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**Autor:** Jauch, J.M. / Lavine, R. / Newton, R.G.

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# Scattering Into Cones

by **J. M. Jauch, R. Lavine**[1]) and **R. G. Newton**[2])

Department of Theoretical Physics, University of Geneva, Switzerland

and

Department of Physics, Indiana University, Bloomington, Indiana, USA

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*Abstract.* Dollard's result concerning scattering into cones is generalized for arbitrary dimension of space and more general Hamiltonians.

## 1. Introduction

In the time-dependent quantum mechanical scattering theory the transition from the theoretical calculation of the scattering amplitude to the observed scattering cross-section requires a theorem which was first stated for the single channel non-relativistic case by T. A. Green and O. E. Lanford III [1] concerning the scattering into cones. An elegant derivation of this theorem under more natural conditions was recently given by J. D. Dollard [2]. The essential part of the theorem is contained in Lemma 4 of Dollard's paper which may be stated as follows:

Suppose $\mathscr{H} = L^2(\boldsymbol{R}^3)$ $\phi \in \mathscr{H}$, $\phi = \{\phi(\boldsymbol{x})\}$ and denote by $\hat{\phi}(\boldsymbol{k})$ the Fourier transform of $\phi(\boldsymbol{x})$, defined by

$$\hat{\phi}(\boldsymbol{k}) = \frac{1}{(2\,\pi)^{3/2}} \int e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \phi(\boldsymbol{x})\, d^3x\,, \tag{1}$$

and define

$$\overset{0}{H} = \frac{\boldsymbol{p}^2}{2\,m}\ (m > 0)\,,$$

where $\boldsymbol{p}$ acts on $\phi(\boldsymbol{x})$ by $\boldsymbol{p}\,\phi(\boldsymbol{x}) = -\,i\,\nabla\,\phi(\boldsymbol{x})$.

Let $C$ be a cone with apex at the origin defined as the set of all points $\boldsymbol{x} \in \boldsymbol{R}^3$ satisfying

$$\boldsymbol{x} \cdot \boldsymbol{n} \geq \alpha\,|\,\boldsymbol{x}\,|\,,$$

where $\boldsymbol{n}$ is a unit vector and $0 < \alpha \leq 1$.

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[1]) Visiting Professor on leave of absence from Cornell University, Department of Mathematics.
[2]) Supported in part by the US and Swiss National Science Foundations and the US Army Research Office, Durham, N.C.

Under these conditions Dollard proved that:

$$\lim_{t\to+\infty} \int_C |\,(e^{-i\overset{0}{H}t}\,\phi)\,(\boldsymbol{x})\,|^2\,d^3x = \int_C |\,\hat{\phi}(\boldsymbol{k})\,|^2\,d^3k\,. \tag{2}$$

This proposition has a very plausible content. Indeed in the context of quantum theory the left-hand side of (2) represents the probability of finding a particle inside the cone $C$ as the time $t \to \infty$. The right-hand side on the other hand represents the probability at the time $t = 0$ to find a particle with momentum vector inside the cone $C$. Equation (2) thus says roughly that every particle with a momentum vector inside the cone $C$ in momentum space will eventually appear inside the same cone $C$ in position space.

Such a general heuristic statement of the content of equation (2) makes it quite plausible that this result cannot depend on the precise form (1) of the generator $\overset{0}{H}$. In fact we should expect a similar result for any generator $\overset{0}{H}$ of the form

$$\overset{0}{H} = F(p)\,,$$

where $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$ and $F$ is a sufficiently regular function with positive derivative: $F'(p) \equiv p\,G(p) > 0$.

Such a generalization is needed if one wants to establish the relation between scattering amplitude and cross-section in a relativistic theory, of for scattering of particles with other dispersion laws (for instance phonons in crystals). For a relativistic scattering system we would have to choose for $\overset{0}{H}$ an expression such as $\overset{0}{H} = \sqrt{p^2 + m^2}$.

