Zeitschrift: Helvetica Physica Acta

Band: 47 (1974)

Heft: 2

Artikel: The maximal kinematical invariance groups of Schrödinger equations with arbitrary potentials

Autor: Niederer, U.

DOI: https://doi.org/10.5169/seals-114565

Helvetica Physica Acta
Vol. 47, 1974. Birkhäuser Verlag Basel

The Maximal Kinematical Invariance Groups of Schrödinger Equations with Arbitrary Potentials

by **U. Niederer**
Institut für Theoretische Physik der Universität Zürich, 8001 Zürich, Switzerland[1])

(21. XII. 73)

Abstract. The general potential-independent form of the maximal kinematical invariance groups of Schrödinger equations is determined. The problem of finding the invariance group for a given potential is reduced to solving a single equation containing the potential and the invariance transformations. For some of the most interesting potentials the invariance groups are given explicitly.

1. Introduction and Results

This paper is devoted to an analysis of the maximal kinematical invariance groups (MKI's) of n-dimensional Schrödinger equations with arbitrary potentials. The MKI [1] of the equation

$$\left[i\partial_t + \frac{1}{2m}\partial_{kk} - V(t,\mathbf{x})\right]\psi(t,\mathbf{x}) = 0 \quad (k = 1,\ldots,n) \tag{1.1}$$

is the group of all coordinate transformations

$$g: \quad (t,\mathbf{x}) \to g(t,\mathbf{x}) \tag{1.2}$$

with the property that there exist companion functions $f_g(t,\mathbf{x})$ such that the map

$$g: \quad \psi \to T_g\psi, \quad (T_g\psi)(t,\mathbf{x}) = f_g[g^{-1}(t,\mathbf{x})]\,\psi[g^{-1}(t,\mathbf{x})] \tag{1.3}$$

sends any solution ψ of equation (1.1) into another solution, $T_g\psi$, of the same equation.

The MKI has already been determined for the free particle [1], the harmonic oscillator [2], and the free fall, i.e. the particle in a constant homogeneous field of force [3]. It was shown that the MKI of all three systems is the Schrödinger group, Sch(n), in three different realizations; this group can be described as the n-dimensional Galilei group with the time-translations replaced by the three-parameter group $SL(2,\mathbb{R})$ of projective time-transformations (see (3.1) below). The purpose of the present paper is to extend these results in two ways, namely, to gain potential-independent information on the general form of the invariance groups, and to give more examples of non-trivial invariance groups.

---

[1]) Work supported by the Swiss National Foundation.

In Section 2 the determination of the MKI of equation (1.1) is carried as far as it is possible without specific knowledge of the potential; the following theorem is proved:

Theorem: The MKI of equation (1.1) is given by the coordinate transformations

$$g(t,\mathbf{x}) = \left(\int dt \frac{1}{d^2(t)}, \frac{R\mathbf{x}+\mathbf{y}(t)}{d(t)}\right) \tag{1.4}$$

and the companion functions

$$f_g(t,\mathbf{x}) = d^{n/2} \exp\left\{-\frac{im}{2}\left[\frac{\dot{d}}{d}\mathbf{x}^2 + 2R\mathbf{x}\cdot\left(\frac{\dot{d}}{d}\mathbf{y} - \dot{\mathbf{y}}\right) + \frac{\dot{d}}{d}\mathbf{y}^2 - \dot{\mathbf{y}}\cdot\mathbf{y} + l(t)\right]\right\}, \tag{1.5}$$

where the scalars d, l and the vector $\mathbf{y}$ are (real) functions of t, and $R \in O(n)$ are constant rotations. All these quantities and the constant of integration in (1.4) have to be determined from the condition

$$V[g(t,\mathbf{x})] - d^2 V(t,\mathbf{x}) = \frac{m}{2} d\,\ddot{d}\,\mathbf{x}^2 + mR\mathbf{x}\cdot(d\,\ddot{d}\,\mathbf{y} - d^2\ddot{\mathbf{y}}) + \frac{m}{2}(d\ddot{d}\mathbf{y}^2 - d^2\ddot{\mathbf{y}}\cdot\mathbf{y} + d^2\dot{l}). \tag{1.6}$$

Comparing the different orders in $\mathbf{x}$ of equation (1.6) for a given potential we obtain several conditions for the unknown quantities and the coordinate transformations (1.4) can then be fully determined. In Section 3 this program is carried out for some of the most interesting potentials.

