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The Maximal Kinematical Invariance Group of the Harmonic Oscillator

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Abstract. The largest group of coordinate transformations leaving invariant the Schrödinger equation of the n-dimensional harmonic oscillator is determined and shown to be isomorphic to the corresponding group of the free particle equation. It can be described as a Galilei group in which the time-translations have been replaced by the group $SL(2, R)$ of projective transformations. The relation between the oscillator group and the spectrum generating algebra of the harmonic oscillator is investigated. The relevance of the oscillator group and the group $SL(2, R)$ for general quantum systems is discussed.

1. Introduction

In a recent paper [1] the maximal kinematical invariance group (MKI) of the free particle Schrödinger equation, i.e. the largest group of coordinate transformations leaving invariant this equation, was determined and it was shown that the Schrödinger group, as it was called, could be described as a Galilei group in which the one-parameter group of time-translations has been changed to the three-parameter group $SL(2, R)$. The present paper is devoted to a similar analysis in the case of the Schrödinger equation of the n-dimensional harmonic oscillator

$$\Delta(t, \mathbf{x})\, \psi(t, \mathbf{x}) \equiv \left(i\, \partial_t + \frac{1}{2m}\, \partial^2_{kk} - \frac{m\omega^2}{2}\, x_k x_k\right) \psi(t, \mathbf{x}) = 0, \tag{1.1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$. The MKI of equation (1.1), denoted by $HO(n)$ and called n-dimensional oscillator group, is defined as the set of all coordinate transformations g,

$$(t, \mathbf{x}) \to g(t, \mathbf{x}), \tag{1.2}$$

with the property that

$$\Delta[g(t, \mathbf{x})]\,[f_g(t, \mathbf{x})\, \psi(t, \mathbf{x})] = 0 \tag{1.3}$$

for some function f_g and all solutions ψ of (1.1).

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At first sight, it might seem that in comparison with the free particle case, the presence of the potential term in equation (1.1), which fixes an origin of space, vigorously limits the number of allowed coordinate transformations. This, however, is not so and the most surprising result of the subsequent analysis is that *the oscillator group $HO(n)$ and the Schrödinger group $Schr(n)$ are isomorphic*. In fact, if appropriate coordinates are used for the harmonic oscillator, the transformations of the two groups are identical.

In Section 2, we first reduce equation (1.1) by splitting it into two separate sets of equations for the coordinate transformations g and the companion functions f_g respectively. The g-equations are then solved and the group $HO(n)$ is determined. The f-equations are solved in Section 3 and it is shown that the harmonic oscillator solutions carry a projective unitary irreducible representation of $HO(n)$. Restricting, for simplicity, our attention to the case $n = 1$ we then produce a plausibility argument for the existence of the invariance group $HO(1)$ by connecting its generators to the creation and annihilation operators of the harmonic oscillator. Section 4 contains a discussion of the relationship between the harmonic oscillator and the free particle which is intimated by the isomorphism of the two groups $HO(1)$ and $Schr(1)$. It is shown that there exists a simple formula connecting the solutions of the two systems. In Section 5, the algebras $SL(2, R)$ and $HO(1)$[2]) are established as spectrum generating algebras of the harmonic oscillator. Finally, in Section 6, we demonstrate that the algebras $SL(2, R)$ and $HO(1)$ not only occur in connection with the harmonic oscillator, but are actually hidden in most quantum systems.

