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Charge-Independent Analysis of Low Energy πN Scattering Data

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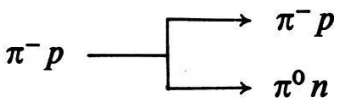
Abstract. We investigate the determination of charge-independent phase shifts from low energy hadron–hadron scattering data. For a many-channel system at low energies we present a simple formulation of a well-known method, which allows us to calculate an isospin-mixing matrix and the phase shift corrections due to both the electromagnetic interaction and the electromagnetic mass differences. A potential model is used together with a relativistically modified Schrödinger equation. In the case of the πN system, explicit formulae are derived for the phase shift corrections and the mixing parameter. It results that a charge-independent phase shift analysis is only possible by including not just the Coulomb corrections but also the electromagnetic mass difference corrections. The results of such an analysis will here be presented.

1. Introduction

It became evident from the work of Carter et al. [1] that, with the usual methods of calculating the electromagnetic corrections, it was not possible to achieve a charge-independent partial-wave analysis of pion–nucleon scattering experiments below 300 MeV. The importance of the electromagnetic corrections has also been emphasized by Woolcock [2], who points out the influence that these effects might have on the evaluation of f^2 and a_1–a_3.

One of the reasons for the unsatisfactory state of affairs in the analysis of the experimental data is certainly the neglect of the electromagnetic mass differences of the nucleons and of the pions. In the present paper we therefore take this effect into account using the model of Oades and Rasche [3, 4]. The result is that the data can indeed be analysed with charge-independent strictly nuclear phases.

The complications in the analysis of the data come mainly from the $\pi^- p$ initial state, which leads to three possible final two-particle states at low energies, namely $\pi^- p$, $\pi^0 n$ and γn. We therefore have to deal with a coupled three-channel problem. The general formalism for including the effect of the γn channel on the analysis of the $\pi^- p$ experiments was worked out for the first time by Rasche and Woolcock [5]. But, given the experimental accuracy presently attainable, there is no hope of including the γn channel in the analysis of $\pi^- p$ experiments. We therefore keep to the two-channel approximation



and neglect the γn channel, except for a phenomenological modification to be described later.

Even in this approximation it is not easy to extract the charge-independent phases from the experiments. To achieve this, one has to understand the electromagnetic corrections, which consist mainly of the Coulomb corrections and the dynamical effect of the mass difference between the $\pi^- p$ and $\pi^0 n$ states. Auvil, in the appendix of Ref. [6], considers the inclusion of mass difference effects in a two-channel problem. He arrives at a Schrödinger-like equation, with some relativistic modifications, which contains terms which take account of mass differences in a very approximate way. However, he makes no attempt to show how to calculate electromagnetic corrections from this equation. On the other hand, Oades and Rasche [3] had previously given a consistent formulation of Coulomb and mass difference corrections in a potential theory model. They show in detail how to calculate these corrections, and we have therefore chosen their model as the basis for the numerical calculations in this paper.

One can also try to separate out the electromagnetic effects in a relativistic, dispersion-theoretic approach. The first steps in this direction were taken by Sauter [7, 8], who handles the Coulomb corrections in a non-relativistic, dispersion-theoretic way. Sauter states that the numerical agreement with the potential model of Oades and Rasche is good in both the one-channel ($\pi^+ p \to \pi^+ p$) and the two-channel ($\pi^- p \to \pi^- p, \pi^0 n$) cases for low energies. Hamilton et al. [9] have pointed out that Sauter's formulae are useful only for small energies and they have given a different non-relativistic dispersion-theoretic treatment of the Coulomb corrections for the one- and two-channel problems. They planned to generalize their method to a relativistic treatment of the electromagnetic effects. This has been achieved for the one-channel problem by Tromborg and Hamilton [10]. But they have not yet treated the coupled-channel case with mass differences in a dispersion-theoretic way, so that there is at present no alternative to the potential model of Ref. [3].

