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On the Approach to Equilibrium in Kinetic Theory. II. Fluid Mechanics

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Abstract. The long time behaviour of solutions to the Enskog–Vlasov type kinetic equations is studied in detail. It is found that for this purpose the Enskog–Vlasov dynamics can be reduced to fluid dynamics. The quantities that are phenomenological in fluid dynamics appear as functions of the quantities that are phenomenological in the Enskog–Vlasov dynamics and of the properties of the fixed point of the Enskog–Vlasov dynamics approached as time goes to infinity. The interesting case where the fixed point corresponds to a near critical state is also discussed.

1. Introduction

McLennan [1] and Scharf [2] have shown that the long time behaviour of solutions to the Boltzmann kinetic equation can be obtained by solving the equations of fluid dynamics whose phenomenological quantities are functions of the phenomenological quantities entering the Boltzmann kinetic equation. By using the results of Ref. [3] we can extend the work of McLennan and Scharf to the Enskog–Vlasov dynamics (hereafter the EV-dynamics). This extension allows us to study the reduction of the kinetic theory dynamics to the fluid dynamics in the situation where the approached (as time goes to infinity) state corresponds to a near critical state. Both thermodynamics and asymptotic dynamics are strictly derived only from the EV-dynamics that serve, in this paper, as the original dynamics with which we start.

The basic ideas of McLennan and Scharf can be formulated, now in the way applicable not only to the Boltzmann kinetic equation dynamics, as follows.

i) The equilibrium fixed points of the EV-dynamics are defined in Ref. [3] as the time-independent solutions to the EV-kinetic equation that are moreover invariant with respect to the transformation $f(\mathbf{r}, \mathbf{v}) \mapsto f(\mathbf{r}, -\mathbf{v})$, where $\mathbf{r}$, $\mathbf{v}$ stands for the position and velocity vector respectively and f, a real-valued function of $\mathbf{r}, \mathbf{v}$, is an element of the set $\mathscr{H}$ on which the EV-dynamics is defined. All possible equilibrium fixed points are classified and the thermodynamic interpretation of the classification is developed in Ref. [3]. It is also proved in Ref. [3] that if the equilibrium fixed point $f_0^{(\mathrm{EV})}$ corresponds, in the thermodynamic interpretation, to the thermodynamically stable equilibrium state then there is a natural Hilbert space structure for the linear space $H_0^{(\mathrm{EV})}$ on which the linearized (around $f_0^{(\mathrm{EV})}$) EV-dynamics is defined such that the infinitesimal generator of the linearized EV-dynamics is an Onsager operator [4]. It means in particular that $f_0^{(\mathrm{EV})}$ is asymptotically stable.

ii) The similar discussion of the fluid dynamics in Ref. [4] (hereafter F-dynamics) leads to the identical result, i.e. if $f_0^{(F)}$ is the equilibrium fixed point of F-dynamics corresponding, in the thermodynamic interpretation, to the thermodynamically stable equilibrium state then there is a natural Hilbert space structure for the linear space $H_0^{(F)}$ on which the linearized (around $f_0^{(F)}$) F-dynamics is defined such that the infinitesimal generator of the linearized F-dynamics is an Onsager operator [4]. It means in particular that $f_0^{(F)}$ is asymptotically stable.

iii) Let us take one $f_0^{(EV)}$ and one $f_0^{(F)}$ that correspond, in the thermodynamic interpretation, to the same – in the sense of thermodynamics – state. The problem is to find two subspaces H_1, H_2 of $H_0^{(EV)}$ satisfying the following properties:

a) H_1 and H_2 are Hilbert spaces having the same dimension as $H_0^{(F)}$.
b) H_1 is invariant with respect to the linearized (around $f_0^{(EV)}$) EV-dynamics.
c) H_1 is asymptotic in the sense that the elements of H_1 characterize the long time behaviour of the trajectories defined by the linearized EV-dynamics. If for example the closest to zero spectral points of the linearized EV-operator are eigenvalues then H_1 will be chosen as their corresponding eigenspaces.
d) H_2 is completely isomorphic to $H_0^{(F)}$ and the linearized EV-dynamics restricted to H_1 and transformed (one-to-one transformation) to H_2 is identical with the linearized F-dynamics. The identification gives the map from the phenomenological quantities in the EV-dynamics to the phenomenological quantities in the F-dynamics. This map depends on the thermodynamical properties of $f_0^{(EV)}$.

