambda^- p a}\, \phi_{m,\rho}(\zeta\, s_\Lambda, \Lambda^{-1} p) \tag{2.11}$$

in virtue of the decomposition

$$U(\Lambda, a) = U[l(\Lambda, a)]\, U(s_\Lambda) = U(s_\Lambda)\, U[l(\Lambda, a)]\,. \tag{2.12}$$

3. The Physical States

We now specify the representations to be associated with the particles of our model. Since there is no internal degree of freedom in a theory with SL(2, C) symmetry, we restrict ourselves to a set of particles all having the same internal quantum numbers like charge, strangeness etc. In the framework of this model, an irreducible representation of the group contains an infinite ladder of particles with spin values

$$s = 1/2\,|\,m\,|, \quad 1/2\,|\,m\,| + 1, \quad 1/2\,|\,m\,| + 2, \ldots$$

where m is the integer characteristic of the representation of SL(2, C). If this ladder is to contain a certain spin, an appropriate choice for $|m|$ has to be made. Since the representations (m, ϱ) and $(-m, -\varrho)$ are equivalent, the sign of m is irrelevant. The second variable, ϱ, is determined by the requirement that there shall exist a parity operator which maps the representation onto itself. RÜHL [7] has shown that this leads to the restriction by which only representations with $\varrho = 0$ are considered.

a) *The Baryon*

In order to have a spin 1/2 particle (briefly called baryon) in the SL(2, C) representation involved with G, we chose $|m| = 1$, viz., most conveniently in the present case, $m = -1$. The representation of G associated with the baryon in this model is therefore the direct product of a P^* representation with the SL(2, C) representation B $(m = -1, \varrho = 0)$. The corresponding state vector will be denoted by

$$\overset{B}{\phi}(\zeta, k)\,.$$

The physical content of this representation of G is found by reducing it with respect to the physical Poincaré-group P defined in (2.1) and (2.12). $\overset{B}{\phi}(\zeta, k)$ contains infinitely many representations of P, all with the same mass M but different spins $s = 1/2, 3/5, 5/2, \ldots$. This reduction is easily carried out with the help of the generating functions introduced in [1]. Those vectors among $\overset{B}{\phi}(\zeta, k)$ which form the SU(2) representation $D^{(1/2)}$ in the little group which belongs to the fixed momentum $p_0(M, \mathbf{0})$ are given by

$$\overset{B}{\phi}(\zeta, k; \frac{1}{2}, \lambda, p_0) = \left(\frac{2}{\pi}\right)^{1/2} \zeta_\lambda(\zeta\, \tilde{\sigma}_0\, \zeta^+)^{-3/2}\, \delta^{(3)}\,(\boldsymbol{k} - \boldsymbol{p}_0) \tag{3.1}$$

$\lambda = 1, 2$ labels the spin states of the particle in the rest system, $(\zeta\, \tilde{\sigma}_0\, \zeta^+) = \zeta_\alpha\, \tilde{\sigma}_0^{\alpha\dot{\beta}}\, \zeta_{\dot{\beta}}$ is a SU(2) invariant, the power of which is determined by the invariance condition under transformations of SU(2). (3.1) displays explicitely the transformation properties of the representation $D^{(1/2)}$.

The full irreducible spin 1/2 representation of the physical Poincaré group P contained in our representation $\overset{B}{\phi}(\zeta, k)$ of G is then found by boosting (3.1) to the momentum $p(\omega_p, \boldsymbol{p})$:

$$\overset{B}{\phi}(\zeta, k; \frac{1}{2}\,\lambda, p) = U(\Lambda_p)\,\overset{B}{\phi}(\zeta, k; \frac{1}{2}\,\lambda, p_0)\,. \tag{3.2}$$

Λ_p is a rotation-free physical Lorentz transformation (2.12) which leads from the momentum p_0 of the little group to the general momentum p. The associated SL(2, C) boost s_r satisfies the Equation [6]

$$s_r\, \tilde{\sigma}_0\, s_r^+ = \frac{p_\mu\, \tilde{\sigma}^\nu}{M} = \frac{\tilde{p}}{M}\,. \tag{3.3}$$

Taking (2.11) and (3.3) into account, one finds for (3.2)

$$\overset{B}{\phi}(\zeta, k; \frac{1}{2}\,\lambda, p) = \left(\frac{2}{\pi}\right)^{1/2} \chi(p, \lambda)\, \zeta_\alpha \left(\zeta\, \frac{\tilde{p}}{M}\, \zeta^+\right)^{-3/2} \frac{\omega_p}{M}\, \delta^{(3)}\, (\boldsymbol{k} - \boldsymbol{p})\,. \tag{3.4}$$

These vectors represent the model baryon moving with momentum p and with either of the two independent spin states labelled by λ. The baryon two component spinor $\chi^\alpha(p, \lambda)$ is constructed in the usual way with the help of the boost matrix s_p:

$$\chi^\alpha(p, \lambda) = s^\alpha_{p\,\lambda}\,. \tag{3.5}$$

The states (3.4) are orthonormalised according to

$$\int d\mu(\zeta)\, d\Omega(k)\, \overset{B}{\phi}{}^*(\zeta, k; \frac{1}{2}\,\lambda_1, p_1)\, \overset{B}{\phi}(\zeta, k; \frac{1}{2}\,\lambda_2, p_2) = \delta_{\lambda_1 \lambda_2}\, \frac{\omega_p}{M}\, \delta^{(3)}\, (\boldsymbol{p}_1 - \boldsymbol{p}_2)\,. \tag{3.6}$$

b) *The Antibaryon*

The antibaryon is best described by the complex conjugate representation of the one used above for the baryon, i.e. by $\overline{B}$ $(m = 1, \varrho = 0)$. The reduction of this representation onto the physical spin 1/2 representation of the Poincaré group P can be carried out in analogy to the method above. The state vector of the moving antiparticle is then given by

$$\overset{\overline{B}}{\phi}(\zeta, k; \frac{1}{2}\,\lambda, p) = \left(\frac{2}{\pi}\right)^{1/2} \psi^{\dot{\alpha}}(p, \lambda)\, \zeta^{\dot{\alpha}} \left(\zeta\, \frac{\tilde{p}}{M}\, \zeta^+\right)^{-3/2} \frac{\omega_p}{M}\, \delta^{(3)}\, (\boldsymbol{k} - \boldsymbol{p})\,. \tag{3.7}$$

The spinor of the antiparticle, $\psi^{\dot{\alpha}}(p, \lambda)$, is defined by the complex conjugate boost matrix

$$\psi^{\dot{\alpha}}(p, \lambda) = s^{\dot{\alpha}}_{p\,\dot{\lambda}}\,. \tag{3.8}$$

c) *The Vector Meson*

In order to contain a spin 1 particle (briefly called vector meson), the representation of G to be used should be characterized by the values $m = 0$ or $|m| = 2$ of the homogeneity variable. Let us chose $|m| = 2$, such as to make the smallest SU(2) representation contained in it to be $D^{(1)}$, and for the sake of convenience let us restrict to $m = -2$. ϱ is again put equal to 0. This meson representation will be denoted by

$$\overset{M}{\phi}(\zeta, p)\,.$$

It is reducible with respect to the physical Poincaré group P. The state vectors which span the representation $D^{(1)}$ of our model meson are again most easily constructed with the help of a generating function of Ref. [1]. In the rest system with $q_0(\mu, \mathbf{0})$ of the meson, the state vectors are given by

$$\overset{M}{\phi}(\zeta, k; 1, r, q_0) = \left(\frac{3}{\pi}\right)^{1/2} m^{\gamma}\,(q_0, r)\,\zeta_{\gamma}\,\zeta^{\delta}\,(\zeta\,\tilde{\sigma}_0\,\zeta^{+})^{-2}\,\delta^{(3)}\,(\boldsymbol{k} - \boldsymbol{q}_0)\,. \tag{3.9}$$