In Ref. [3] a distinction is made between 'inner' and 'outer' Coulomb corrections. There were two reasons for doing this. First, the outer Coulomb corrections are to a large extent model independent and can be calculated to all orders in the Coulomb parameter without solving a differential equation. Second, one can apply a prescription of van Hove [11] to take into account relativistic effects in the outer corrections; this is not possible for the inner corrections. In the present calculations we do not make explicit the distinction between inner and outer corrections and try rather to formulate the problem in a way more suitable for practical calculations. Furthermore, in Ref. [3] the so-called 'additive electromagnetic amplitude' is that corresponding to a pure point-charge Coulomb potential. But the analysis in Ref. [1] uses an additive electromagnetic amplitude corresponding to a form factor for the charge distribution and we have to adapt the results of Ref. [3] slightly to take this fact into account. In a previous paper [12] we considered this problem in the single-channel case and pointed out an inconsistency connected with it in the calculations by Bugg [13] of the Coulomb corrections which were used in the analysis of Ref. [1].

In Section 2 we therefore describe briefly the general formalism for the coupled-channel problem, assuming that the given data correspond to an arbitrary additive electromagnetic amplitude. In Section 3 the electromagnetic corrections to the charge-independent phase shifts are given. In Section 4 the question of relativistic effects is considered, while Section 5 gives the perturbation expressions for the one- and two-channel cases. Section 6 is devoted to a review of the special potentials used in the numerical calculations and Section 7 describes the details of our phase shift analysis and gives the corrections for energies up to 250 MeV.

2. Many-Channel Formalism

2.1. The total scattering amplitude

We present here the general formalism for n coupled two-particle states. We assume that one of the particles has spin 0 and the other has spin $\frac{1}{2}$ and we work in the c.m. frame. The scattering and reaction processes are described by a generalized Schrödinger equation in matrix form, with a separate hadronic potential $U^{(lJ)}$ for each partial wave of orbital angular momentum l and total angular momentum j. In what follows we drop the superscripts (lj) and carry the calculations through for the s-wave $(0\frac{1}{2})$. The generalizations to arbitrary (lj) are evident and are stated without derivation. The hadronic potential U is then an $n*n$ matrix and the same is true for the electromagnetic potential, which we call V.

Throughout the paper we assume rotation invariance, parity conservation and time-reversal invariance. The last condition implies that U is a real and symmetric matrix. It would be possible, without difficulty, to generalize the results to the case of complex potentials; this would be a phenomenological way to take inelasticities into account. But in our case the inelasticities are small even at the highest energy considered, so that it is justified to neglect them in calculating the phase shift corrections.

We assume that the hadronic potential is charge independent. The total potential $U+V$ thus consists of a part U, which decomposes in the basis of isospin eigenstates into submatrices with definite isospin, and a part V, which is diagonal in the physical basis (where each particle in every two-particle state has a definite charge). This is the basic physical assumption made in [3] and [4] and we refer to [4] and [14] for a more detailed discussion and for the application to K^-p scattering.

It is convenient to assume that

$$U(r)=0 \qquad r \geqslant r_N$$

$$V_i(r)=q_i\gamma_i(m_i r)^{-1} \quad r \geqslant r_C. \tag{2.1}$$

Here $V_i(r)$ is the diagonal element of $V(r)$ for the ith physical channel, with q_i the corresponding c.m. momentum, m_i the corresponding reduced mass and γ_i the corresponding Coulomb parameter; thus

$$\gamma_i = z_i e^2 m_i q_i^{-1},$$

where z_i is the product of the charge numbers in the ith channel (e.g. $z_i=-1$ for π^-p, $z_i=0$ for $\pi^0 n$). Equation (2.1) facilitates some general statements and is used throughout the numerical calculations. We define

$$r_0=\max(r_N, r_C). \tag{2.2}$$

For $r<r_C$ we assume some spherically symmetric charge distribution for the particles which takes into account their form factors. The case $r_C=0$ gives the pure Coulomb potential corresponding to the electromagnetic interaction of point charges.