2. Determination of the Group HO(n)

To solve equation (1.3) for the unknown coordinate transformations g and companion functions f_g, we use the same method as in the appendix of [1]. We first define the derivatives

$$u(t, \mathbf{x}) = \frac{\partial t}{\partial t'} \neq 0, \quad c_i(t, \mathbf{x}) = \frac{\partial t}{\partial x'_i},$$

$$b_i(t, \mathbf{x}) = \frac{\partial x_i}{\partial t'}, \quad d_{ik}(t, \mathbf{x}) = \frac{\partial x_i}{\partial x'_k}, \quad \det d_{ik} \neq 0, \tag{2.1}$$

where $(t', \mathbf{x}') = g(t, \mathbf{x})$. The differential operator $\Delta(t', \mathbf{x}')$ of (1.3) is now expressed as an operator in (∂_t, ∂_i) and by comparing different orders of space derivatives of the harmonic oscillator solutions ψ we obtain from (1.3) the equations

$$c_i = 0, \tag{2.2}$$

$$d_{il}\, d_{kl} = u \delta_{ik}, \tag{2.3}$$

$$2u\, \partial_i f_g + (d_{lk}\, \partial_l\, d_{ik} + 2imb_i)\, f_g = 0, \tag{2.4}$$

$$u\, \partial_{ii}^2 f_g + (d_{lk}\, \partial_l\, d_{ik} + 2imb_i)\, \partial_i f_g + 2imu\dot{f}_g + m^2 \omega^2 (u\mathbf{x}^2 - \mathbf{x}'^2)\, f_g = 0. \tag{2.5}$$

Equation (2.3) tells us that the matrix d_{ik} can be expressed as

$$d_{ik} = u^{1/2} R_{ki}, \tag{2.6}$$

[2]) A group and its Lie algebra are denoted by the same symbol.

where $R \in O(n)$ is an n-dimensional rotation. The integrability conditions for (2.4), (2.5) and the derivatives (2.1) then imply the relations

$$u = u(t), \quad R_{ik} = \text{constant}, \quad \partial_i b_k = \tfrac{1}{2}\dot{u}\,\delta_{ik}, \tag{2.7}$$

$$2u\dot{b}_i - \dot{u}b_i + 2\omega^2(u^2 x_i - u^{1/2} x'_k R_{ki}) = 0, \tag{2.8}$$

and as an intermediate result we obtain, after some rearrangement, the equations

$$t' = \int dt u^{-1}(t), \quad \mathbf{x}' = u^{-1/2}[R\mathbf{x} + \mathbf{y}(t)],$$

$$\ddot{\mathbf{y}} + \omega^2 \mathbf{y} = 0, \quad u\ddot{u} - \tfrac{1}{2}\dot{u}^2 + 2\omega^2(u^2 - 1) = 0,$$

$$\partial_i f_g = im[R^{-1}\dot{\mathbf{y}} - \tfrac{1}{2}\frac{\dot{u}}{u}(\mathbf{x} + R^{-1}\mathbf{y})]_i f_g, \tag{2.9}$$

$$\dot{f}_g = \frac{im}{2u^2}[(\tfrac{1}{4}\dot{u}^2 - \omega^2)(R\mathbf{x} + \mathbf{y})^2 + \omega^2 u^2 \mathbf{x}^2 + u^2\dot{\mathbf{y}}^2 - u\dot{u}(R\mathbf{x} + \mathbf{y})\cdot\dot{\mathbf{y}}] f_g + \frac{n}{4}\frac{\dot{u}}{u} f_g,$$

where the more convenient vector $\mathbf{y}$ has replaced the vector $\mathbf{b}$. The set (2.9) of equations has the advantage that it decomposes into equations for the coordinate transformations g and, for known g, into equations for the companion functions f_g. We first solve the g-equations; the solution of the f-equations is given in the next section.