The transformation property is directly visible; $r = 1, 2, 3$ labels the three independent spin states of the particle. The vector meson matrix introduced in (3.9) is defined by

$$m^{\gamma}_{\delta}(q_0, r) = \varepsilon^{\mu}(q_0, r)\,e^{\gamma}_{\mu\,\delta}(q_0) \qquad e^{\gamma}_{\mu\,\delta}(q_0) = \sigma_{0\,\mu\,\delta}^{\;\;\gamma}\,\delta\,, \quad \varepsilon^{\mu}(q_0, r) = \delta^{\mu}_{r}\,.$$

(Clearly $e_{0\,\delta}^{\,\gamma}(q_0) = 0$ as a consequence of the definition $\sigma_{\mu\nu} = 1/2\,(\tilde{\sigma}_{\mu}\,\sigma_{\nu} - \tilde{\sigma}_{\nu}\,\sigma_{\mu})$.)

The full irreducible spin 1 representation of P, which can be interpreted as the moving meson, is found by boosting (3.9) to the momentum q. Using again (3.3), one finds (instead of the meson mass μ we introduce the energy variable $s = q^2$):

$$\overset{M}{\phi}(\zeta, k; 1, r, q) = \left(\frac{3}{\pi}\right)^{1/2} m^{\gamma}\,\delta\,(q, r)\,\zeta_{\gamma}\,\zeta^{\delta}\left(\zeta\,\frac{\tilde{q}}{\sqrt{s}}\,\zeta^{+}\right)^{-2}\frac{\omega_q}{\sqrt{s}}\,\delta^{(3)}\,(\boldsymbol{k} - \boldsymbol{q})\,, \tag{3.10}$$

where the new meson wave function now reads

$$m^{\gamma}_{\delta}(q, r) = \varepsilon^{\mu}(q, r)\,e^{\gamma}_{\mu\,\delta}(q) \qquad e^{\gamma}_{\mu\,\delta}(q) = (s_q^T\,\sigma_{0\,\mu}\,\varepsilon\,s_q\,\varepsilon^T)^{\gamma}_{\delta} = \sigma^{\gamma}_{\mu\nu\delta}\,\frac{q^{\nu}}{\sqrt{s}} \tag{3.11}$$

(τ stands for transposition, ε is the SL(2, C)-metric[2]).)

The matrix $e_{\mu}{}^{\gamma}_{\delta}(q)$ transforms the original SU(2) basis labelled by γ and δ to a basis labelled by vectorindices in a cartesian coordinate system, whereas the polarisation vector $\varepsilon^{\mu}(q, r)$, ($r = 1, 2, 3$), combines out of it on orthogonal basis which corresponds to the three physical polarisation states of the moving meson.

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[2]) Notation: Greek indices from the first half of the alphabet label components of SL(2, C) spinors and run over two values: $\alpha\,\beta, \dot{\alpha}, \dot{\beta} = 1, 2$. Indices from the second half of the alphabet label components of Lorentz four vectors and run over four values: $\mu, \nu = 0, 1, 2, 3$.

Metric in spin space: $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot{\alpha}\dot{\beta}}$; $\varepsilon^{12} = \varepsilon^{\dot{1}\dot{2}} = \varepsilon_{12} = \varepsilon_{\dot{1}\dot{2}} = 1$.

σ-matrices: $\sigma_{\mu\,\dot{\alpha}\beta} = (1, \tau)_{\dot{\alpha}\beta}$, $\tilde{\sigma}_{\mu}^{\alpha\dot{\beta}} = (\varepsilon\,\sigma_{\mu}\,\varepsilon^{+})^{\alpha\dot{\beta}} = (1, -\tau)^{\alpha\dot{\beta}}$ (τ are the usual Pauli matrices).

Spinor indices are as often as possible suppressed. Scalar contractions are put in parentheses, e.g.

$$(M\,\sigma_{\mu\nu}\,\tilde{N}) = M_{\dot{\alpha}\beta}\,\sigma^{\beta}_{\mu\nu\gamma}\,\tilde{N}^{\gamma\dot{\alpha}}.$$

Contractions of four-vectors with σ-matrices are written as SL(2, C)-tensors; e.g.

$$\tilde{p}:\ \tilde{p}^{\alpha\dot{\beta}} = p^{\mu}\,\tilde{\sigma}_{\mu}^{\alpha\dot{\beta}};\quad p:\ p_{\dot{\alpha}\beta} = p_{\mu}\,\sigma^{\mu}_{\dot{\alpha}\beta}\,.$$

Metric in Minkowski space: $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

The antisymmetric Levi-Cività tensor $\varepsilon^{\mu\nu\varrho\sigma}$ is defined by $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.

4. The Baryon Meson Vertex

We shall now analyse the baryon meson vertex with the help of the symmetry group G, i.e. with the help of the particle states constructed in Section 3. These states were generated by boosts out of the states of the little group and hence describe real particles on the mass shell. Thus they can only be used in vertices where all particles participate with timelike momenta. This is the case for the production channel of the baryon meson vertex, $M \to B + \bar{B}$, upon which we restrict ourselves.

Let $p_{1\mu}$ and λ_1 denote the physical four-momentum and spin of the baryon, $p_{2\mu}$ and λ_2 the corresponding variables of the antibaryon and q_μ and r those of the meson, and call the homogeneous variables and the momenta involved in the representations of G ξ and $k_{1\mu}$ in the case of the baryon, η and $k_{2\mu}$ in the case of the antibaryon and ζ and q_μ in the case of the meson. Then the problem consists in finding a coupling V for the representations $\overset{B}{\phi}(\xi, k_1; 1/2, \lambda_1, p_1)$, $\overset{\bar{B}}{\phi}(\eta, k_2; 1/2, \lambda_2, p_2)$ and $\overset{M}{\phi}(\zeta, k; 1, r, q)$ in such a way that this coupling transforms according to the trivial one-dimensional representation of G. This is achieved by means of a Clebsch Gordan kernel $K(\xi, k_1; \eta, k_2 \mid \zeta, q)$:

$$V(p_1, \lambda_1; p_2, \lambda_2 \mid q, r) = \int d\Omega(k_1)\, d\Omega(k_2)\, d\Omega(k)\, d\mu(\zeta)\, d\mu(\eta)\, d\mu(\zeta)$$
$$\times \overset{B}{\phi}\left(\xi, k_1; \frac{1}{2}\lambda_1, p_1\right) \overset{M}{\phi}\left(\eta, k_2; \frac{1}{2}\lambda_2, p_2\right) \overset{M}{\phi}(\zeta, k; 1, r, q)\, K(\xi, k_1; \eta, k_2 \mid \zeta, k)\,. \tag{4.1}$$

The group structure (2.2) indicates that K splits up into a product of a kernel in the unphysical Poincaré group P^* and akernel in SL(2, C),

$$K(\xi, k_1; \eta, k_2 \mid \zeta, k) = G(k_1, k_2 \mid k)\, H(\xi, \eta \mid \zeta)\,. \tag{4.2}$$