The radial Schrödinger equation for the s-wave then reads (see e.g. Refs. [3, 4])

$$(D+Q^2-2MU-2MV)|R\rangle=0, \tag{2.3}$$

where

$$D=\mathbb{1}_n\frac{d^2}{dr^2} \tag{2.4}$$

and Q^2 and M are diagonal matrices in the physical basis, with diagonal elements q_i^2 and m_i respectively. Equation (2.3) has n linearly independent regular solutions $|R\rangle_\alpha$ ($\alpha = 1,\ldots n$) vanishing at $r = 0$. Calling the vectors of the basis consisting of the physical states $|i\rangle$ ($i = 1,\ldots n$), the components $R_{i\alpha}(r)$ of the solution $|R\rangle_\alpha$ are defined by

$$|R\rangle_\alpha = \sum_{i=1}^{n} R_{i\alpha}|i\rangle \quad \alpha = 1, \ldots n. \tag{2.5}$$

Let us now define the functions

$$\mathring{F}_i(r) = (m_i/q_i)^{1/2} \sin(q_i r - \gamma_i \ln(2q_i r))$$

$$\mathring{G}_i(r) = (m_i/q_i)^{1/2} \cos(q_i r - \gamma_i \ln(2q_i r)). \tag{2.6}$$

It is well known that, due to the short range of the hadronic potential, the $R_{i\alpha}$ can be expanded asymptotically in terms of the $\mathring{F}_i$ and $\mathring{G}_i$ as

$$R_{i\alpha}(r) \underset{r\to\infty}{\sim} \mathring{F}_i(r)\,\mathring{\kappa}_{i\alpha} + \mathring{G}_i(r)\,\mathring{\sigma}_{i\alpha} \quad i, \alpha = 1, \ldots n. \tag{2.7}$$

The matrices $\mathring{\kappa}$ and $\mathring{\sigma}$ corresponding to the expansion coefficients determine the partial-wave scattering matrix $\mathring{a}$ for the total potential (see e.g. Ref. [14]). The connection is established most easily in terms of the matrix

$$\mathring{K} = \mathring{\sigma}\mathring{\kappa}^{-1}. \tag{2.8}$$

$\mathring{K}$ is the matrix which transforms the expansion coefficients $\mathring{\kappa}_{i\alpha}$ for a given regular solution of (2.3) into the corresponding $\mathring{\sigma}_{i\alpha}$. It is a consequence of time-reversal invariance that $\mathring{K}$ is a real symmetric matrix. In Appendix I we show explicitly how this property follows from (2.3) and (2.7). The partial-wave scattering matrix $\mathring{a}$ is given in terms of $\mathring{K}$ by

$$\mathring{a} = Q^{-1/2}\,\mathring{K}(\mathbb{1} - i\mathring{K})^{-1}\,Q^{-1/2}. \tag{2.9}$$

Appendix II gives the differential cross-sections in terms of the elements of the matrix $\mathring{a}$.

2.2. The additive electromagnetic and the nuclear scattering amplitudes

Let $\hat{V}$ be any electromagnetic potential with the same behaviour as V for $r \geqslant r_C$; $\hat{V}$ might differ from V for $r < r_C$. The Schrödinger equation

$$(D + Q^2 - 2M\hat{V})|\hat{R}\rangle = 0 \tag{2.10}$$

has n regular solutions $|\hat{R}\rangle_\alpha$ ($\alpha = 1,\ldots n$). The components $\hat{R}_{i\alpha}(r)$ ($i = 1,\ldots n$) of these solutions in the physical basis can be taken to have the asymptotic behaviour

$$\hat{R}_{i\alpha}(r) \underset{r\to\infty}{\sim} (m_i/q_i)^{1/2} \sin(q_i r - \gamma_i \ln(2q_i r) + \hat{v}_i) \quad \alpha = i$$

$$\hat{R}_{i\alpha}(r) = 0 \qquad \alpha \neq i. \tag{2.11}$$

For later reference we define

$$\hat{F}_i = \hat{R}_{ii}. \tag{2.12}$$

The irregular solution corresponding to $\hat{F}_i$ we call $\hat{G}_i$; it has the asymptotic behaviour

$$\hat{G}_i(r) \underset{r\to\infty}{\sim} (m_i/q_i)^{1/2} \cos(q_i r - \gamma_i \ln(2q_i r) + \hat{v}_i). \tag{2.13}$$

In the case $r_C = 0$ of a pure Coulomb potential we have

$$\hat{v}_i(r_C = 0) = \arg \Gamma(1 + i\gamma_i) \tag{2.14}$$

and $\hat{F}_i$ and $\hat{G}_i$ are multiples of the usual Coulomb wave functions (as tabulated e.g. in [15]) or the spherical Bessel and Neumann functions when $\gamma_i = 0$.