The reader who is familiar with the recent Brussels school concept of subdynamics in the theory of the Liouville dynamics [5] will find that the relation between the F-dynamics and the Boltzmann-dynamics in the works of McLennan and Scharf is analogical to the relation between a subdynamics of the Liouville dynamics and the Liouville dynamics in the Brussels school theory. This observation stimulated our interest in the extension of the McLennan and Scharf idea to the EV-dynamics since, as far as we know, the Brussels school theory (based on the Liouville dynamics) has not yet been extended to include thermodynamical phenomena like, for example, the critical phenomena.

In the following sections of this paper the point iii) above is concretely realized (the points i) and ii) are realized in Refs. [3, 4] and reviewed in Sections 2, 3 respectively). The map from the phenomenological quantities in the EV-dynamics to the phenomenological quantities in the F-dynamics is obtained explicitly. We would like to point out that this map is obtained purely from the EV-dynamics. No extra information from thermodynamics, equilibrium statistical mechanics etc. is used in this paper. The final formulas that we obtain are very similar to the formulas derived from the van der Waal equilibrium theory (this is not surprising, since the theory of the equilibrium fixed points is equivalent to the van der Waal equilibrium theory [3]) and from the Enskog dynamics by using the standard methods for calculations of the kinetic coefficients, e.g. Ref. [6]. It is well known that the critical phenomena derived from these formulas are not realistic, which is in our case just a consequence of the fact that the starting EV-dynamics in not realistic. We believe, however, that the extension of the McLennan and Scharf idea developed here can be applied to more realistic dynamical models and in this way results interesting from the point of view of the theory of critical phenomena can be obtained.

2. The Enskog–Vlasov Dynamics

The EV-dynamics has been introduced and discussed in Ref. [3]. In order to establish the notation and the basis (the point i) in the Introduction) for this paper we mention a few results derived in Ref. [3].

An element $f_{\text{seqs}}^{(\text{EV})}$ of the two-dimensional manifold – parametrized by (α, β) – of the equilibrium fixed points of the EV-dynamics is called a single-phase thermodynamically stable equilibrium state if $f_{\text{seqs}}^{(\text{EV})} = n_0(\alpha, \beta)\, e^{-\beta v^2}$, where n_0 (depending on (α, β)) is a positive constant determined by the condition that the function

$$G^{(\text{EV})} = -n \ln(n) - c_3 \tfrac{2}{3}\pi\sigma^3 n \int \vartheta(n)\, dn - c_4 \tfrac{1}{2}\beta W n^2 + \alpha n - \tfrac{3}{2}\ln\left(\frac{\beta}{2\pi}\right) n \tag{1}$$

reaches its non-degenerate maximum at $n = n_0$. The parameters (α, β) have the following thermodynamic meaning: $\beta = 1/T$, where T is the temperature, $\alpha = \beta\mu$, where μ is the chemical potential. We define

$$\gamma = \max_n G^{(\text{EV})} = g^{(\text{EV})}(\alpha, \beta) \tag{2}$$

that has the thermodynamic meaning $\gamma = \beta p$, where p is the pressure.

The phenomenological quantities in the EV-dynamics, denoted $\mathscr{P}^{(\text{EV})}$, are $\{c_3, c_4, \sigma, \vartheta(n), V_{\text{pot}}(|\mathbf{r} - \mathbf{r}_1|)\}$. With respect to the notation used in Ref. [3] we put $c_1 \equiv c_2 \equiv 1$, the functions ϑ and V_{pot} are identical with the functions η and V introduced in Ref. [3]. The quantity W appearing in (1) equals $\int_\Omega d^3r' V_{\text{pot}}(|\mathbf{r} - \mathbf{r}'|)$, where Ω is the bounded space region in which the system considered is confined. We assume that the volume of Ω equals one. The physical meaning of $\mathscr{P}^{(\text{EV})}$ is explained in Ref. [3]. From the mathematical point of view c_3, c_4 and σ are non-negative real constants, ϑ is a real-valued twice-differentiable function of $n(\mathbf{r})$, $\mathbf{r} \in \Omega$, the function V_{pot} is a once-differentiable function from $|\mathbf{r} - \mathbf{r}'|$, $\mathbf{r}, \mathbf{r}' \in \Omega$ to the negative real line. Their further properties are determined by requiring that the geometric properties of the manifold of the equilibrium fixed points and (2) give the thermodynamics identical with the thermodynamics of a van der Waals gas.