The usual invariance requirements imply for G

$$G(k_1, k_2 \mid k) = F(s)\, \delta^{(4)}(k - k_1 - k_2)\,; \qquad s = k^2 = (k_1 + k_2)^2\,,$$

where $F(s)$ is the undetermined form factor of this theory, whereas for H the invariance condition

$$H(\xi\, s, \eta\, s \mid \zeta\, s) = H(\xi, \eta \mid \zeta)$$

can be satisfied by assuming a product of the powers of all the possible SL(2, C) invariants. This leads (see general formula in [1]) to:

$$H(\xi, \eta \mid \zeta) = (\xi^+ \varepsilon^+ \eta^+)^{-1} (\zeta\, \varepsilon\, \xi)^{-1/2} (\xi^+ \varepsilon^+ \zeta^+)^{-1/2} (\zeta\, \varepsilon\, \eta)^{-3/2} (\eta^+ \varepsilon^+ \zeta^+)^{1/2}\,. \tag{4.4}$$

Inserting the explicite states (3.4), (3.7) and (3.10) and carrying out the trivial 3-dimensional momentum integrations, one finds for V from (4.1)

$$V(p_1, \lambda_1; p_2, \lambda_2 \mid q, r) = \chi^{\dot{\alpha}}(p_1, \lambda_1)\, \psi^\beta(p_2, \lambda_2)\, m^\gamma_\delta(q, r)$$
$$\times\, \delta^{(4)}(q - p_1 - p_2)\, J_{\dot{\alpha}\beta\gamma}^{\ \ \ \ \delta}(p_1, p_2, q) \tag{4.5}$$

with the characteristic integral

$$J_{\dot{\alpha}\beta\gamma}^{\ \ \ \ \delta} = s\, M^3 \int d\mu(\xi)\, d\mu(\eta)\, d\mu(\zeta)\, \xi_{\dot{\alpha}}\, \eta_\beta\, \zeta_\gamma\, \zeta^\delta (\xi\, \tilde{p}_1\, \xi^+)^{-3/2}$$
$$\times\, (\eta\, \tilde{p}_2\, \xi^+)^{-3/2} (\zeta\, \tilde{q}\, \zeta^+)^{-2} H(\xi, \eta \mid \zeta)\,. \tag{4.6}$$

Here, constant factors (like powers of π) have been absorbed in the undetermined function $F(s)$.

The integral (4.6) can be written in a more symmetric way which helps towards a much easier handling of the next steps. This transcription will bring the usefullness of the homogeneous variables into the right light. Although the method is explained at length in [1], let us repeat some of the arguments.

Consider the integral

$$I = \int d\mu(\xi)\, f(\xi)\, (\xi \tilde{N} \xi^+)^{-k/m} \qquad (k, m \text{ integer} \neq 0)\,.$$

The typical denominator occuring in it can be transferred to an exponential by making use of a Gaussian integral

$$I = \int d\mu(\xi)\, f(\xi)\, (\xi \tilde{N} \xi^+)^{-k/m} = \frac{m}{\Gamma(k/m)} \int d\mu(\xi) \int_0^\infty d\alpha\, e^{-\alpha^m (\xi \tilde{N} \xi^+)}\, \alpha^{k-1}\,. \tag{4.7}$$

Furthermore we can get rid of the measure condition $d\mu(\xi) = d\xi\, \delta^{(2)}\, (\xi_2 - 1)$ which constrains the integration to run effectively only over the first (complex) component z of the homogeneous spinor ξ and keeps the second component to be one. Let us first assume that $f(\xi)$ in (4.7) is homogeneous of the degree n in ξ

$$f(a\,\xi) = a^n\, f(\xi) \qquad (a \text{ complex, } n \text{ integer})$$

We then carry out in (4.7) the integration over the second component of ξ and multiply artificially with a constant factor $2\, m\, \pi$ written in the form of a phase integral

$$I = C \int dz \int_0^\infty d\alpha \int_0^{2\pi} d\varphi\, e^{-\alpha^m (\xi_0 \tilde{N} \xi_0^+)}\, f(\xi_0)\, \alpha^{k-1}\, e^{i m \varphi} \tag{4.8}$$

ξ_0 stands for the spinor with the components $(z, 1)$, C contains numerical factors. (4.8) suggests to introduce a new spinor ξ defined by

$$\xi = \alpha^{m/2}\, e^{i\varphi}\, \xi_0\,.$$

With this, the integral can be expressed as

$$I = C \int dz\, d\alpha\, d\varphi\, e^{-\xi \tilde{N} \xi^+}\, f(\xi)\, \alpha^{k-n-1}\,. \tag{4.9}$$

It is now easy to transform the differential $dz\, d\alpha\, d\varphi$ over the four real numbers $Re(z)$, $Im(z)$, α and φ into the symmetric differential $d\xi$ over the variables $Re(\xi_1)$, $Im(\xi_1)$, $Re(\xi_2)$ and $Im(\xi_2)$. The jacobian of this transformation is

$$\frac{m}{2}\, a^{2m-1}$$

and depends only on the variable m. Since m can always be choosen in a way such that α^{k-n-1} in (4.9) equals the jacobian necessery to transform the differentials, we have then

$$I = C \int d\xi\, e^{-\xi \tilde{N} \xi^+}\, f(\xi)\,.$$

Such manipulations are now used for the transcription of the first three denominators in (4.6) and bring the integral to the simple symmetric form

$$J_{\dot{\alpha}\beta\gamma}^{\ \ \delta} = s\, M^3\, F(s) \int d\xi\, d\eta\, d\zeta\, \xi_{\dot{\alpha}}\, \eta_\beta\, \zeta_\gamma\, \zeta^\delta\, H(\xi, \eta \mid \zeta)\, e^{-\xi \tilde{p}_1 \xi^+ - \eta \tilde{p}_2 \eta^+ - \zeta \tilde{q} \zeta^+}. \tag{4.10}$$

For a moment we turn back to (4.5) and keep (3.11) for the meson wave function $m_\delta^\gamma(q, r)$ in mind. The coefficient of the polarisation vector $\varepsilon_\mu(q, r)$ therein defines the matrix element of the meson current,

$$K_\mu(p_1, p_2, q) = \chi^{\dot{\alpha}}(p_1, \lambda_1)\, \psi^\beta(p_2, \lambda_2)\, \sigma_{\mu\nu\delta}^{\ \ \gamma}\, \frac{q^\nu}{\sqrt{s}}\, \delta^{(4)}(q - p_1 - p_2)\, J_{\dot{\alpha}\beta\gamma}^{\ \ \delta}(p_1, p_2, q). \tag{4.11}$$

This matrix element is very closely related to the form factors as will be shown in Section 6. If the integral (4.10) iscomputed, the form factors are in principle known up to the free function $F(s)$. We shall therefore be concerned next with the calculation of this integral.