Solution of the g-equations

The solution of the equation for $u(t)$ is

$$u(t) = (1 + \eta^2)^{-1}[(\alpha\eta + \beta)^2 + (\gamma\eta + \delta)^2], \tag{2.10}$$

where $\eta \equiv \tan \omega t$ and $\alpha\delta - \beta\gamma = 1$. If (2.10) is used in the expressions for $(t', \mathbf{x}')$ we obtain the result that the group $HO(n)$ is the set of all coordinate transformations of the form

$$g = (S, \mathbf{a}, \mathbf{v}, R),$$

$$g(t, \mathbf{x}) = \left(\frac{1}{\omega}\arctan\frac{\alpha\tan\omega t + \beta}{\gamma\tan\omega t + \delta}, \left[\frac{1 + \tan^2\omega t}{(\alpha\tan\omega t + \beta)^2 + (\gamma\tan\omega t + \delta)^2}\right]^{1/2} \times [R\mathbf{x} + \mathbf{v}\sin\omega t + \mathbf{a}\cos\omega t]\right), \tag{2.11}$$

where

$$S \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2, R); \quad \mathbf{a}, \mathbf{v} \in \mathbb{R}^n, \quad R \in O(n).$$

Note that the time-translations, a symmetry which is always present for time-independent potentials, are contained in (2.11) if S is a two-dimensional rotation.

HO(n) in oscillator coordinates

The transformations (2.11) take on a much simpler form if a new set of coordinates, called oscillator coordinates, is used, namely

$$\eta = \tan\omega t, \quad \vec{\xi} = (1 + \tan^2\omega t)^{1/2}\mathbf{x} = (\cos\omega t)^{-1}\mathbf{x}. \tag{2.12}$$

In these coordinates the $HO(n)$ transformations are

$$g(\eta, \vec{\xi}) = \left(\frac{\alpha\eta + \beta}{\gamma\eta + \delta}, \frac{R\vec{\xi} + \mathbf{v}\eta + \mathbf{a}}{\gamma\eta + \delta} \right), \tag{2.13}$$

and the group $HO(n)$ appears as an enlarged Galilei group in which the one-parameter subgroup of time-translations has been replaced by the three-parameter group $SL(2, R)$ of projective transformations. The product and the inverse in $HO(n)$ can easily be worked out from (2.13) with the result

$$\begin{aligned} g_3 = g_2 g_1: \quad & S_3 = S_2 S_1, \quad R_3 = R_2 R_1, \\ & \mathbf{a}_3 = R_2 \mathbf{a}_1 + \beta_1 \mathbf{v}_2 + \delta_1 \mathbf{a}_2, \\ & \mathbf{v}_3 = R_2 \mathbf{v}_1 + \alpha_1 \mathbf{v}_2 + \gamma_1 \mathbf{a}_2, \end{aligned} \tag{2.14}$$

$$\begin{aligned} g' = g^{-1}: \quad & S' = S^{-1}, \quad R' = R^{-1}, \\ & \mathbf{a}' = -R^{-1}(\alpha \mathbf{a} - \beta \mathbf{v}), \\ & \mathbf{v}' = -R^{-1}(\delta \mathbf{v} - \gamma \mathbf{a}), \end{aligned} \tag{2.15}$$

where S and R are multiplied as matrices in $\mathbb{R}^2$ and $\mathbb{R}^n$ respectively.

The most surprising fact about the group $HO(n)$ is the finding that it coincides with the Schrödinger group $Schr(n)$, i.e. the MKI of the n-dimensional free particle [1][3]). More precisely, the group $HO(n)$ acts on the oscillator coordinates $(\eta, \vec{\xi})$ exactly as the group $Schr(n)$ acts on the Cartesian coordinates $(t, \mathbf{x})$ of the free particle. This relationship between harmonic oscillator and free particle is discussed further in Section 4.

3. The Oscillator Representation of HO(n)

The general form (2.11) of the coordinate transformations g only solves part of the equations (2.9) and it now remains to determine the functions f_g accompanying a given transformation g. The two differential equations for f_g in (2.9) are easily solved with the result

$$\begin{aligned} f_g(t, \mathbf{x}) &= u_g(t)^{n/4} \exp\left[-\frac{im}{4u_g(t)} h_g(t, \mathbf{x}) \right], \\ h_g(t, \mathbf{x}) &= \dot{u}_g(t)\,[R\mathbf{x} + \mathbf{y}_g(t)]^2 - 2u_g(t)\, \dot{\mathbf{y}}_g(t)\, [2R\mathbf{x} + \mathbf{y}_g(t)], \end{aligned} \tag{3.1}$$

where $u_g(t)$ is given by (2.10), and

$$\mathbf{y}_g(t) = (1 + \eta^2)^{-1/2}(\mathbf{v}\eta + \mathbf{a}) = \mathbf{v} \sin \omega t + \mathbf{a} \cos \omega t. \tag{3.2}$$