The critical state $f_c^{(\text{EV})}$ corresponding to (α_c, β_c) is defined as the equilibrium fixed point with smallest β at which $G^{(\text{EV})}$ reaches its degenerate extremal value. It means that (α_c, β_c) are determined by two equations

$$r = 0 \tag{3.1}$$

$$\frac{1}{n_0^2} = c_3 \tfrac{2}{3}\pi\sigma^3 \left(3\frac{d\vartheta}{dn} + n\frac{d^2\vartheta}{dn^2}\right)\Bigg|_{n=n_0}, \tag{3.2}$$

where

$$r = 1 + w n_0; \quad w = c_3 \tfrac{2}{3}\pi\sigma^3 \left(n\frac{d\vartheta}{dn} + 2\sigma\right)\Bigg|_{n=n_0} + c_4 \beta W.$$

The solution of (3.2) is called the critical density and is denoted by n_c.

The Fourier transform (with respect to $\mathbf{r}$) of the local linearized (around $f_{\text{seqs}}^{(\text{EV})}$) EV-dynamics (in other words the Fourier transform of the Hessian of the EV-vector field evaluated at $f_{\text{seqs}}^{(\mathrm{EV})}$) with $\mathbf{k} \equiv (0, 0, k)$ fixed is

$$\frac{\partial \varphi^{(\mathrm{EV})}}{\partial t} = Q^{(\mathrm{EV})} \varphi^{(\mathrm{EV})}, \tag{4}$$

where $\varphi^{(\mathrm{EV})} \in H_0^{(\mathrm{EV})}$, $H_0^{(\mathrm{EV})}$ is the complex Hilbert space whose elements are $\varphi^{(\mathrm{EV})}(\mathbf{v})$, $\mathbf{v} \in \mathbb{R}^3$. The scalar product $\langle \varphi_1^{(\mathrm{EV})}, \varphi_2^{(\mathrm{EV})} \rangle$ in $H^{(\mathrm{EV})}$ is defined by

$$\langle \varphi_1^{(\mathrm{EV})}, \varphi_2^{(\mathrm{EV})} \rangle = \int d^3\mathbf{v} \left(\frac{\beta}{2\pi} \right)^{3/2} e^{-1/2\beta v^2} (\varphi_1^{(\mathrm{EV})}(\mathbf{v}))^* A^{(\mathrm{EV})} \varphi_2^{(\mathrm{EV})}(\mathbf{v}), \tag{5}$$

where φ^* is the complex conjugate of φ and

$$A^{(\mathrm{EV})} \varphi = \varphi + w \int d^3\mathbf{v}\, n_0 \left(\frac{\beta}{2\pi} \right)^{3/2} e^{-1/2\beta v^2} \varphi(\mathbf{v}). \tag{6}$$

The mathematical and physical reasons for introducing this particular scalar product in $H^{(\mathrm{EV})}$ are in Ref. [3]. One of its advantages is that we do not need to deal with the complicated operator $Q^{(\mathrm{EV})}$ but with the much simpler operator $A^{(\mathrm{EV})} Q^{(\mathrm{EV})}$

$$\begin{aligned} A^{(\mathrm{EV})} Q^{(\mathrm{EV})} = {} & n_0 \vartheta(n_0) R_{B,l} - ik \left(v_3 + c_3 \tfrac{2}{3} \pi \sigma^3 \left(n_0 \frac{d\vartheta}{dn} \bigg|_{n=n_0} + \vartheta(n_0) \right) + c_4 \beta W \right) \\ & \times \int d^3\mathbf{v}_I\, f_{\text{seqs}}(v_I)\, (v_{I,3} + v_3) \varphi(\mathbf{v}_I) + c_3 \vartheta(n_0) \frac{\sigma^3}{2} \int d^2 \boldsymbol{\varkappa} \\ & \times \int d^3\mathbf{v}_I\, f_{\text{seqs}}(v_I)\, (v_{I\delta} - v_\delta) \varkappa_\delta \varkappa_3 \varphi(\mathbf{v}_I - \boldsymbol{\varkappa}(v_{I\chi} - v_\chi)\varkappa_\chi), \end{aligned} \tag{7}$$