Another generalization might be mentioned: The dimension of the configuration space is irrelevant. Thus we might as well state a theorem for any finite dimension. This will include the 1-dimensional case which is of some interest both physically and mathematically.

## 2. The Main Theorem

We adopt the notation of the introduction except that we leave the dimension of the configuration space general but finite. The Hilbert space is the space $L^2(\boldsymbol{R}^n)$. A normalized element of this space is the function $\phi(\boldsymbol{x})$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and we denote its Fourier transform by

$$\hat{\phi}(\boldsymbol{k}) \equiv (\Phi\,\phi)\,(\boldsymbol{k}) = \frac{1}{(2\,\pi)^{n/2}} \int_{\mathrm{R}^n} d^n x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\,\phi(\boldsymbol{x})\,. \tag{3}$$

We shall assume that the generator $H$ of the one-parameter group $U_t = e^{-iHt}$ is given by

$$H = F(p)\,,\quad p = \sqrt{\sum_j p_j^2}\,,$$

where $p_j$ are the momentum operators

$$(p_j\phi)\,(\boldsymbol{x}) = -\,i\,\frac{\partial}{\partial x_j}\,\phi(\boldsymbol{x})\,.$$

In momentum space the operator $H$ is multiplication operator

$$(H\,\hat{\phi})\,(\boldsymbol{k}) = F(k)\,\hat{\phi}(\boldsymbol{k}) \quad \text{where} \quad k = \sqrt{\sum_j k_j^2}\,.$$

We shall assume that the partial derivatives $\boldsymbol{\nu}\cdot\nabla F$ in each direction $\boldsymbol{\nu}$ (unit vector) are locally square integrable and of polynomial growth at infinity. Let $C$ be the cone

$$C = \left\{ \boldsymbol{x} \in \boldsymbol{R}^n \mid \frac{\boldsymbol{x}\cdot\boldsymbol{n}}{|\boldsymbol{x}|} > \alpha \right\}. \tag{4}$$

Writing $\phi_t = U_t\,\phi$, we have the following

*Theorem:*

For every $\phi \in L^2(\boldsymbol{R}^n)$ we have

$$\lim_{t\to+\infty} \int_C |\,\phi_t(\boldsymbol{x})\,|^2\,d^n\,\boldsymbol{x} = \int_C |\,\hat{\phi}(\boldsymbol{k})\,|^2\,d^n\boldsymbol{k}\,. \tag{5}$$

*Proof:*

Denote by $\mathscr{S}$ the space of testfunctions of Schwartz. Suppose $\phi \in \mathscr{S}$; then $\hat{\phi} \in \mathscr{S}$ also. Let $\boldsymbol{\nu}$ be a unit vector in $\boldsymbol{R}^n$ and $(Q_{\boldsymbol{\nu}}\,\phi)\,(\boldsymbol{x}) = \boldsymbol{x}\cdot\boldsymbol{\nu}\,\phi(x)$ and $Q_{\boldsymbol{\nu}}\,\phi \in \mathscr{S}$. $Q_{\boldsymbol{\nu}}$ is essentially selfadjoint on this domain and its Fourier transform is given by $(Q_{\boldsymbol{\nu}}\,\phi)\,\hat{\phi}(\boldsymbol{k}) = i\,\boldsymbol{\nu}\cdot\nabla\,\hat{\phi}(\boldsymbol{k})$.