The theorem may also be used to find the maximal number $N(n)$ of dimensions any MKI can have. Since the functions $d(t)$ and $\mathbf{y}(t)$ are determined by second-order differential equations, they contain at most 2 and $2n$ free parameters, respectively; together with the constant of integration and the rotations we find

$$N(n) = 3 + 2n + \tfrac{1}{2}n(n-1) = \tfrac{1}{2}(n+1)(n+2) + 2. \tag{1.7}$$

The Schrödinger group Sch(n) is an example of an MKI for which this maximal number is actually attained.

2. Proof of the Theorem

The equation which determines the MKI is given by

$$\left[i\partial'_t + \frac{1}{2m}\partial'_{kk} - V(t',\mathbf{x}')\right][f_g(t,\mathbf{x})\psi(t,\mathbf{x})] = 0, \tag{2.1}$$

where $(t',\mathbf{x}') \equiv g(t,\mathbf{x})$, and ψ is any solution of the original Schrödinger equation (1.1). In this section equation (2.1) is solved as far as possible for an unspecified potential. The technique used is similar to that in the appendix of [1].

Defining the derivatives

$$d^2(t,\mathbf{x}) = \partial t/\partial t', \qquad c_i(t,\mathbf{x}) = \partial t/\partial x'_i,$$

$$b_i(t,\mathbf{x}) = \partial x_i/\partial_t', \quad d_{ik}(t,\mathbf{x}) = \partial x_i/\partial x'_k, \tag{2.2}$$

we convert equation (2.1) into an equation with the differential operators ∂_t, ∂_k and, using the fact that ψ is an arbitrary solution of (1.1), we obtain the following set of equations by comparing the different orders of space-derivatives of ψ:

$$c_i = 0, \tag{2.3}$$

$$d_{ir}\, d_{kr} = d^2\, \delta_{ik}, \tag{2.4}$$

$$2d^2\, \partial_i f_g + (d_{rk}\, \partial_r\, d_{ik} + 2imb_i) f_g = 0, \tag{2.5}$$

$$d^2\, \partial_{ii} f_g + (d_{rk}\, \partial_r\, d_{ik} + 2imb_i)\, \partial_i f_g + 2imd^2 \dot{f}_g + 2m[d^2\, V(t,\mathbf{x}) - V(t',\mathbf{x}')] f_g = 0. \tag{2.6}$$

The derivatives (2.2) are easily inverted to

$$\partial t'/\partial t = d^{-2}, \qquad \partial t'/\partial x_i = 0,$$

$$\partial x'_i/\partial t = -d^{-3}\, R_{ik}\, b_k, \quad \partial x'_i/\partial x_k = d^{-1}\, R_{ik}, \tag{2.7}$$

where (2.4) is used to write d_{ik} as a rotation,

$$d_{ik} = d\, R^{-1}_{ik} = d\, R_{ki}. \tag{2.8}$$

The integrability conditions for (2.7) imply

$$d = d(t), \quad R_{ik} = R_{ik}(t), \quad \partial_i\, b_k = d\,\dot{d}\,\delta_{ik} - d^2\, R_{lk}\, \dot{R}_{li}. \tag{2.9}$$

With these relations equations (2.5) and (2.6) now read

$$\partial_i \ln f_g = -\, im\, \frac{b_i}{d^2}, \quad \partial_t \ln f_g = im\, \frac{b_i\, b_i}{2d^4} + \frac{\partial_i\, b_i}{2d^2} + i\left[V(t,\mathbf{x}) - \frac{1}{d^2}\, V(t',\mathbf{x}')\right]. \tag{2.10}$$

The integrability conditions for the first of these equations are

$$\partial_i\, b_k = \partial_k\, b_i, \tag{2.11}$$

hence, together with (2.9), we obtain

$$R_{ik} = \text{const.}, \quad \partial_i\, b_k = d\dot{d}\, \delta_{ik}. \tag{2.12}$$

The last equation in (2.7) is now integrated to

$$\mathbf{x}' = d^{-1}[R\mathbf{x} + \mathbf{y}(t)], \tag{2.13}$$

where we have replaced the vector $\mathbf{b}(t)$ by the more convenient vector $\mathbf{y}(t)$ by putting

$$\mathbf{b} = d\dot{d}(\mathbf{x} + R^{-1}\mathbf{y}) - d^2\, R^{-1}\dot{\mathbf{y}}. \tag{2.14}$$