where the summation convention has been used, v_3 means the third component of the vector $\mathbf{v}$, $\boldsymbol{\varkappa}$ is a unit vector, the operator $R_{B,l}$ is the well-known linear Boltzmann operator. It has been proved in Ref. [3] that the operator $A^{(\mathrm{EV})} Q^{(\mathrm{EV})}$ in $\mathbb{C}L_2$ (or $Q^{(\mathrm{EV})}$ in $H_0^{(\mathrm{EV})}$) is an Onsager operator [4]. It means in particular that its real part is a self-adjoint operator and its imaginary part multiplied by the imaginary unit i is also a self-adjoint operator.

3. The Fluid Dynamics

The fluid dynamics (hereafter the F-dynamics) has been discussed in Ref. [4]. In order to establish the notation and the basis for this paper (the point ii) in the Introduction) we mention a few results from Ref. [4].

An element of $F_{\text{seqs}} \equiv (N_0, E_0, 0, 0, 0)$ of the two-dimensional manifold of the equilibrium fixed points (parametrized by (γ, β)) is called a single-phase thermodynamically stable equilibrium state if N_0, E_0 are constants ($N_0 > 0$) determined by the condition that the function

$$G^{(\mathrm{F})} = \int_\Omega d^3\mathbf{r} (-S(E, V) + \gamma V + \beta E) \tag{8}$$

restricted to the manifold $\tau(E, V) = 1/\beta$ reaches its non-degenerate minimum at N_0, E_0. In (8) we used $V = 1/N$, the function $S(E, V)$ is defined by the following relations

$$\frac{\partial S}{\partial V} = \frac{p}{\tau}, \quad \frac{\partial S}{\partial E} = \frac{1}{\tau}. \tag{9}$$

We define

$$\alpha = \min_{(E, V)|_{\tau=\beta^{-1}}} G^{(F)} = g^{(F)}(\gamma, \beta) \tag{10}$$

The parameters α, β, γ have the same thermodynamic interpretation as in the preceding section.

The phenomenological quantities in F-dynamics $\mathscr{P}^{(F)}$ are $\{p(E, V), \tau(E, V), \lambda(E, V), \eta(E, V), \eta_v(E, V)\}$. For their physical meaning see Ref. [4], from the mathematical point of view we assume that p and τ are twice differentiable functions of E and V, λ, η and η_v are continuous, $\lambda \neq 0$, $\tau|_{\partial\Omega} = T$, λ, η, η_v evaluated at $f^{(F)}_{\text{seqs}}$ are positive.

The critical state $f^{(F)}_c$ corresponding to (γ_c, β_c) is defined as the equilibrium fixed point with smallest β at which $G^{(F)}$ reaches its degenerate extremal point. It means that (γ_c, β_c) are determined by the condition that the determinant of the matrix $A^{(F)}$ equals zero and by the minimality of β.

$$A^{(F)} = \begin{pmatrix} \frac{1}{N_0^2}(p_n - \gamma\tau_n), & \beta\tau_n, & 0, & 0, & 0 \\ \beta\tau_n, & \beta\tau_e, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{pmatrix}, \tag{11}$$

where we use the notation $\tau_n = (\partial\tau/\partial N)|_{(E_0, N_0)}$ etc.

The Fourier transform (with respect to $\mathbf{r}$) of the local linearized (around $f^{(F)}_{\text{seqs}}$) F-dynamics (in other words the Fourier transform of the Hessian of the F-vector field evaluated at $f^{(F)}_{\text{seqs}}$) with $\mathbf{k} \equiv (0, 0, k)$ fixed is

$$\frac{\partial \varphi^{(F)}}{\partial t} = Q^{(F)} \varphi^{(F)} \tag{12}$$

where $\varphi^{(F)} \in H_0^{(F)}$, $\varphi^{(F)} \equiv (n, e, u_1, u_2, u_3)$, $H_0^{(F)}$ is the complex five-dimensional Hilbert space with the scalar product

$$G\langle \varphi_1^{(F)}, \varphi_2^{(F)} \rangle = (n_1^*, e_1^*, \mathbf{u}_1^*)\, A^{(F)} \begin{pmatrix} n_2 \\ e_2 \\ \mathbf{u}_2 \end{pmatrix}. \tag{13}$$