5. Reduction of the Integral

The aim of this section is to show in some detail how the integral (4.10) over 12 real variables can be reduced to one over only two variables. We start with a new, SL(2, C)-scalar integral

$$J(A) = F(s) \int d\xi\, d\eta\, d\zeta\, e^{-\xi \tilde{p}_1 \xi^+ - \eta \tilde{p}_2 \eta^+ - \zeta \tilde{q} \zeta^+}\, H'_A(\xi, \eta \mid \zeta)$$

$$H'_A(\xi, \eta \mid \zeta) = [(\xi\, \varepsilon\, \eta)\, (\eta^+\, \varepsilon^+\, \xi^+)]^{-1}\, [(\xi\, \varepsilon\, \zeta)\, (\zeta^+\, \varepsilon^+\, \xi^+)]^{-1/2}\, [(\eta\, A\, \eta)\, (\zeta^+\, A^+\, \eta^+)]^{1/2}. \tag{5.1}$$

A is an arbitrary SL(2, C)-matrix introduced in such a way, that differentiation with respect to its elements gives (4.10) again:

$$J_{\dot{\alpha}\beta\gamma}^{\ \ \delta}(p_1, p_2, q) = s\, M^3\, \frac{\delta}{\delta \tilde{p}_1^{\,i\dot{\alpha}}}\, \frac{\delta}{\delta A^{\varkappa\gamma}}\, \frac{\delta}{\delta A^{\beta\lambda}}\, J(\varepsilon)\, \varepsilon^{\lambda\delta}\, \varepsilon^{i\varkappa}. \tag{5.2}$$

After differentiation, A is replaced by ε. Note that some constant numbers have been – and will again be – absorbed in $F(s)$.

a) ξ-*Integration*

It has proved to be most convenient to start with the ξ-integration. That part of (5.1) which contains the ξ-spinors is

$$X = F(s) \int d\xi\, e^{-\xi \tilde{p}_1 \xi^+}\, [(\xi\, \varepsilon\, \eta)\, (\eta^+\, \varepsilon^+\, \xi^+)]^{-1}\, [(\xi\, \varepsilon\, \zeta)\, (\zeta^+\, \varepsilon^+\, \xi^+)]^{-1/2}. \tag{5.3}$$

By making use again of (4.7) the denominators may be brought to exponentials and the ξ-integration can be done:

$$X = F(s) \int d\xi \int_0^\infty d\mu\, d\nu\, e^{-\xi \tilde{N} \xi^+} = F(s) \int d\mu\, d\nu\, \frac{1}{\det N}$$

$$\tilde{N} = \tilde{p}_1 + \mu\, \varepsilon\, \eta\, \eta^+\, \varepsilon^+ + \nu^2\, \varepsilon\, \zeta\, \zeta^+\, \varepsilon^+.$$

The determinant of the matrix $\tilde{N}$ appearing here can be computed most generally and yields

$$\det N = p_1^2 \left[1 + \mu(\eta^+ \varepsilon^+ p_1^{-1} \varepsilon \eta) + \nu^2(\zeta^+ \varepsilon^+ p_1^{-1} \varepsilon \zeta) + \mu \nu^2 \Delta\right]$$

$$\Delta = (\eta^+ \varepsilon^+ p_1^{-1} \varepsilon \eta \zeta^+ \varepsilon^+ p_1^{-1} \varepsilon \zeta) - (\eta^+ \varepsilon^+ p_1^{-1} \varepsilon \zeta \zeta^+ \varepsilon^+ p_1^{-1} \varepsilon \eta) .$$

Because of $\varepsilon^+ p_1^{-1} \varepsilon = p_1^T / p_1^2$ ($_1^T$ stands for transposition), we get $\triangle = (\eta \varepsilon \zeta)(\zeta^+ \varepsilon^+ \eta^+)$ and

$$\det N = p_1^2 + \mu(\eta \tilde{p}_1 \eta^+) + \nu(\zeta \tilde{p}_1 \zeta^+) - \mu \nu^2 p_1^2 (\eta \varepsilon \zeta)(\zeta^+ \varepsilon^+ \eta^+) .$$

(Only after differentiation with respect to p_1, but not before, we shall put $p_1^2 = M^2$.)

Next, the ν-integration must be done, else an immediate μ-integration would lead to divergent constants. The result is, after using (4.7) again for a denominator:

$$X = F(s) \left(\frac{p_1^2}{\eta \tilde{p}_1 \eta^+}\right)^{1/2} \int \frac{d\mu}{\sqrt{1+\mu}} \int d\varrho \, e^{-\varrho^2[(\eta \tilde{p}_1 \eta^+)(\zeta \tilde{p}_1 \zeta^+) + \nu p_1^2 (\eta \varepsilon \zeta)(\zeta^+ \varepsilon^+ \zeta^+)]} .$$

b) *ζ-Integration*

Because of the additional factor $(\eta \tilde{p}_1 \eta^+)^{-1/2}$ we do not proceed with the η^-, but with the ζ-integration. The remaining terms in (5.1) containing ζ-spinors are

$$Z = \int d\zeta \, e^{-\zeta \tilde{q} \zeta^+} [(\eta A \zeta)(\zeta^+ A^+ \eta^+)]^{1/2} X .$$

The factor $B = [(\eta A \zeta)(\zeta^+ A^+ \eta^+)]$ can be brought to an exponential with the help of a formula analogous to (4.7), viz.

$$B^{1/2} = -\frac{1}{2\sqrt{\pi}} \int^{\Omega} e^{-\lambda^2 B} \frac{d\lambda}{\lambda^2} ,$$

where the path of integration Ω along the real λ-axis avoids the pole at $\lambda = 0$. With this the ζ-integration can be done:

$$Z = F(s) \left(\frac{p_1^2}{\eta \tilde{p}_1 \eta^+}\right)^{1/2} \int_0^\infty \frac{d\mu}{\sqrt{1+\mu}} \int^{\Omega} \frac{d\lambda}{\lambda^2} \int_0^\infty d\varrho \, \frac{1}{\det M} .$$

A matrix often to be encountered in the following is

$$\tilde{Q} = \tilde{q} + \varrho^2 \nu \tilde{p}_1 .$$

With it, the matrix M can be written in the form

$$\tilde{M} = \tilde{Q} + \varrho^2 \nu p_1^2 \varepsilon \eta \eta^+ \varepsilon^+ + \lambda^2 A^\tau \eta \eta^+ A^{\tau+} .$$

Its determinant does not simplify before A can be put equal to ε. Therefore we now carry out the differentiation with respect to A. Using

$$\frac{\delta}{\delta A^{\varkappa\lambda}} \frac{\delta}{\delta A^{\beta\gamma}} \frac{1}{\det M} = \frac{1}{\det M} \left(\tilde{M}^{-1\mu\dot{\alpha}} \tilde{M}^{-1\nu\dot{\delta}} + \tilde{M}^{-1\mu\dot{\delta}} \tilde{M}^{-1\gamma\dot{\alpha}}\right) \frac{\delta M_{\dot{\alpha}\mu}}{\delta A^{\varkappa\lambda}} \frac{\delta M_{\dot{\gamma}\nu}}{\delta A^{\beta\gamma}} ,$$

one finds in the limit $A = \varepsilon$

$$M = Q + (\lambda^2 + \varrho^2 \nu\, p_1^2)\, \varepsilon\, \eta\, \eta^+\, \varepsilon^+$$

$$\det M = Q^2 + (\lambda^2 + \varrho^2 \nu\, p_1^2)\, (\eta\, \tilde{Q}\, \eta^+)$$

$$Q^2 = Q_\mu\, Q^\mu = q^2 + 2\, \varrho^2 \nu\, (q\, p_1) + \varrho^4 \nu^2\, p_1^2\,.$$