It follows that $e^{-iHt}\,\phi \in D_{Q_{\boldsymbol{\nu}}}$ and

$$(U_t{}^*\,Q_{\boldsymbol{\nu}}\,U_t\,\phi)\,\hat{}\,\,(\boldsymbol{k}) = i\,\boldsymbol{\nu}\cdot\nabla\,\hat{\phi}(\boldsymbol{k}) + t\,\boldsymbol{\nu}\cdot\nabla\,F(\boldsymbol{k})\,\hat{\phi}(\boldsymbol{k})\,. \tag{6}$$

Since $F$ is a function of $k = \sqrt{k_1^2 + \cdots + k_n^2}$ only, we may also write

$$\nabla F(k) = \frac{F'(k)}{k}\,\boldsymbol{k} = G(k)\,\boldsymbol{k},\, G(k) = \frac{1}{k}\,F'(k)\,. \tag{7}$$

It follows from (6) that for all $\phi \in \mathscr{S}$ we have

$$\frac{1}{t}\,U_t^*\,Q_{\boldsymbol{\nu}}\,U_t\,\phi = \frac{1}{t}\,Q_{\boldsymbol{\nu}}\,\phi + \boldsymbol{\nu}\cdot\boldsymbol{p}\,G(p)\,\phi\,. \tag{8}$$

From this result we conclude first that

$$\left\| \frac{1}{t}\,U_t^*\,Q_{\boldsymbol{\nu}}\,U_t\,\phi - \boldsymbol{\nu}\cdot\boldsymbol{p}\,G(p)\,\phi \right\| = \frac{1}{t}\,\|\,Q_{\boldsymbol{\nu}}\,\phi\,\| \to \text{for } t \to 0 \text{ for all } \phi \in \mathscr{S}\,.$$

If we denote by $\boldsymbol{\nu}\cdot\boldsymbol{p}\,G(p)$ also the smallest closed extension of this operator defined on $\mathscr{S}$ we have established

$$\frac{1}{t}\,U_t^*\,Q_{\boldsymbol{\nu}}\,U_t \overset{s}{\to} \boldsymbol{\nu}\cdot\boldsymbol{p}\,G(p)\,. \tag{9}$$

We can now use the Theorem of Section 135 in Ref. [3] according to which the property (9) entails the convergence of the respective spectral projections. Thus if we let

$$h(s) = \begin{cases} 0 & \text{if} \quad s \le 0 \\ 1 & \text{if} \quad s > 0 \end{cases}$$

then we obtain

$$U_t^* \, h\left(\frac{Q_{\boldsymbol{\nu}}}{t}\right) U_t \overset{s}{\to} h\big(\boldsymbol{\nu} \cdot \boldsymbol{p} \; G(p)\big) . \tag{10}$$

At this point we use the invariance property of the particular spectral projection under the operation of expansion and contraction of the space $\boldsymbol{R}^n$. For all values of $t$ we have indeed

$$h\left(\frac{Q_{\boldsymbol{\nu}}}{t}\right) = h(Q_{\boldsymbol{\nu}}) ,$$

so that (10) is equivalent to

$$U_t^* \, h(Q_{\boldsymbol{\nu}}) \, U_t \overset{s}{\to} h\big(\boldsymbol{\nu} \cdot \boldsymbol{p} \; G(p)\big) . \tag{11}$$

From this follows for any $\phi \in \mathscr{H}$

$$\big(\phi_t , h(Q_{\boldsymbol{\nu}}) \, \phi_t\big) \to \big(\phi, h(\boldsymbol{\nu} \cdot \boldsymbol{p} \; G(p)) \, \phi\big) . \tag{12}$$

Since $G(p) > 0$ by hypothesis we find that (12) leads to the result

$$\int\limits_{\boldsymbol{\nu} \cdot \boldsymbol{x} > 0} | \, \phi_t(\boldsymbol{x}) \, |^2 \, d^n x \to \int\limits_{\boldsymbol{\nu} \cdot \boldsymbol{k} > 0} | \, \hat{\phi}(\boldsymbol{k}) \, |^2 \, d^n k . \tag{13}$$

This is the desired result for the special cones consisting of the half-spaces on one side of planes passing through the origin.