The linear operator $Q^{(F)}$ is given by the matrix

$$Q^{(F)} = \begin{pmatrix} 0 & 0 & -ik\,N_0 & 0 & 0 \\ -\lambda\frac{\tau_n}{N_0}k^2 & -\lambda\frac{\tau_e}{N_0}k^2 & -\frac{\gamma}{\beta\,N_0}ik & 0 & 0 \\ 0 & 0 & -\frac{1}{N_0}(\frac{4}{3}\eta+\eta_v)k^2 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\eta}{N_0}k^2 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\eta}{N_0}k^2 \end{pmatrix} \tag{14}$$

It has been proved in Ref. [4] that $Q^{(F)}$ in $H_0^{(F)}$ is an Onsager operator.

4. The Subspace H_1

We have to find the five-dimensional subspace H_1 of $H_0^{(EV)}$ satisfying the properties a), b), c) in the point iii) of the Introduction. With the use of Theorem 3 in Ref. [3] we can follow closely Refs. [1], [2].

Let H_0 be the nullspace of the operator $R_{B,l}$ with the basis $\chi_1 = 1$, $\chi_2 = (\frac{1}{2}v^2 - (3/2\beta))$, $\chi_3 = v_3$, $\chi_4 = v_2$, $\chi_5 = v_2$ We shall assume that k is small. The imaginary part of $A^{(EV)}Q^{(EV)}$ is considered as a small perturbation of the real part and, as was proved in Theorem 3 of Ref. [3], for k sufficiently small, the eigenvalue 0 of $R_{B,l}$ will split smoothly into five eigenvalues ω_i, $i = 1, \ldots, 5$ that are the closest to zero spectral points of $A^{(EV)}Q^{(EV)}$.

The corresponding eigenfunctions are ψ_i, $i = 1, \ldots, 5$. The eigenvalues ω_i and the eigenfunctions ψ_i can be calculated by using the standard perturbation method and the corresponding infinite series is, for k sufficiently small, convergent. We shall write explicitly the results for ω_i up to the order k^2 and ψ_i up to the order k.

$$\begin{aligned} \omega_1 &= \omega k^2 + ik\Lambda \\ \omega_2 &= \omega k^2 - ik\Lambda \\ \omega_3 &= -\frac{2r}{q}\beta^2\kappa_2(1+\tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))^2 k^2 \\ \omega_4 &= -\beta\kappa_1(1+\tfrac{4}{15}\pi n_0\sigma^3\vartheta(n_0))^2 k^2 \\ \omega_5 &= \omega_4, \end{aligned} \tag{15}$$

where

$$\omega = -\tfrac{2}{3}\beta(\kappa_1(1+\tfrac{4}{15}\pi n_0\sigma^3\vartheta(n_0))^2) + \beta\frac{\kappa^2}{q}s^2(1+\tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))^2),$$

$$\Lambda = \left(\frac{q}{3\beta}\right)^{1/2}, \quad s = 1 + e_0, \quad e_0 = c_3\tfrac{2}{3}\pi n_0\sigma^3\vartheta(n_0), \quad q = 3r + 2s^2.$$

The quantities κ_1 and κ_2 are defined by

$$\left(\left(\frac{v^2}{2}-\frac{5}{2\beta}\right)v_i, S_j\right)_{\mathbb{C}L_2} = \kappa_2\,\delta_{ij},$$

$$((v_i v_j - \tfrac{1}{3}\delta_{ij}v^2), T_{lm})_{\mathbb{C}L_2} = \kappa_1(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}-\tfrac{2}{3}\delta_{ij}\delta_{lm}),$$

where $(\ ,\ )_{\mathbb{C}L_2}$ means the $\mathbb{C}L_2$ scalar product. The functions S_i and T_{ij} are defined as the solutions of the integral equations

$$-n_0\vartheta(n_0)R_{B,l}\,S_i = \left(\frac{v^2}{2}-\frac{5}{2\beta}\right)v_i,$$

$$-n_0\vartheta(n_0)R_{B,l}\,T_{ij} = (v_i v_j - \tfrac{1}{3}v^2\delta_{ij}),$$

that are orthogonal to H_0.