With this the original integral now reads

$$J_{\dot{\alpha}\beta\gamma}^{\ \ \ \delta} = s\, M^\delta\, F(s)\, \frac{\delta}{\delta \tilde{p}_1^{\varkappa\dot{\alpha}}} \int d\eta \left(\frac{p_1}{\eta\, \tilde{p}_1\, \eta^+}\right)^{1/2} e^{-\eta\, \tilde{p}_2\, \eta^+}$$

$$\times \int_0^\infty \frac{d\mu}{\sqrt{1+\mu}}\, \lambda^2\, d\lambda\, d\varrho\, [Q^2 + (\lambda^2 + \varrho^2 \nu\, p_1^2)\, (\eta\, \tilde{Q}\, \eta^+)]^{-3}\, \eta^\varkappa\, \eta_{\beta\ \gamma}(\varepsilon\, \tilde{Q}\, \eta^+)\ ^\delta(\tilde{Q}\, \eta^+)\,.$$

The substitutions

$$\varrho(\eta\, \tilde{p}_1\, \eta^+) = \sqrt{\omega}\,, \qquad \chi \left(\frac{\eta\, \tilde{p}_1\, \eta^+}{p_1^2}\right)^{1/2} = \nu$$

enable one to carry out firstly the ν-integration (which contributes only with a constant number) and secondly the μ-integration. The result is

$$J_{\dot{\alpha}\beta\gamma}^{\ \ \ \delta} = s^{3/2}\, M^3\, F(s)\, \frac{\delta}{\delta \tilde{p}_1^{\varkappa\dot{\alpha}}} \int d\eta\, \frac{d\omega}{\omega}\, (\eta\, \tilde{p}_1\, \eta^+)^{1/2}\, e^{-\eta\, \tilde{p}_2\, \eta^+}\, (\eta\, \tilde{Q}\, \eta^+)^{-2}$$

$$\frac{\sqrt{H}-1}{G}\, \eta^\varkappa\, \eta^\beta_{\ \ \gamma}(\varepsilon\, Q\, \eta^+)\ ^\delta(\tilde{Q}\, \eta^+) \qquad (5.5)$$

with the new variables

$$\tilde{Q} = \tilde{q} + \omega\, \tilde{p}_1$$

$$H = p_1^2\, \omega\, \frac{(\eta\, \tilde{Q}\, \eta^+)}{Q^2 (\eta\, \tilde{p}_1\, \eta^+)}$$

$$G = p_1^2\, \omega (\eta\, \tilde{Q}\, \eta^+) - Q^2 (\eta\, \tilde{p}_1\, \eta^+)\,.$$

c) *Derivative with Respect to* $\tilde{p}_1$

For the remaining differentiation the following formula is useful

$$\frac{\delta}{\delta \tilde{p}_{\alpha\dot{\beta}}} = \frac{\delta p^\mu}{\delta \tilde{p}_{\alpha\dot{\beta}}}\, \frac{\partial}{\partial p^\mu} = \frac{1}{2}\, \sigma^\mu_{\dot{\beta}\alpha}\, \frac{\partial}{\partial p^\mu}\,.$$

The resulting expression for (5.5) is (after differentiation, p_1^2 can be put equal to M^2 again):

$$J_{\dot{\alpha}\beta\gamma}^{\ \ \ \delta} = s^4\, M^3\, F(s) \int d\eta\, d\omega (\eta\, \tilde{p}_1\, \eta^+)^{1/2}\, e^{-\eta\, \tilde{p}_2\, \eta^+} (\eta\, \tilde{Q}\, \eta^+)^{-2}$$

$$[R_1\, M_{\dot{\alpha}\beta\gamma}^{\ \ \ \delta} + R_2\, N_{\dot{\alpha}\beta\gamma}^{\ \ \ \delta} + R_3\, Q_{\dot{\alpha}\beta\gamma}^{\ \ \ \delta}] \qquad (5.6a)$$

$$R_1 = \frac{\sqrt{H}-1}{G}$$

$$R_2 = (\eta\, \tilde{p}_1\, \eta^+)\, \frac{1/2\, \sqrt{H}\,(3-H)-1}{G^2}$$

$$R_3 = (\eta\, \tilde{Q}\, \eta^+)\, \frac{1-1/2\,(\sqrt{H}+1/\sqrt{H})}{G^2} \tag{5.6b}$$

$$M_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta} = \eta_{\dot\alpha}\, \eta^\beta\, [-\,\eta_\gamma{}^\delta (Q\,\eta^+) + {}_\gamma(\varepsilon\, \tilde{Q}\, \eta^+)\, {}^\delta(\varepsilon\,\eta)]$$

$$N_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta} = (\eta\, Q\, \varepsilon^+)_{\dot\alpha}\, \eta_\beta\, {}_\gamma(\varepsilon\, \tilde{Q}\, \eta^+)\, {}^\delta(Q\,\eta^+)$$

$$Q_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta} = (\eta\, \tilde{p}_1\, \varepsilon^+)_{\dot\alpha}\, \eta_\beta\, {}_\gamma(\varepsilon\, \tilde{Q}\, \eta^+)\, {}^\delta(Q\,\eta^+)\,. \tag{5.6c}$$

d) *Reduction of σ-Products*

According to (4.11) we are interested in the meson current and therefore form the quantity

$$J_{\dot\alpha\beta\mu} = J_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta}\, \sigma_{\mu\nu\delta}^{\ \ \ \gamma}\, \frac{q^\nu}{\sqrt{s}}\,. \tag{5.7}$$

The tensors $M_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta}$, $N_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta}$ and $O_{\dot\alpha\beta\gamma}^{\ \ \ \ \delta}$ lead, after contraction with $\sigma_{\mu\nu}$, to expressions with an odd number of σ-matrices. Such products can be reduced with the help of the formula

$$\sigma_\varrho\, \tilde{\sigma}_\mu\, \sigma_\nu = g_{\varrho\mu}\, \sigma_\nu + g_{\mu\nu}\, \sigma_\varrho - g_{\varrho\nu}\, \sigma_\mu - i\, \varepsilon_{\varrho\mu\nu\tau}\, \sigma^\tau \tag{5.8}$$

to a sum of terms containing only one σ-matrix. One finds

$$J_{\dot\alpha\beta\mu} = s^{3/2}\, M^3\, F(s) \int d\eta\, d\omega (\eta\, \tilde{p}_1)^{1/2}\, e^{-(\eta\, p_2)} (\eta\, Q)^{-2}$$
$$\times\, \eta_\varrho\, \eta_\sigma\, [R_1\, M_{\mu\nu}^{\varrho\sigma} + R_2\, N_{\mu\nu}^{\varrho\sigma} + R_3\, O_{\mu\nu}^{\varrho\sigma}]\, \sigma^\nu_{\dot\alpha\beta}\,. \tag{5.9}$$

Here we have introduced, besides of the spinors η used so far, the associated four null vectors

$$\eta_\varrho = \eta_\alpha\, \tilde{\sigma}_\varrho^{\alpha\dot\beta}\, \eta_{\dot\beta} = (\eta\, \tilde{\sigma}_\varrho\, \eta^+)\,. \tag{5.10}$$

Remember[2]) that greek indices from the first part of the alphabet label spinor components, whereas those from the latter part label the associated four vectors. $(\eta\, p_1)$ stands for the four product $\eta_\varrho\, p_1^\varrho$.