The functions f_g satisfy the important relation

$$f_{g_2}[g_1(t, \mathbf{x})]\, f_{g_1}(t, \mathbf{x}) = \omega(g_2, g_1)\, f_{g_2 g_1}(t, \mathbf{x}), \tag{3.3}$$

$$\begin{aligned} \omega(g_2, g_1) = \exp[-\tfrac{1}{2} im\omega(&-\alpha_1 \mathbf{v}_2 \cdot R_2 \mathbf{a}_1 + \beta_1 \mathbf{v}_2 \cdot R_2 \mathbf{v}_1 - \gamma_1 \mathbf{a}_2 \cdot R_2 \mathbf{a}_1 \\ &+ \delta_1 \mathbf{a}_2 \cdot R_2 \mathbf{v}_1)]. \end{aligned} \tag{3.4}$$

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3) In [1] the subgroup $SL(2, R)$ has been decomposed into the three one-parameter groups of time-translations, dilations and expansions; but the form (2.13) where the group $SL(2, R)$ is left as a whole is easier to handle. The generalization of $Schr(3)$ to arbitrary dimension is obvious.

The verification is straightforward but tedious. We can now define a projective representation $g \to T_g$ of $HO(n)$ on the solutions ψ of the oscillator equation (1.1) by simply putting

$$(T_g \psi)(t, \mathbf{x}) = f_g[g^{-1}(t, \mathbf{x})]\, \psi[g^{-1}(t, \mathbf{x})], \tag{3.5}$$

where, by the definition (1.3) of the MKI, the new functions $T_g \psi$ are again solutions of equation (1.1). The map $g \to T_g$ satisfies, as can easily be proved from (3.3), the relation

$$T_{g_2} T_{g_1} = \omega(g_2, g_1)\, T_{g_2 g_1} \tag{3.6}$$

and hence defines a projective representation of $HO(n)$. Furthermore, the representation (3.5) is unitary with respect to the inner product

$$(\psi_1, \psi_2) = \int d^n x \psi_1^*(t, \mathbf{x})\, \psi_2(t, \mathbf{x}), \tag{3.7}$$

because the factor $u_g^{n/4}$ in (3.1) exactly corrects for the non-invariance of the measure $d^n x$. Finally, the irreducibility of the oscillator representation follows from the fact that in oscillator coordinates, it coincides with the Schrödinger representation [1] of $Schr(n)$ which is irreducible.

The equations (2.11) for the transformations g and (3.1) for the companion functions f_g represent the complete solution of (2.9) and hence of the condition (1.3) for the MKI of the n-dimensional harmonic oscillator. In the remainder of the paper we may, for simplicity, confine ourselves to the case $n = 1$ because the influence of the dimension n is comparatively trivial.

The Lie algebra $HO(1)$

The generators of $HO(1)$ are defined by

$$T(\mathbf{s}, a, v) = 1 - i\mathbf{s}\cdot\mathbf{I} - iaP + ivK + o(2), \tag{3.8}$$

where the parameters $\mathbf{s} = (s_1, s_2, s_3)$ of $SL(2, R)$ are chosen according to

$$S(\mathbf{s}) = \cosh\frac{s}{2} + \frac{1}{s}\sinh\frac{s}{2}\begin{pmatrix} s_1 & s_2 + s_3 \\ s_2 - s_3 & -s_1 \end{pmatrix}, \tag{3.9}$$