The eigenfunctions ψ_i are

$$\psi_i = \psi_i^{(0)} + k\psi_i^{(1)} + k\psi_i^{(2)}, \tag{16}$$

where $\psi_i^{(0)} \in H_0$ and in the basis χ_i

$$\psi_1^{(0)} = (1, \tfrac{2}{3}\beta s, -\beta\Lambda, 0, 0)$$

$$\psi_2^{(0)} = (1, \tfrac{2}{3}\beta s, \beta\Lambda, 0, 0)$$

$$\psi_3^{(0)} = \left(1, -\frac{\beta r}{s}, 0, 0, 0\right)$$

$$\psi_4^{(0)} = (0,0,0,1,0)$$

$$\psi_5^{(0)} = (0,0,0,0,1),$$

$$\psi_1^{(1)} = -i(\beta\Lambda(1+\tfrac{4}{15}\pi n_0\sigma^3\vartheta(n_0))T_{33} - \tfrac{2}{3}\beta s(1+\tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))S_3)$$

$$\psi_2^{(1)} = -i(\beta\Lambda(1+\tfrac{4}{15}\pi n_0\sigma^3\vartheta(n_0))T_{33} + \tfrac{2}{3}\beta s(1+\tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))S_3)$$

$$\psi_3^{(1)} = -i\frac{\beta r}{s}(1+\tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))S_3$$

$$\psi_4^{(1)} = -iT_{32}$$

$$\psi_5^{(1)} = -iT_{31},$$

$$\psi_1^{(2)} = i(g_2^1\psi_2^{(0)} + g_3^1\psi_3^{(0)})$$

$$\psi_2^{(2)} = i(g_1^2\psi_2^{(0)} + g_3^2\psi_3^{(0)})$$

$$\psi_3^{(2)} = i(g_1^3\psi_1^{(0)} + g_2^3\psi_2^{(0)})$$

$$\psi_4^{(2)} = 0$$

$$\psi_5^{(2)} = 0,$$

where the coefficients g^i_j are arbitrary and will be specified in the next section. The eigenfunctions ψ_i, $i = 1, \ldots, 5$, are orthogonal up to the order k, i.e.

$$\langle \psi_i, \psi_j \rangle = N_i \delta_{ij} + O(k^2),$$

$$N_1 = \tfrac{2}{3} q, \quad N_2 = \quad q, \tfrac{2}{3} N_3 = \frac{rq}{2s^2}, \quad N_4 = N_5 = \frac{1}{\beta}.$$

The calculations leading to (15), (16) are rather lengthy but straightforward.

5. The Fluid Dynamics as the Asymptotic Enskog–Vlasov Dynamics

We shall now find H_2 and the dynamics in H_1 transformed into H_2. The following observations help to find H_2 and its basis vectors φ_i, $i = 1, 2, \ldots 5$.

1) $\langle \varphi_i, \varphi^{(EV)} \rangle$ has to have the transformation properties ot a vector.
2) The inverse of the matrix $B^{(2)}_{ij} = \langle \varphi_i, \varphi_j \rangle$ has to coincide with the matrix $A^{(F)}$ (see (11)).
3) The straightforward calculation shows that

$$\frac{\partial}{\partial t} \langle 1, \varphi \rangle = -ikr \langle v_3, \varphi \rangle, \tag{17}$$

where $\varphi \in H_0^{(EV)}$.

We take

$$\varphi_1 = \frac{n_0}{r}, \quad \varphi_2 = K_1 \left(\frac{1}{2} v^2 - \frac{3}{2\beta} \right) + K_2, \quad \varphi_3 = v_3, \quad \varphi_4 = v_2, \quad \varphi_5 = v_1. \tag{18}$$