For the sake of brevity we do not give here the rather lengthy expression for the tensors $M_{\mu\nu}^{\varrho\sigma}$, $N_{\mu\nu}^{\varrho\sigma}$ and $O_{\mu\nu}^{\varrho\sigma}$ occuring in (5.9). We merely state that they do not contain the vectors η_ϱ but are built up by the physical momentum vectors of the vertex. In the last section we shall choose a special coordinante system in which these tensors simplyfy very much. Only this final simplified expression will be given explicitely.

e. *Integration over the four-vector η_e*

Consider the integral (5.9). Because of (5.10), an over all phase $e^{i\varphi}$ of the spinor η is unsensible for the four vector η_ϱ. Hence the phase integration can be done and contributes only a constant number. For the spinor differential there remain three real numbers (e.g. $Re(\eta_1)$, $Im(\eta_1)$ and $|\eta_2|$). Now this differential can be transcribed into a differential of the null vector η_ϱ: let us split from η_ϱ a constant factor Y such that the space components of the remaining vector form a unit three-vector,

$$\eta_\varrho = Y\, y_\varrho \begin{cases} y_0 = 1 \\ y_k = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)\,. \end{cases}$$

The new differential then contains the three real parameters Y, ϑ and φ in the combination

$$Y\, dY\, d\Omega_y$$

where $d\Omega_y$ indicates integration over the direction (ϑ, φ) of the unit vector y_k. Introducing this into (5.9) and carrying out the Y-integration with the help of (4.7) brings (5.9) into the form

$$J_{\dot\alpha\beta} = s^{3/2} M^3 F(s) \int d\Omega_y\, d\omega (y\,p_1)^{-1/2} (y\,p_2)^{-3/2} (y\,Q)^{-2} \times y_\varrho\, y^\sigma [R_1 M^{\varrho\sigma}_{\mu\nu} + R_2 N^{\varrho\sigma}_{\mu\nu} + R_3 O^{\varrho\sigma}_{\mu\nu}]\, \sigma^{\nu}_{\dot\alpha\beta}\,. \tag{5.11}$$

These are all the integrations which we could solve algebraically. Let us summarize the results of this section by giving the final expression for the meson current defined in (4.11):

$$K_\mu(p_1, p_2, q) = \delta^{(4)}(q - p_1 - p_2)\, T_{\mu\nu}(p_1, p_2, q)\, X^{\dot\alpha}(p_1, \lambda_1)\, \sigma^{\nu}_{\dot\alpha\beta}\, \psi^\beta(p_2, \lambda_2)\,, \tag{5.12a}$$

where

$$T_{\mu\nu}(p_1, p_2, q) = s^{3/2} M^3 F(s) \int d\Omega_y\, d\omega (y\,p_1)^{1/2} (y\,p_2)^{-3/2} (y\,Q)^{-2} \times y_\varrho\, y_\sigma [R_1 M^{\delta\sigma}_{\mu\nu} + R_2 N^{\varrho\sigma}_{\mu\nu} + R_3 O^{\varrho\sigma}_{\mu\nu}] \tag{5.12b}$$

$$R_1 = \frac{\sqrt{H} - 1}{G}$$

$$R_2 = (y\,p_1)\, \frac{1/2\sqrt{H}(3 - H) - 1}{G^2} \qquad H = \omega M^2 \frac{(y\,Q)}{Q^2 (y\,p_1)}$$

$$R_3 = (y\,Q)\, \frac{1 - 1/2\,(\sqrt{H} + 1/\sqrt{H})}{G^2} \qquad G = \omega M^2 (y\,Q) - Q^2 (y\,p_1)\,. \tag{5.12c}$$

Let the reader be reminded that the tensors $M^{\varrho\sigma}_{\mu\nu}$, $N^{\varrho\sigma}_{\mu\nu}$ and $O^{\varrho\sigma}_{\mu\nu}$ have not yet been defined explicitely and will be given only at the final stage of the calculations.

6. The Electromagnetic Form Factors

The meson current constructed in Section 5 is now to be related to the usual electromagnetic form factors. This is done by discussing the covariant transformation properties of such a current under the physical Poincaré group. A most general expression for K_μ can be constructed with the two independent momentum vectors of the vertex, the baryon and antibaryon momentum $p_{1\mu}$ and $p_{2\mu}$, or, more conveniently, with the orthogonal combinations

$$q_\mu = (p_1 + p_2)_\mu \quad \text{and} \quad k_\mu = (p_2 - p_1)_\mu\,, \quad (q\,k) = 0\,. \tag{6.1}$$

We require for the construction of K the following three conditions:

i) K_μ is a four vector under the orthochronous Lorentz group $L_+^\uparrow$,

ii) K_μ is the Fourier transform of a conserved current, i.e. $(q\,K) = 0$ (both baryon and antibaryon are in the final state!), and

iii) the matrix elements of K_μ are contracted with the Dirac four component spinors $u(p_1, \lambda_1)$ and $v(p_2, \lambda_2)$.

Note that we do not put a parity condition for the current, i.e. we leave the question open whether K is an axial or a polar vector. A consequence of this will be that we shall have to discuss four form factors. The appropriate parity can then be checked on the numerical results.

A convenient expression for K_μ satisfying (i) to (iii) is

$$K_\mu(p_1, p_2, q) = \bar{u}(p_1, \lambda_1) \Big[F_1(s)\,\gamma_\mu - F_2(s)\,\frac{\gamma_{\mu\nu}\,q^\nu}{2\,M} + F_3(s)\left(q_\mu - \frac{\gamma_\mu\,q^2}{2\,M}\right)\frac{\gamma_5}{2\,M} + F_4(s)\,\frac{k_\mu\,\gamma_5}{2\,M}\Big]\,v(p_2, \lambda_2)\,. \tag{6.2}$$

$F_1(s)$ and $F_2(s)$ are the well known form factors of the polar current, $F_3(s)$ and $F_4(s)$ are their analogues for the axial current. (The minus sign in front of the second term issues in the fact that both baryons are in the final state.)

In order to compare K_μ with (5.12), it must be reduced to a form which contains only two-component-spinors and σ-matrices. To this end we have to specify the representation of four-spinors and γ-matrices. We have choosen

$$u(p, \lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi^\alpha(p, \lambda)\\ \frac{p_{\dot\alpha\beta}}{M}\,\chi^\beta(p, \lambda)\end{pmatrix} \qquad v(p, \lambda) = \frac{1}{\sqrt{2}}\begin{pmatrix}\psi^\alpha(p, \lambda)\\ -\frac{p_{\dot\alpha\beta}}{M}\,\psi^\beta(p, \lambda)\end{pmatrix}$$

$$\bar{u}(p, \lambda) = u^+(p, \lambda)\,A$$

$$\gamma_\mu = \begin{pmatrix}0 & \tilde{\sigma}_\mu\\ \sigma_\mu & 0\end{pmatrix}, \qquad A = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \begin{array}{l}\gamma_{\mu\nu} = \frac{1}{2}\,(\gamma_\mu\,\gamma_\nu - \gamma_\nu\,\gamma_\mu)\\ \gamma_5 = i\,\gamma_0\,\gamma_1\,\gamma_2\,\gamma_3\,.\end{array}$$