The companion function f_g is calculated from (2.10) with the result

$$f_g(t,\mathbf{x}) = d^{n/2} \exp\left\{-\frac{im}{2}\left[\frac{\dot{d}}{d}\mathbf{x}^2 + 2R\mathbf{x}\cdot\left(\frac{\dot{d}}{d}\mathbf{y} - \dot{\mathbf{y}}\right) + \frac{\dot{d}}{d}\mathbf{y}^2 - \dot{\mathbf{y}}\cdot\mathbf{y} + l(t)\right]\right\}, \tag{2.15}$$

and the unknown quantities $d(t)$, $\mathbf{y}(t)$, R, $l(t)$ have to satisfy the condition

$$V(t',\mathbf{x}') - d^2 V(t,\mathbf{x}) = \frac{m}{2} d\ddot{d}\,\mathbf{x}^2 + mR\mathbf{x}\cdot(d\ddot{d}\,\mathbf{y} - d^2\ddot{\mathbf{y}}) + \frac{m}{2}[d\ddot{d}\,\mathbf{y}^2 - d^2\ddot{\mathbf{y}}\cdot\mathbf{y} + d^2\dot{l}), \tag{2.16}$$

which follows from (2.10) and (2.15).

3. Examples of Invariance Groups

Equation (1.6) trivially admits time-translations for any static potential, rotations for any spherically symmetric potential, and translations $\mathbf{y} = \mathbf{v}t + \mathbf{a}$ for any potential which is constant along the direction of $\mathbf{y}$. There are, however, more interesting cases, and a look at (1.6) tells us that, among the static potentials, the widest possibilities are offered by the potentials belonging to one of the following classes:

1) $V(\mathbf{x}) = A\mathbf{x}^2 + \mathbf{B}\cdot\mathbf{x} + C$. In this case the difference on the left-hand side of (1.6) may be balanced by the quadratic right-hand side.
2) $V(\mathbf{x})$ is homogeneous of degree (–2). The difference then vanishes for $\mathbf{y} = 0$, $R = 1$.
3) $V(\mathbf{x}) = A_{ik} x_i x_k$. For $d = 1$, $R = 1$, the difference is linear as is now the right-hand side.

Below we give the coordinate transformations $(t',\mathbf{x}') \equiv g(t,\mathbf{x})$ of the MKI for some of these potentials.

Potentials admitting the full Schrödinger group

The free particle, the free fall, and the harmonic oscillator have already been treated in [3]. The coordinate transformations of the corresponding invariance groups are given as follows:

$V = 0$:

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \mathbf{x}' = \frac{1}{\gamma t + \delta}(R\mathbf{x} + \mathbf{v}t + \mathbf{a}). \tag{3.1}$$

$V = -m\mathbf{g}\cdot\mathbf{x}$:

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \mathbf{x}' = \frac{1}{\gamma t + \delta}\left(R\mathbf{x} + \mathbf{v}t + \mathbf{a} + \frac{1}{2}\frac{(\alpha t + \beta)^2}{\gamma t + \delta}\mathbf{g} - \tfrac{1}{2}t^2 R\mathbf{g}\right). \tag{3.2}$$

$V = \tfrac{1}{2}m\omega^2\mathbf{x}^2$:

$$t' = \frac{1}{\omega}\arctan\frac{\alpha\tan\omega t + \beta}{\gamma\tan\omega t + \delta},$$

$$\mathbf{x}' = \left[\frac{1 + \tan^2\omega t}{(\alpha\tan\omega t + \beta)^2 + (\gamma\tan\omega t + \delta)^2}\right]^{\frac{1}{2}}(R\mathbf{x} + \mathbf{v}\sin\omega t + \mathbf{a}\cos\omega t). \tag{3.3}$$

In all three cases the MKI is the n-dimensional Schrödinger group Sch(n) which is defined by the group elements

$$g=(S,\mathbf{a},\mathbf{v},R);\quad S=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,\mathbb{R});\quad \mathbf{a},\mathbf{v}\in\mathbb{R}^n;\quad R\in O(n), \tag{3.4}$$

and the multiplication law $g_3=g_2g_1$:

$$S_3=S_2S_1,\quad \mathbf{a}_3=\delta_1\mathbf{a}_2+\beta_1\mathbf{v}_2+R_2\mathbf{a}_1,$$
$$R_3=R_2R_1,\quad \mathbf{v}_3=\gamma_1\mathbf{a}_2+\alpha_1\mathbf{v}_2+R_2\mathbf{v}_1. \tag{3.5}$$

The group Sch(n) can be considered as the n-dimensional Galilei group with the time-translations replaced by the projective transformations of $SL(2,\mathbb{R})$ (which include the time-translations as a subgroup).