The coefficients K_1, K_2 will be specified later. As follows from (3.1) $\beta \to \beta_c$ implies $r \to 0$. It means that the changes arising if approaching the critical point can be read from the dependence on r. It follows from (18) that the inverse $A^{(2)}$ of the matrix $B^{(2)} \equiv (\langle \varphi_i, \varphi_j \rangle)$ is

$$A^{(2)} \equiv \begin{pmatrix} \frac{2}{3} \frac{\beta}{n_0} \left(\frac{K_2}{K_1} \right)^2 r^2 + \frac{r}{n_0^2 \beta} & -\frac{2}{3} \frac{K_2 r \beta}{n_0 K_1^2} & 0 & 0 & 0 \\ -\frac{2}{3} \frac{K_2 r \beta}{n_0^2 K_1^2} & \frac{2}{3} \frac{\beta}{K_1^2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

If we introduce α^i_j and θ^i_j by

$$\varphi_i = \sum_{j=1}^{5} \alpha^i_j \psi_j \tag{20}$$

and

$$\psi_i = \sum_{j=1}^{5} \theta^i_j \varphi_j$$

then we can write the EV-dynamics in H_2 as

$$\frac{da_i}{dt} = Q^{(2)}_{ij}\, a_j, \tag{21}$$

where

$$a_i = \langle \varphi_i, \varphi^{(\mathrm{EV})} \rangle$$

and

$$Q^{(2)}_{ij} = \left[\sum_l \alpha^i_l \omega^*_l \theta^l_j \right]^*.$$

We shall write

$$\begin{aligned} \alpha &= \alpha^{(0)}{}^i_j + k\alpha^{(1)}{}^i_j \\ \theta &= \theta^{(0)}{}^i_j + k\theta^{(1)}{}^i_j. \end{aligned} \tag{22}$$

By using (20), (18) and (16) one finds that

$$\alpha^{(0)} \equiv \begin{pmatrix} \dfrac{3n_0}{2q} & \dfrac{3}{2}\dfrac{n_0}{q} & \dfrac{2n_0 s^2}{rq} & 0 & 0 \\ \dfrac{3}{2q}\left(K_2 r + \dfrac{K_1 s}{\beta}\right) & \dfrac{3}{2q}\left(K_2 r + \dfrac{K_1 s}{\beta}\right) & \dfrac{2}{q}\left(K_2 s^2 + \dfrac{3}{2\beta} K_1 s\right) & 0 & 0 \\ -\dfrac{1}{2\beta\Lambda} & \dfrac{1}{2\beta\Lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

$$\alpha^{(1)}{}^i_j = -i \sum_{l \neq j} \alpha^{(0)}{}^i_l g^l_j, \tag{24}$$

$$\theta^{(0)} = \begin{pmatrix} -\dfrac{2}{3}\dfrac{\beta r}{K_1 n_0}\left(K_2 s - \dfrac{3}{2\beta} K_1\right) & \dfrac{2}{3}\dfrac{\beta s}{K_1} & -\dfrac{q}{3\Lambda} & 0 & 0 \\ -\dfrac{2}{3}\dfrac{\beta r}{K_1 n_0}\left(K_2 s - \dfrac{3}{2\beta} K_1\right) & \dfrac{2}{3}\dfrac{\beta s}{K_1} & \dfrac{q}{3\Lambda} & 0 & 0 \\ \dfrac{\beta r}{K_1 n_0 s}\left(K_2 r + \dfrac{K_1}{\beta s}\right) & -\dfrac{\beta r}{s K_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\theta^{(1)}{}^i_j = i \sum_{j \neq l} \theta^{(0)}{}^l_j g^i_l.$$

By identifying $A^{(2)}$ with $A^{(F)}$ and $Q^{(2)}$ with $Q^{(F)}$ we get the so-far unspecified coefficients K_1, K_2, g_j^i, $i, j = 1, 2, 3$ and $\mathscr{P}^{(F)}$ expressed in terms of $\mathscr{P}^{(EV)}$.

First we compare $A^{(2)}$ with $A^{(F)}$. We get

$$\beta\tau_e = \frac{2}{3}\frac{\beta}{K_1^2}$$

$$\tau_n = -\frac{2}{3}\frac{K_2 r}{n_0 K_1^2}$$

$$p_n - \gamma\tau_n = \frac{2}{3}\left(\frac{K_2}{K_1}\right)^2 r^2 + \frac{r}{\beta}. \tag{25}$$