In order to generalize this to arbitrary circular cones we note that any circular cone can be inscribed and circumscribed by polygonal cones in such a manner that if $\chi_C(\boldsymbol{x})$ is the characteristic function of the cone

$$\chi_C(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{x} \in C \\ 0 & \text{if} \quad \boldsymbol{x} \notin C \end{cases}$$

then there exist unit vectors $\mu_1, \ldots, \mu_K ; \nu_1, \ldots, \nu_L$ such that

$$\chi_{C^i}(\boldsymbol{x}) = \prod_{j=1}^{K} h(\boldsymbol{x} \cdot \mu_j) \le \chi_C(x) \le \prod_{j=1}^{L} h(\boldsymbol{x} \cdot \nu_j) = \chi_{C^e}(\boldsymbol{x})$$

and for any fixed $\psi(\boldsymbol{x})$ and arbitrary $\varepsilon > 0$

$$\int \{ \chi_{C^e}(\boldsymbol{x}) - \chi_{C^i}(\boldsymbol{x}) \} \, | \, \psi(\boldsymbol{x}) \, |^2 \, d^n x < \varepsilon . \tag{14}$$

Now from (11) we have

$$\begin{aligned}\chi_{C^i}\big(\boldsymbol{p}\ G(\boldsymbol{p})\big) &= s - \lim_{t\to+\infty} U_t^*\ \chi_{C^i}(Q_1, \ldots, Q_n)\ U_t \\ &\leqslant s - \lim_{t\to+\infty} U_t^*\ \chi_C(Q_1, \ldots, Q_n)\ U_t \\ &\leqslant s - \lim_{t\to+\infty} U_t^*\ \chi_{C^e}(Q_1, \ldots, Q_n)\ U_t = \chi_{C^e}\big(\boldsymbol{p}\ G(\boldsymbol{p})\big)\,.\end{aligned}$$

But from (14) we have for any $\phi$ and arbitrary $\varepsilon > 0$ by suitable choice of $\mu_j$ and $\boldsymbol{\nu}_j$

$$0 < \big(\phi, \chi_{C^e}\big(\boldsymbol{p}\ G(\boldsymbol{p})\big)\,\phi\big) - \big(\phi, \chi_{C^i}\big(\boldsymbol{p}\ G(\boldsymbol{p})\big)\,\phi\big) < \varepsilon\,.$$

It follows that

$$\lim_{t\to\infty}\,(\phi_t, \chi_C\ \phi_t) = \big(\phi, \chi_C\big(\boldsymbol{p}\ G(\boldsymbol{p})\big)\,\phi\big)\,.$$

Since $G > 0$ by hypothesis this leads to

$$\lim_{t\to\infty} \int_C |\ \phi_t(\boldsymbol{x})\ |^2\, d^n\boldsymbol{x} = \int_C |\ \hat{\phi}(k)\ |^2\, d^n k$$

and this proves the theorem.

## 3. Supplementary Remarks

We have stated theorem 1 for circular cones only but it is fairly obvious that it can be generalized to more general convex cones, with the same procedure. Since it is only a matter of technical detail without requiring any new idea we shall not do this here.

A further generalization is obtained by considering countable unions of disjoint convex cones. This gives a theorem of sufficient generality for all practical applications.

Another remark concerns the limit $t \to -\infty$. If we reverse the sign of $t$ then the only equation which changes in the preceding section is (10) which becomes

$$h\left(\frac{Q_{\boldsymbol{\nu}}}{t}\right) = h(Q_{-\boldsymbol{\nu}})\,. \qquad (10)^1$$

This leads to the conclusion that there is a corresponding theorem which says that

$$\lim_{t\to-\infty} \int_C |\ \phi_t(\boldsymbol{x})\ |^2\, d^n x = \int_{C'} |\ \hat{\phi}(\boldsymbol{k})\ |^2\, d^n k\,, \qquad (16)$$

where $C'$ is the complementary cone defined by

$$C' = \{\,\boldsymbol{x} \in \boldsymbol{R}^n\ |\ \boldsymbol{x}\cdot\boldsymbol{n} \leqslant -\alpha\,|\ \boldsymbol{x}\,|\,\}\,.$$

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