$$s \equiv +(s_1^2 + s_2^2 + s_3^2)^{1/2}.$$

In the oscillator representation (3.5) the generators are found to be

$$\begin{aligned}
I_1 &= -\frac{i}{2\omega}\sin 2\omega t\, \partial_t - \frac{i}{2}\cos 2\omega t x\, \partial_x - \frac{i}{4}\cos 2\omega t + \tfrac{1}{2} m\omega x^2 \sin 2\omega t, \\
I_2 &= -\frac{i}{2\omega}\cos 2\omega t\, \partial_t + \frac{i}{2}\sin 2\omega t x\, \partial_x + \frac{i}{4}\sin 2\omega t + \tfrac{1}{2} m\omega x^2 \cos 2\omega t, \\
I_3 &= -\frac{i}{2\omega}\partial_t,
\end{aligned} \tag{3.10}$$

$$P = -i\cos\omega t\, \partial_x + m\omega x \sin\omega t,$$
$$K = i\sin\omega t\, \partial_x + m\omega x \cos\omega t,$$

and the corresponding commutators are

$$\begin{aligned}
&[I_1, I_2] = iI_3, \quad [I_1, P] = \frac{i}{2}P, \quad [I_1, K] = -\frac{i}{2}K, \\
&[I_2, I_3] = -iI_1, \quad [I_2, P] = \frac{i}{2}K, \quad [I_2, K] = \frac{i}{2}P, \\
&[I_3, I_1] = -iI_2, \quad [I_3, P] = -\frac{i}{2}K, \quad [I_3, K] = \frac{i}{2}P, \\
&[P, K] = -im\omega.
\end{aligned} \tag{3.11}$$

Note that in a true representation the generators P and K commute and that the non-vanishing of the commutator $[P, K]$ in (3.11) is due to the projective nature of the oscillator representation.

The oscillator representation on energy states

We obtain a discrete description of the oscillator representation of $HO(1)$ if instead of arbitrary solutions of the harmonic oscillator equation we use eigenstates of the energy operator $i\,\partial_t$. An orthonormal energy basis is given by

$$\psi_n(t, x) = (n!)^{-1/2} \left(\frac{m\omega}{\pi}\right)^{1/4} \exp[-i\omega(n + \tfrac{1}{2})\,t - \tfrac{1}{2}m\omega x^2]\, H_n(\sqrt{2m\omega}\, x), \tag{3.12}$$

where $n = 0, 1, 2, \ldots$ and H_n are the Hermite polynomials. On these states the generators of $HO(1)$ are the ladder operators

$$\begin{aligned}
&I_+\psi_n \equiv (I_1 + iI_2)\,\psi_n = \frac{i}{2}\sqrt{(n+1)\,(n+2)}\,\psi_{n+2}, \\
&I_-\psi_n \equiv (I_1 - iI_2)\,\psi_n = -\frac{i}{2}\sqrt{n(n-1)}\,\psi_{n-2}, \\
&I_3\psi_n = -\tfrac{1}{2}(n + \tfrac{1}{2})\,\psi_n, \\
&P_+\psi_n \equiv (P + iK)\,\psi_n = i\sqrt{2m\omega(n+1)}\,\psi_{n+1}, \\
&P_-\psi_n \equiv (P - iK)\,\psi_n = -i\sqrt{2m\omega n}\,\psi_{n-1}.
\end{aligned} \tag{3.13}$$

The form (3.13) of the oscillator representation suggests an argument to make plausible the existence of the invariance group $HO(1)$ by relating it to the existence of the creation and annihilation operators $(a^\dagger, a)$ of the harmonic oscillator. Indeed, we have

$$P_+ = i\sqrt{2m\omega}\, e^{-i\omega t}\, a^\dagger, \quad P_- = -i\sqrt{2m\omega}\, e^{i\omega t}\, a, \tag{3.14}$$

$$I_+ = \frac{i}{2} e^{-2i\omega t}\, a^{\dagger 2}, \quad I_- = -\frac{i}{2} e^{2i\omega t}\, a^2, \quad I_3 = -\tfrac{1}{4}(aa^\dagger + a^\dagger a). \tag{3.15}$$