The functions $\hat{R}_{i\alpha}$ can be expanded asymptotically in terms of $\mathring{F}_i$ and $\mathring{G}_i$. This gives certain expansion coefficients as in (2.7), the matrices of which we call $\hat{\kappa}$ and $\hat{\sigma}$. Again we define as in (2.8) and (2.9) the symmetric matrix

$$\hat{K} = \hat{\sigma} \cdot \hat{\kappa}^{-1} \tag{2.15}$$

and the electromagnetic partial-wave scattering matrix

$$\hat{a} = Q^{-1/2} \hat{K}(\mathbb{1} - i\hat{K})^{-1} Q^{-1/2}. \tag{2.16}$$

We see from (2.11) that $\hat{\sigma}$ and $\hat{\kappa}$ are diagonal matrices in the physical basis, with matrix elements

$$\hat{\sigma}_{ii} = \sin \hat{v}_i \quad \hat{\kappa}_{ii} = \cos \hat{v}_i. \tag{2.17}$$

It follows from (2.13) that

$$\hat{G}_i(r) \underset{r\to\infty}{\sim} -\mathring{F}_i \hat{\sigma}_{ii} + \mathring{G}_i \hat{\kappa}_{ii}. \tag{2.18}$$

In view of (2.1) and (2.2) the functions $R_{i\alpha}(r)$ can be expanded for $r \geqslant r_0$ in terms of the $\hat{F}_i(r)$ and $\hat{G}_i(r)$

$$R_{i\alpha}(r) = \hat{F}_i(r) \kappa_{i\alpha} + \hat{G}_i(r) \sigma_{i\alpha}, \quad r \geqslant r_0. \tag{2.19}$$

The expansion coefficients are the elements of nuclear matrices κ and σ. They are distinct from the hadronic (purely nuclear), charge-independent matrices $\hat{\kappa}$ and $\hat{\sigma}$ to be defined later. From κ and σ we define the real symmetric matrix

$$K = \sigma \cdot \kappa^{-1}. \tag{2.20}$$

To get the connection between $\mathring{\kappa}$, $\hat{\kappa}$, κ, $\mathring{\sigma}$, $\hat{\sigma}$ and σ we compare (2.7) with (2.19), using (2.11), (2.12) and (2.18); the result is

$$\begin{aligned} \mathring{\kappa} &= \hat{\kappa}\kappa - \hat{\sigma}\sigma \\ \mathring{\sigma} &= \hat{\sigma}\kappa + \hat{\kappa}\sigma. \end{aligned} \tag{2.21}$$

Using (2.21) we can separate the electromagnetic partial-wave amplitude from the total partial-wave amplitude; we have

$$\begin{aligned} \mathring{K}(\mathbb{1} - i\mathring{K})^{-1} &= \mathring{\sigma}(\mathring{\kappa} - i\mathring{\sigma})^{-1} \\ &= (\hat{\sigma} + \hat{\kappa}K)(\hat{\kappa} - \hat{\sigma}K - i\hat{\sigma} - i\hat{\kappa}K)^{-1} \\ &= (\hat{\sigma} + \hat{\kappa}K)(\mathbb{1} - iK)^{-1}(\hat{\kappa} - i\hat{\sigma})^{-1} \\ &= [\hat{\sigma} + (\hat{\kappa}K + i\hat{\sigma}K)(\mathbb{1} - iK)^{-1}](\hat{\kappa} - i\hat{\sigma})^{-1} \\ &= \hat{K}(\mathbb{1} - i\hat{K})^{-1} + (\hat{\kappa} + i\hat{\sigma})K(\mathbb{1} - iK)^{-1}(\hat{\kappa} - i\hat{\sigma})^{-1}. \end{aligned} \tag{2.22}$$

Using (2.22), (2.9) and (2.16) we obtain

$$\mathring{a} = \hat{a} + Q^{-1/2}(\hat{\kappa} + i\hat{\sigma})\, K(\mathbb{1} - iK)^{-1}(\hat{\kappa} - i\hat{\sigma})^{-1}\, Q^{-1/2}. \tag{2.23}$$

Appendix II gives (2.23) in explicit form and shows how to get the electromagnetic scattering amplitude corresponding to $\hat{a}$.