That Sch(n) is the MKI for the free fall and the harmonic oscillator is surprising because the fixed origin and the fixed vector $\mathbf{g}$, in (3.2), would ordinarily seem to exclude translations and rotations. It was shown in [3] that the two realizations (3.2) and (3.3) of Sch(n) and the corresponding Schrödinger equations can be obtained from the free particle case by appropriate coordinate transformations.

The inverse square potential

A short calculation leads to the tollowing result:

$V=k\dfrac{1}{\mathbf{x}^2}$:

$$t'=\frac{\alpha t+\beta}{\gamma t+\delta},\quad \mathbf{x}'=\frac{1}{\gamma t+\delta}R\mathbf{x}. \tag{3.6}$$

Thus the MKI is the direct product of $SL(2,\mathbb{R})$ with $O(n)$.

The anisotropic harmonic oscillator

Assuming that all frequencies are different and non-vanishing we obtain

$V=\frac{1}{2}m\sum_{i=1}^{n}\omega_i^2x_i^2$:

$$t'=t,\quad x_i'=x_i+v_i\sin\omega_i t+a_i\cos\omega_i t, \tag{3.7}$$

where $\omega_i^2\neq\omega_k^2\neq 0(i\neq k)$. The MKI is the $2n$-dimensional Abelian group $\mathbb{R}^{2n}$.

The time-dependent Kepler problem

Finally, to give an example for a time-dependent potential, we consider the Kepler problem with a time-dependent gravitational 'constant':

$V=-\kappa\dfrac{1}{tr}$:

$$t'=\frac{t}{1+\gamma t},\quad \mathbf{x}'=\frac{1}{1+\gamma t}R\mathbf{x}. \tag{3.8}$$

$$V = -\kappa \frac{1}{t^{1/2} r}:$$

$$t' = \frac{1}{\delta^2} t, \quad \mathbf{x}' = \frac{1}{\delta} R\mathbf{x}. \tag{3.9}$$

For comparison we note that the MKI of the true (static) Kepler problem is given by

$$V = -\kappa \frac{1}{r}:$$

$$t' = t + \beta, \quad \mathbf{x}' = R\mathbf{x}. \tag{3.10}$$

Thus in all three cases the MKI is the direct product of the one-dimensional Abelian group $\mathbb{R}$ with $O(n)$; only its realizations are different. It is interesting to note that the three time-transformations are three different one-parameter subgroups of $SL(2, \mathbb{R})$ which, taken together, generate the full $SL(2, \mathbb{R})$; in other words, $SL(2, \mathbb{R})$ can be decomposed into the three groups (3.8), (3.9), (3.10) of time-transformations.

In a recent paper [4] it was shown that, for space-dimension one, the only static potentials with non-trivial MKI are sums of powers $(x + c_\nu)^\nu$ with $\nu = -2, 0, 1, 2$. The list at the beginning of this section might suggest that the same situation holds for arbitrary dimensions. To show that this is not the case, we give the following, somewhat pathological, counter-example for $n \geqslant 2$:

$$V(\mathbf{x}) = H_{-2}[e^{\frac{1}{2}\ln(\mathbf{x}+\mathbf{b})^2 J}(\mathbf{x} + \mathbf{b})], \tag{3.11}$$

where $H_{-2}[\mathbf{z}]$ is a homogeneous function of degree (-2), J is a generator of $O(n)$, i.e. a real antisymmetric matrix, and $\mathbf{b}$ is an arbitrary vector. Apart from the time-translations the potential (3.11) admits the one-parameter group

$$t' = \frac{1}{\delta^2} t, \quad \mathbf{x}' = \frac{1}{\delta}[R(s)\,\mathbf{x} + R(s)\mathbf{b} - \delta\mathbf{b}]. \tag{3.12}$$

where $\delta = e^s$ and $R(s) = e^{sJ}$.

REFERENCES

[1] U. Niederer, Helv. Phys. Acta *45*, 802 (1972).
[2] U. Niederer, Helv. Phys. Acta *46*, 191 (1973).
[3] U. Niederer, *The group theoretical equivalence of the free particle, the harmonic oscillator and the free fall*, Proceedings of the 2nd International Colloquium on Group Theoretical Methods in Physics, 1973, University of Nijmegen, The Netherlands.
[4] C. P. Boyer, *The maximal kinematical invariance group for an arbitrary potential*, preprint CIMAS 1973/53, Mexico.