The operators $P_\pm$ defined in (3.14) act on solutions of equation (1.1) in the same way as $(a^\dagger, a)$ act on solutions of the time-independent equation, hence they generate a symmetry and the only question is whether it is a kinematical symmetry, as opposed to internal symmetries which are not considered here. The symmetry is of the kinematical type, i.e. is a coordinate transformation, if the operators $P_\pm$ can be represented as first-order differential operators in the coordinates (t,x), because the corresponding coordinate transformations are then obtained from the Lie differential equations [2]. Since $(a^\dagger, a)$ are constructed linearly from ∂_x and x the operators $P_\pm$ do generate a kinematical symmetry. Furthermore, the second-order operators $\mathbf{I}$ of (3.15) again generate a kinematical symmetry because the wave equation (1.1) serves to convert the second-order operator ∂_x^2 into the first-order operator ∂_t. The fourth independent expression quadratic in $(a^\dagger, a)$, namely $[a, a^\dagger]$, is a c-number; it does not generate a symmetry but is responsible for the oscillator representation to be a projective representation. Neither do higher order expressions in $(a^\dagger, a)$ generate kinematical symmetries because they definitely are of at least second order in ∂_x and ∂_t.

4. Harmonic Oscillator Versus Free Particle

Having seen in Section 2 that the MKI of the harmonic oscillator is the same group as the MKI of the free particle we now proceed to analyze the connection between these two systems. We first note that an oscillator motion in Cartesian coordinates implies a free motion in oscillator coordinates and vice versa:

$$\ddot{x} + \omega^2 x = 0 \Leftrightarrow \frac{d^2 \xi}{d\eta^2} = 0. \tag{4.1}$$

Thus, apart from the fact that the oscillator coordinates only describe half a period of the oscillator motion, the two systems are, on the level of classical mechanics, transformed into each other by a change of the coordinate system. The implication (4.1) also holds in the framework of quantum mechanics if the coordinates x and ξ are replaced by the expectation values

$$\begin{aligned}(\psi, x\psi) &= \int dx x \psi^*(t,x)\, \psi(t,x),\\ (\varphi, \xi\varphi) &= \int d\xi (1+\eta^2)^{-1/2}\, \xi \varphi^*(\eta,\xi)\, \varphi(\eta,\xi),\end{aligned} \tag{4.2}$$

respectively, where ψ is a solution of (1.1) and φ is a solution of the corresponding wave equation in oscillator coordinates, namely

$$\left(i\,\partial_\eta + \frac{1}{2m\omega}\partial_\xi^2 + \frac{i\eta}{1+\eta^2}\,\xi\,\partial_\xi - \tfrac{1}{2}m\omega\,\frac{1}{(1+\eta^2)^2}\,\xi^2\right)\varphi(\eta,\xi) = 0. \tag{4.3}$$

How closely related the two systems indeed are is shown by the existence of a simple formula connecting the harmonic oscillator solutions to the free particle solutions in oscillator coordinates. Let $\chi(\eta,\xi)$ be a solution of the free particle equation with mass $m\omega$,

$$\left(i\,\partial_\eta + \frac{1}{2m\omega}\partial_\xi^2\right)\chi(\eta,\xi) = 0, \tag{4.4}$$

normalizable with respect to the inner product

$$(\chi_1, \chi_2) = \int d\xi \chi^*(\eta,\xi)\, \chi(\eta,\xi). \tag{4.5}$$

Then to each of these solutions χ there corresponds a normalizable solution ψ of the harmonic oscillator equation (1.1), given by

$$\psi(t,x) = (1+\eta^2)^{1/4} \exp\left(-\tfrac{1}{2} im\omega \frac{\eta}{1+\eta^2}\,\xi^2\right)\chi(\eta,\xi). \tag{4.6}$$

Note that the one-to-one correspondence (4.6) between the solutions ψ and χ is local and does not depend on the solutions itself. It can also be shown that (4.6) is invariant under $HO(1)$, i.e. that the mapping $\psi \leftrightarrow \chi$ commutes with the transformations T_g of the oscillator and the Schrödinger [1] representation.