With this identification and the formula (5.8) the reduction of (6.2) reads

$$\begin{aligned}K_\mu(p_1, p_2, q) = \frac{1}{2\,M^2}\Big[&F_1(s)\,(q^2\,g_{\mu\nu} - q^\mu\,q^\nu + k^\mu\,k^\nu - i\,\varepsilon_{\mu\nu\varrho\sigma}\,q^\varrho\,k^\sigma)\\ &+ F_2(s)\,(q^2\,g_{\mu\nu} - q_\mu\,q_\nu - i\,\varepsilon_{\mu\nu\varrho}\,\sigma\,q^\varrho\,k^\sigma)\\ &+ F_3(s)\left\{q_\mu\,q_\nu + \frac{q^2}{4\,M^2}\,((q^2 - 4\,M^2)\,g_{\mu\nu} - q_\mu\,q_\nu + k_\mu\,k_\nu - i\,\varepsilon_{\mu\nu\varrho\sigma}\,q^\varrho\,k^\sigma)\right\}\\ &+ F_4(s)\,k_\mu\,q_\nu\Big]\,\chi^{\dot\alpha}(p_1, \lambda_1)\,\sigma^\nu_{\dot\alpha\beta}\,\psi^\beta(p_2, \lambda_2)\,.\end{aligned} \tag{6.3}$$

Upon comparing (6.3) with (5.12) the electromagnetic form factors can be projected out of the integral:

$$F_1(s) = \left[\left(\frac{1}{4} - \frac{M^2}{k^2}\right) g^{\mu\nu} - \left(\frac{1}{4} - \frac{3\,M^2}{k^2}\right) \frac{k^\mu\,k^\nu}{k^2} + \frac{i}{4\,k^2}\,k_\varrho\,q_\sigma\,\varepsilon^{\varrho\sigma\mu\nu}\right] T_{\mu\nu}$$

$$F_2(s) = \left[\frac{M^2}{k^2}\left(g^{\mu\nu} - \frac{3\,k^\mu\,k^\nu}{k^2}\right) - \frac{i\,M^2}{q^2\,k^2}\,q_\varrho\,k_\sigma\,\varepsilon^{\varrho\sigma\mu\nu}\right] T_{\mu\nu}$$

$$F_3(s) = \left[-\frac{M^2}{q^2}\left(g^{\mu\nu} - \frac{k^\mu\,k^\nu}{k^2}\right) - \frac{i\,M^2}{q^2\,k^2}\,q_\varrho\,k_\sigma\,\varepsilon^{\mu\sigma\mu\nu}\right] T_{\mu\nu}$$

$$F_4(s) = \frac{2\,M^2}{q^2\,k^2}\,k^\mu\,q^\nu\,T_{\mu\nu}\,. \tag{6.4}$$

These projections can be carried out covariantly on the tensors $y_\varrho\,y_\sigma\,M^{\varrho\sigma}_{\mu\nu}$, $y_\varrho\,y_\sigma\,N^{\varrho\sigma}_{\mu\nu}$ and $y_\varrho\,y_\sigma\,O^{\varrho\sigma}_{\mu\nu}$. But the result is considerably simplified in the centre of mass system of the meson, if one choses the baryon to move in the $+\,z$-direction and the antibaryon in the $-\,z$-direction:

$$\begin{aligned}
p_{1\,\mu} &: \tfrac{1}{2}\sqrt{s}\,(1, 0, 0, \alpha) && \\
p_{2\,\mu} &: \tfrac{1}{2}\sqrt{s}\,(1, 0, 0, -\alpha) && s = q^2 \\
q_\mu &: \sqrt{s}\,(1, 0, 0, 0) && \alpha = \left(\frac{s - 4\,M^2}{s}\right)^{1/2} \\
k_\mu &: \sqrt{s}\,(0, 0, 0, -\alpha) && u = 1 + \tfrac{1}{2}\,\omega \\
Q_\mu &: \sqrt{s}\,(u, 0, 0, v) && v = \tfrac{1}{2}\,\alpha\,\omega\,.
\end{aligned} \tag{6.5}$$

Since the momentum vectors in this coordinate system contain only a zeroth and a third component and since the form factors are functions of scalar expressions in these and the vector y_ϱ, the integration $d\Omega_y$ over the angles (ϑ, φ) involves only the components $y_0 = 1$ and $y_3 = \cos\vartheta = x$, such that the angular integration reduces to one over the real variable x alone, extended from $-\,1$ to $+\,1$. Hence the final result for the electromagnetic form factors contains only a twofold integration:

$$F_i(s) = s^{-3/2}\,M^3\,F(s)\int_{-1}^{1} dx \int_0^\infty d\omega\,E(x, \omega)\,[R_1\,A_i + R_2\,B_i + R_3\,C_i]$$

$$(i = 1, 2, 3, 4) \tag{6.6a}$$

$$\begin{aligned}
E(x, \omega) &= (1 - \alpha\,x)^{1/2}\,(1 + \alpha\,x)^{-3/2}\,(u - v\,x)^{-2} \\
R_1 &= \frac{\sqrt{H} - 1}{G} \\
R_2 &= \frac{1}{2}\,(1 - \alpha\,x)\,\frac{1/2\,\sqrt{H}\,(3 - H) - 1}{G^2} \\
R_3 &= (u - v\,x)\,\frac{1 - 1/2\,(\sqrt{H} + 1/\sqrt{H})}{G^2} \\
H &= \frac{M^2}{s}\,\frac{2\,\omega + \omega^2\,(1 - \alpha\,x)}{(u^2 - v^2)\,(1 - \alpha\,x)} \\
G &= \omega - \frac{s}{2\,M^2}\,(1 + \omega)\,(1 - \alpha\,x)
\end{aligned} \tag{6.6b}$$

$$A_1 = \frac{s\,u}{4\,M^2}\,a_1 - \frac{s\,v}{4\,\alpha^2\,M^2}\,a_2 + \frac{1}{\alpha^2}\,a_3$$

$$A_2 = -\frac{1}{\alpha^2}\,a_3 + \frac{v}{\alpha}\,a_2$$

$$A_3 = -u\,a_1 + \frac{v}{\alpha}\,a_2$$

$$A_4 = \frac{2}{\alpha}\,a_4 \tag{6.6c}$$

$$B_1 = \frac{s}{2\,M^2}\left(-\frac{s}{4\,M^2}\,b_1 + \frac{s}{4\,\alpha\,M^2}\,b_2 + \frac{1}{\alpha^2}\,b_3\right)$$

$$B_2 = -\frac{s}{2\,\alpha\,M^2}\left(\frac{1}{\alpha}\,b_3 + b_2\right)$$

$$B_3 = \frac{s}{2\,M^2}\left(b_1 - \frac{1}{\alpha}\,b_2\right)$$

$$B_4 = \frac{s}{\alpha\,M^2}\,b_4 \tag{6.6d}$$

$$C_1 = \frac{s}{4\,M^2}\left(\frac{s}{4\,M^2}\,c_1 + \frac{s}{4\,\alpha\,M^2}\,c_2 + \frac{1}{\alpha^2}\,c_3\right)$$