The nuclear quantities κ, σ and K obviously depend on the choice of $\hat{V}$; in other words, they depend on the choice of the additive electromagnetic amplitude used in the analysis of the experiments. This makes it quite clear that they are not the hadronic (purely nuclear) quantities $\bar{\kappa}$, $\bar{\sigma}$ and $\bar{K}$ which we are going to define in the next section.

3. Corrections to Charge-Independent Quantities

We now define the charge-independent quantities $\bar{\kappa}$, $\bar{\sigma}$ and $\bar{K}$. To do this, we start from the Schrödinger equation

$$(D + \bar{Q}^2 - 2\bar{M}U)|\bar{R}\rangle = 0. \tag{3.1}$$

Here again $\bar{Q}$ and $\bar{M}$ are diagonal matrices in the physical basis. In addition the matrix elements corresponding to states belonging to the same isospin multiplet are put equal; this means that the electromagnetic mass differences have been neglected. From this and the charge independence of U it follows that, in the basis of isospin eigenstates, the system of coupled equations (3.1) decomposes into decoupled subsystems corresponding to definite isospin.

Equation (3.1) has n independent regular solutions $|\bar{R}\rangle_\alpha$ ($\alpha = 1,\ldots n$). Defining

$$\bar{F}_i(r) = (\bar{m}_i \bar{q}_i)^{1/2}\, r j_0(\bar{q}_i r)$$

$$\bar{G}_i(r) = -(\bar{m}_i \bar{q}_i)^{1/2}\, r n_0(\bar{q}_i r) \tag{3.2}$$

we can expand the components $\bar{R}_{i\alpha}$ of $|\bar{R}\rangle_\alpha$ in the charge basis for $r \geqslant r_0$ in the form

$$\bar{R}_{i\alpha}(r) = \bar{F}_i(r)\, \bar{\kappa}_{i\alpha} + \bar{G}_i(r)\, \bar{\sigma}_{i\alpha} \quad r \geqslant r_0. \tag{3.3}$$

Again we define a real symmetric matrix $\bar{K}$ by

$$\bar{K} = \bar{\sigma}\bar{\kappa}^{-1}. \tag{3.4}$$

In the basis of isospin eigenstates, $\bar{\sigma}$, $\bar{\kappa}$ and $\bar{K}$ decompose into direct sums of submatrices corresponding to definite isospin.

It is our aim to calculate $\bar{K}$ from K, thus obtaining the charge-independent (i.e. purely nuclear) scattering quantities from the nuclear ones. For this purpose we use the fact that every real symmetric matrix can be diagonalized by a real orthogonal transformation. Calling the transformation matrices τ and $\bar{\tau}$ for K and $\bar{K}$ respectively, and calling the eigenvalues $\tan\delta_i$ and $\tan\bar{\delta}_i$ ($i = 1,\ldots n$), we have

$$K_{fi} = \sum_{\alpha=1}^{n} \tau_{f\alpha} \tan\delta_\alpha\, \tau_{i\alpha} \quad \tau\tau^t = \mathbb{1} \tag{3.5}$$

$$\bar{K}_{fi} = \sum_{\alpha=1}^{n} \bar{\tau}_{f\alpha} \tan\bar{\delta}_\alpha\, \bar{\tau}_{i\alpha} \quad \bar{\tau}\bar{\tau}^t = \mathbb{1} \tag{3.6}$$

Comparing (3.5), (2.19) and (2.20), we see that the most suitable way of fixing

the n linearly independent column vectors $(R_{i\alpha})$ is to put

$$\kappa_{i\alpha} = \tau_{i\alpha} \cos \delta_\alpha \quad \sigma_{i\alpha} = \tau_{i\alpha} \sin \delta_\alpha \tag{3.7}$$

and correspondingly

$$\bar{\kappa}_{i\alpha} = \bar{\tau}_{i\alpha} \cos \bar{\delta}_\alpha \quad \bar{\sigma}_{i\alpha} = \bar{\tau}_{i\alpha} \sin \bar{\delta}_\alpha. \tag{3.8}$$