What is different for the two systems is, of course, the form of the energy operator which for the system (4.4), considered as a harmonic oscillator, is not $i\partial_\eta$ as it would be for a free particle, but

$$i\omega[(1+\eta^2)\,\partial_\eta + \eta\xi\,\partial_\xi + \tfrac{1}{2}\eta - \tfrac{1}{2} im\omega\xi^2]. \tag{4.7}$$

It can be checked that the solutions of (4.4) which are eigenstates of the operator (4.7) lead back, with (4.6), directly to the solutions (3.12) of the harmonic oscillator.

5. HO(1) and the Spectrum Generating Algebra

It is well known [3] that $SL(2,R)$ is the spectrum generating algebra of the one-dimensional harmonic oscillator in the following sense: Consider the three Hermitean operators

$$\begin{aligned} L_1 &= -\frac{i}{4}(2x\,\partial_x + 1), \\ L_2 &= \frac{1}{4m\omega}(\partial_x^2 + m^2\omega^2 x^2), \\ L_3 &= \frac{1}{4m\omega}(\partial_x^2 - m^2\omega^2 x^2) = -\frac{1}{2\omega}H, \end{aligned} \tag{5.1}$$

which are obtained by closing with respect to commutation the two operators ∂_x^2 and x^2 appearing in the Hamiltonian H of the harmonic oscillator. They form the Lie algebra $SL(2,R) \simeq SO(2,1) \simeq SU(1,1)$ and the corresponding Casimir operator is a c-number, namely

$$-L_1^2 - L_2^2 + L_3^2 = -3/16. \tag{5.2}$$

Hence the Hilbertspace of the harmonic oscillator is composed of one or more of the spaces of those unitary irreducible representations of $SL(2,R)$ which belong to the Casimir value $-3/16$ and whose L_3-spectrum is bounded from above (to guarantee a lower bound for the Hamiltonian). There are only two representations satisfying these criteria [4] and if both are combined they reproduce the correct spectrum of H. For this reason $SL(2,R)$ is called the spectrum generating algebra of the harmonic oscillator.

Now the question naturally arises whether the subalgebra $SL(2,R) \subset HO(1)$ is related to the spectrum generating $SL(2,R)$. The Casimir operator of the former $SL(2,R)$ is obtained from (3.10) and is given by

$$-I_1^2 - I_2^2 + I_3^2 = \tfrac{1}{2}mx^2\,\Delta(t,x) - 3/16, \tag{5.3}$$

where $\Delta(t,x)$ is the differential operator of the harmonic oscillator equation. Hence the operators **I**, when acting on oscillator solutions, also generate the spectrum. The bases **I** and **L** of the two spectrum generating algebras are related by a rotation in the 1-2-plane:

$$\begin{aligned} L_1 &= \cos 2\omega t I_1 - \sin 2\omega t I_2, \\ L_2 &= \sin 2\omega t I_1 + \cos 2\omega t I_2, \\ L_3 &= I_3, \end{aligned} \tag{5.4}$$

where in **I** the operator $i\partial_t$ has to be replaced by $-(\partial_x^2 - m^2\omega^2 x^2)/2m$.

Thus we have obtained the interesting result that the MKI of the harmonic oscillator incorporates the spectrum generating algebra as a subalgebra. Let us finally mention an important difference in the spectrum generating power of $SL(2,R)$ and $HO(1)$. To cover the full energy spectrum of the harmonic oscillator, we need two irreducible representations of $SL(2,R)$, corresponding to the fact that the generators **I** of (3.13) are two-step ladder operators, whereas in the case of $HO(1)$ one irreducible representation is sufficient because the additional generators $P_\pm$ are one-step ladder operators. In other words, $HO(1)$ may with even better reason be called the spectrum generating algebra of the harmonic oscillator.