$$C_2 = -\frac{s}{4\,M^2\,\alpha}\left(\frac{1}{\alpha}\,c_3 + c_2\right)$$

$$C_3 = -\frac{s}{4\,M^2}\left(c_1 + \frac{1}{\alpha}\,c_2\right)$$

$$C_4 = \frac{s}{2\,M^2}\,c_4 \tag{6.6e}$$

$$a_1 = 1 - x^2 = a_2$$

$$a_3 = u + 2\,v\,x - 3\,u\,x^2$$

$$a_4 = -v + u\,x$$

$$b_1 = u\,(u^2 + 3\,v^2) - 2\,v\,(3\,u^2 + v^2)\,x + u\,(u^2 + 3\,v^2)\,x^2$$

$$b_2 = v\,(3\,u^2 + v^2) - 2\,u\,(u^2 + 3\,v^2)\,x + v\,(3\,u^2 + v^2)\,x^2$$

$$b_3 = u\,(u^2 - 5\,v^2) + 2\,v\,(3\,u^2 + v^2)\,x - u\,(3\,u^2 + v^2)\,x^2$$

$$b_4 = v\,(u^2 - v^2)\,(x^2 - 1)$$

$$c_1 = -(u^2 + 2\,\alpha\,u\,v + v^2) + 2\,(\alpha\,u^2 + 2\,u\,v + \alpha\,v^2)\,x - (u^2 + 2\,\alpha\,u\,v + v^2)\,x^2$$

$$c_2 = (\alpha\,u^2 + 2\,u\,v + \alpha\,v^2) - 2\,(u^2 + 2\,\alpha\,u\,v + v^2)\,x + (\alpha\,u^2 + 2\,u\,v + \alpha\,v^2)\,x^2$$

$$c_3 = (u^2 - 2\,\alpha\,u\,v - 3\,v^2) + 2\,(\alpha\,u^2 + 2\,u\,v + \alpha\,v^2)\,x - (3\,u^2 + 2\,\alpha\,u\,v - v^2)\,x^2$$

$$c_4 = (u^2 - v^2)\,(x^2 - 1)\,. \tag{6.6f}$$

At this stage numerical computation of the form factors becomes possible. The results will be presented in Section 7. To conclude this section we merely state that the integral converges for all physical values of s. For energies near the threshold $s = 4\,M^2$, the variable α tends to zero. Care must therefore be taken in handling factors like $1/\alpha^2$ appearing in the integrand; in the limit $\alpha \to 0$ these factors are always multiplied by $(1 - 3\,x^2)$ which vanishes upon integration over x. This keeps the form factors finite even at the threshold.

7. Numerical Results

We have computed the form factors (6.6) in the production channel above the threshold $s = 4\,M^2$ and received the following results:

a) *Axial Form Factors*

Within the accuracy of the numerical data the axial form factors $F_3(s)$ and $F_4(s)$ vanish identically. (The computer data show that they are up to a factor 10^{-5} smaller than the vector form factors at the threshold and that they remain constant trough out the whole energy domain.) From this we conclude that SL(2, C) symmetry fixes once and for ever the parity of a vertex. Crossing symmetry which would lead to the scattering channel is therefore violated since it would transform the antiparticle into a particle and thereby change the over all parity of the vertex.

b) *The Dirac Form Factors*

The vector form factors $F_1(s)$ and $F_2(s)$ could be computed rather accurately with iterated Romberg method. Since the ratio $F_2(s)/F_1(s)$ for the threshold energy $s = 4\,M^2$ is found equal to 3/2, we normalise $F_2(4\,M^2)$ to 1.5 and $F_1(4\,M^2)$ to 1 and choose the free function $F(s)$ to be a constant $= 1$. A characteristic falling off towards higher energies is found as shown in Figure 1. The ratio $F_2(s)/F_1(s)$ for our energies is illustrated in Figure 2.

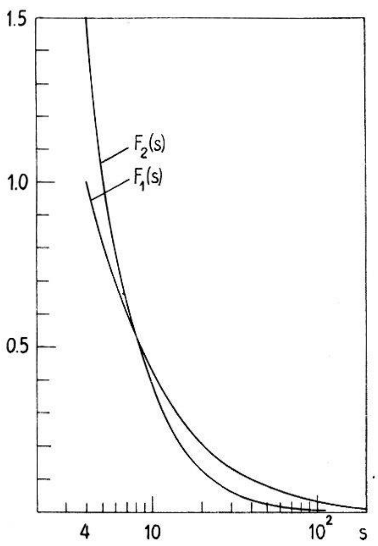

Figure 1
The vector form factors $F_1(s)$ and $F_2(s)$. $(M^2 = 1)$.

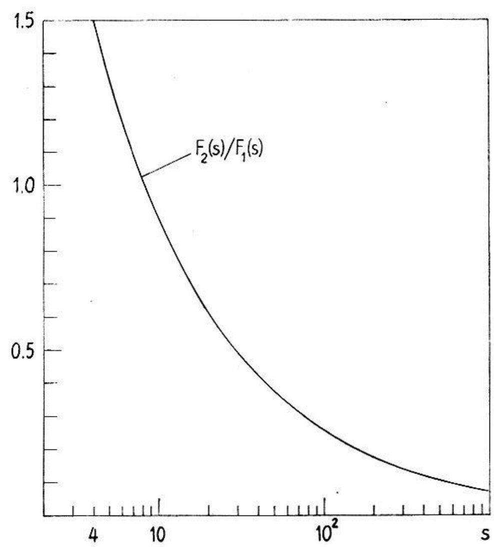

Figure 2
The ratio $F_2(s)/F_1(s)$. $(M^2 = 1)$.

c) *The Sachs Form Factors*

The Sachs form factors in production channel are defined by

$$G_E(s) = F_1(s) + \frac{s}{4\,M^2}\,F_2(s)$$

$$G_M(s) = F_1(s) + F_2(s)\,.$$

They have been calculated and represented graphically in Figure 3 (again under the assumption that the free function $F(s) = 1$ for all s). Figure 4 shows the ratio $G_M(s)/G_E(s)$. It should be noted that this ratio falls off from the value 1 at the threshold $s = 4\,M^2$ approximately like $2\,M/\sqrt{s}$. (This latter function is dotted in Figure 4.)

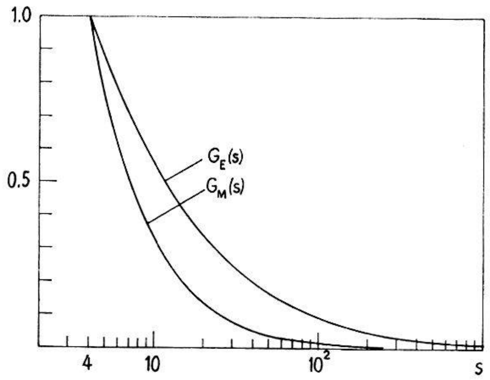

Figure 3

The Sachs form factors $G_E(s)$ and $G_M(s)$. ($M^2 = 1$).

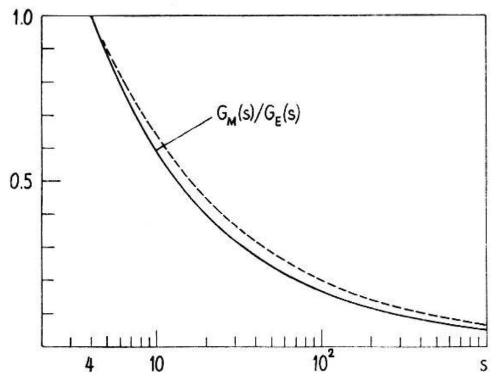

Figure 4

The ratio $G_M(s)/G_E(s)$. The dotted line shows the approximation $2\,M/\sqrt{s}$. ($M^2 = 1$).

Bebié [3] has given a closed expression for the ratio $G_M(s)/G_E(s)$. Our numerical values are in agreement with his within less than 1% (even up to as high energies as $s = 1024\,M^2$). Figure 4 may therefore also be taken as a graphical representation of his formula. This throws some light on the reliability and accuracy of the numerical calculations which one can expect from the integration technique used in this model.

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