By identifying the corresponding matrix elements $Q_{ij}^{(2)}$ and $Q_{ij}^{(F)}$ that are proportional to k we obtain trivial identities for $i = 1, j = 1, 2, 3$; $i = 2, j = 1, 2$; $i = 3, j = 3$. For $i = 2$; $j = 3$

$$\frac{\gamma}{n_0\beta} = K_2 r + \frac{K_2 s}{\beta}. \tag{26}$$

For $i = 3, j = 1$

$$p_n = -r\left(\frac{2}{3}\frac{K_2 s}{K_1} - \frac{1}{\beta}\right),$$

which, by using (25) and (26), appears to be a trivial identity. For $i = 3, j = 2$

$$n_0\beta\tau_n + \frac{1}{n_0}\gamma\tau_e = \frac{2}{3}\frac{s}{K_1},$$

which turns out again to be a trivial identity if using (25) and (26). By comparison of (26) and (2), (1) one gets

$$K_1 = 1$$

$$K_2 r = \tfrac{1}{2} c_4 W n_0. \tag{27}$$

Thus

$$\tau_e = \tfrac{2}{3}$$

$$\tau_m = -\tfrac{1}{3} c_4 W$$

$$p = \frac{r}{\beta} - \tfrac{1}{3} c_4 s W r. \tag{28}$$

By identifying the corresponding matrix element of $Q^{(2)}$ and $Q^{(F)}$ that are proportional to k^2 one obtains, for $j = 3$, $i = 1, 2$, $i = 3, j = 1, 2$,

$$g_1^2 + g_2^1 = 0$$

$$g_1^3 + g_2^3 = 0$$

$$g_3^1 + g_3^2 = 0, \tag{29}$$

for $i = 1, j = 1, 2$

$$\Xi_i^j = 0, \tag{30}$$

where

$$\Xi_i^j = 2\alpha_1^{(0)j}\theta_i^{(0)1}(\omega - \Lambda(g_1^2 - g_2^1)) + \Lambda(\alpha_3^{(0)j}\theta_i^{(0)1}(g_2^3 - g_1^3) + \alpha_1^{(0)j}\theta_i^{(0)3}(g_3^1 - g_3^2)) + \alpha_3^{(0)j}\theta_i^{(0)3}\omega_3.$$

From (14) and (26) we have

$$\frac{Q_{2,1}^{(F)}}{Q_{2,2}^{(F)}} = \frac{\tau_n}{\tau_e} = -\frac{K_2 r}{n_0}.$$

From

$$\frac{Q_{2,1}^{(F)}}{Q_{2,2}^{(F)}} = \frac{Q_{2,1}^{(2)}}{Q_{2,2}^{(2)}}$$

one gets

$$\Xi_1^2 = -\frac{K_2 r}{n_0}\Xi_2^2. \tag{31}$$

The solution of (29), (30), (31) is

$$g_3^2 = \frac{2}{3}\frac{s^2}{r}\Lambda^{-1}\omega_3,$$

$$g_3^1 = -g_3^2,$$

$$g_1^3 = \frac{1}{2s^2}\Lambda^{-1}\omega_3$$

$$g_2^3 = -g_1^3$$

$$g_2^1 = \left(\frac{\omega_3}{3r} - \tfrac{1}{2}\omega\right)\Lambda^{-1}$$

$$g_1^2 = -g_2^1 \tag{32}$$

From $Q_{44}^{(F)} = Q_{55}^{(F)} = Q_{44}^{(2)} = Q_{55}^{(2)}$ one gets

$$\eta = \beta\kappa_1(1 + \tfrac{4}{15}\pi n_0\sigma^3\vartheta(n_0))^2 \tag{33}$$

From $Q_{22}^{(F)} = Q_{22}^{(2)}$

$$\lambda = \beta^2\kappa_2(1 + \tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))^2 \tag{34}$$

and from $Q_{33}^{(F)} = Q_{33}^{(2)}$ we get (34) and

$$\eta_v = \beta^2\kappa_2(1 + \tfrac{2}{5}\pi n_0\sigma^3\vartheta(n_0))^2(s^2 - 1). \tag{35}$$

The map from $\mathscr{P}^{(EV)}$ into $\mathscr{P}^{(F)}$ – depending on (α, β) – is now explicit in (28), (33), (34) and (35). If $\beta \to \beta_c$ then $r \to 0$ (see 3.1) so that the critical phenomena are read from the dependence on r.

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