6. The Relevance of SL(2,R) and HO(1) for Quantum Systems

In the preceding sections the groups $SL(2,R)$ and $HO(1)$ were established as invariance groups of a wave equation describing a special quantum system, namely the one-dimensional harmonic oscillator. In the present section, we want to point out that, independent of a special system and its wave equation, the two groups occur in a natural way, 1) in all one-dimensional non-relativistic quantum systems, and 2) in all Boson systems. In both cases the pattern is the same: there exists a given algebra of two basic quantities and $SL(2,R)$ is the algebra of all expressions which are of second order in these quantities (hence called the *strictly quadratic algebra*), whereas $HO(1)$ is the algebra of all expressions of first and second order (called the *quadratic algebra*). More precisely:

(1) $SL(2,R)$ is the strictly quadratic algebra of the Heisenberg algebra $[Q,P]=i$, i.e. the three Hermitean operators

$$I_1 = \tfrac{1}{4}(QP+PQ), \quad I_2 = \tfrac{1}{4}(Q^2-P^2), \quad I_3 = -\tfrac{1}{4}(Q^2+P^2), \tag{6.1}$$

satisfy the commutation relations of $SL(2,R)$.

(1′) The algebra (6.1) can be completed to the algebra $HO(1)$ by inclusion of the Heisenberg algebra itself because the five operators (**I**, P, $K=Q$) satisfy the commutation relations (3.11) for $m\omega=1$. Thus $HO(1)$ is the quadratic algebra of the Heisenberg algebra.

(2) $SL(2,R)$ is the strictly quadratic algebra of the pair $(a^\dagger, a)$ of Boson creation and annihilation operators, i.e. the three Hermitean operators

$$I_1 = -\frac{i}{4}(a^2 - a^{\dagger 2}), \quad I_2 = \tfrac{1}{4}(a^2 + a^{\dagger 2}), \quad I_3 = -\tfrac{1}{4}(aa^\dagger + a^\dagger a), \tag{6.2}$$

satisfy the commutation relations of $SL(2,R)$. A similar relation was exploited in (3.15).

(2′) $HO(1)$ is the quadratic algebra of the Boson operators $(a^\dagger, a)$, the additional generators being defined by

$$P = \frac{i}{\sqrt{2}}(a^\dagger - a), \quad K = \frac{i}{\sqrt{2}}(a^\dagger + a). \tag{6.3}$$

There is a different interpretation of $SL(2, R)$ and $HO(1)$ in connection with Boson systems, namely, $SL(2, R)$ and $HO(1)$ are both spectrum generating algebras of the Boson number operator $a^\dagger a$. This is clear from (6.2) where $a^\dagger a = -\frac{1}{2}I_3 - \frac{1}{4}$ and from the fact that the Casimir operator of the algebra (6.2) has the value $-3/16$. The two irreducible representations of $SL(2, R)$ with this value of the Casimir and I_3-spectrum bounded from above yield the correct spectrum of $a^\dagger a$. In the case of the larger algebra $HO(1)$, these two representations together form a single irreducible representation.

Note that in both the cases (1) and (2) there exists besides the operators **I** a fourth linearly independent non-vanishing quadratic expression, namely the commutator of the basic quantities. This commutator is a c-number hence, strictly speaking, it is not $HO(1)$ itself but a projective representation (or, in different terms, a central extension) of $HO(1)$ which is the quadratic algebra of the basic quantities.

We conclude these remarks by pointing out that in the case of the harmonic oscillator the two statements (1) and (2) collapse into one because the operators $(a^\dagger, a)$ are linear expressions of the operators (Q, P).

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