The next step consists in rewriting (2.3) for the regular solution $|R\rangle_\alpha$ in the form

$$\bar{M}^{-1}(D + \bar{Q}^2 - 2\bar{M}U)|R\rangle_\alpha = \bar{M}^{-1} \Delta |R\rangle_\alpha \tag{3.9}$$

with

$$\Delta = 2MV + 2(M - \bar{M})\, U - (Q^2 - \bar{Q}^2) \tag{3.10}$$

and rewriting (3.1) for the regular solution $|R\rangle_\beta$ in the form

$$\bar{M}^{-1}(D + \bar{Q}^2 - 2\bar{M}U)|\bar{R}\rangle_\beta = 0. \tag{3.11}$$

Multiplying (3.9) by ${}_\beta\langle \bar{R}|$ and (3.11) by ${}_\alpha\langle R|$ and subtracting the resulting equations from each other gives

$$\sum_{i=1}^{n} \bar{m}_i^{-1} \frac{d}{dr} W_r[\bar{R}_{i\beta}, R_{i\alpha}] = {}_\beta\langle \bar{R}|\bar{M}^{-1} \Delta |R\rangle_\alpha \tag{3.12}$$

where $W_r[\bar{R}_{i\beta}, R_{i\alpha}]$ is defined as in equation (A2). Integrating (3.12) from 0 to r_0 gives

$$\sum_{i=1}^{n} \bar{m}_i^{-1}\, W_{r_0}[\bar{R}_{i\beta}, R_{i\alpha}] = \int_0^{r_0} {}_\beta\langle \bar{R}|\bar{M}^{-1} \Delta |R\rangle_\alpha. \tag{3.13}$$

Using (2.19) and (3.3), the left-hand side of (3.13) can be expressed in terms of the known functions $\bar{F}_i$, $\bar{G}_i$, $\hat{F}_i$, $\hat{G}_i$, the measured nuclear quantities σ and κ and the charge-independent quantities $\bar{\sigma}$ and $\bar{\kappa}$, which are to be determined. Equation (3.13) is the basic equation which will be used to relate the nuclear quantities to the charge-independent ones. These relations then depend on some model-dependent integral over the nuclear region.

The general procedure for calculating δ_α and $\tau_{i\alpha}$ can be described as follows. Starting from the charge-independent parameters $\bar{\delta}_\alpha$ and $\bar{\tau}_{i\alpha}$, one chooses some hadronic potential U which reproduces these parameters via equations (3.1), (3.3), (3.4) and (3.6). In the next step one calculates a set of n linearly independent regular solutions $R_{i\alpha}$ of (2.3) at $r = r_0$ by integrating (2.3) from 0 to r_0; one also calculates their derivatives at $r = r_0$. This set of linearly independent regular solutions for $r \leqslant r_0$ will match smoothly at $r = r_0$ to the solutions of the form (2.19) for $r \geqslant r_0$ if $\kappa_{i\alpha}$ and $\sigma_{i\alpha}$ are chosen correctly. This determines δ_α and $\tau_{i\alpha}$ via the equations (2.20) and (3.5).

For later reference we define the electromagnetic corrections c_α to $\bar{\delta}_\alpha$ by

$$\delta_\alpha = \bar{\delta}_\alpha + c_\alpha. \tag{3.14}$$

Similarly we define the electromagnetic correction matrix C to $\bar{\tau}$ by

$$\tau = \bar{\tau}(\mathbb{1} + C). \tag{3.15}$$

Since τ and $\bar{\tau}$ are both orthogonal matrices it follows that

$$C + C^t = -C^t C. \tag{3.16}$$

If one is interested only in electromagnetic perturbation results (3.16) becomes

$$C \doteq -C^t. \tag{3.17}$$

Because the matrix $\bar{M}^{-1}\Delta$ is not symmetric it is convenient for later purposes to derive an antisymmetry relation. Starting from (3.9) and using (2.19) and (3.7), one finds by calculating $\sum_i W_{r_0}[R_{i\alpha}, R_{i\beta}]$

$$\int_0^{r_0} dr({}_\alpha\langle R|\bar{M}^{-1}\Delta|R\rangle_\beta - {}_\beta\langle R|\bar{M}^{-1}\Delta|R\rangle_\alpha) = \sin(\delta_\alpha - \delta_\beta)\sum_i \Delta m_i \tau_{i\alpha}\tau_{i\beta} \tag{3.18}$$

where

$$\Delta m_i = (m_i - \bar{m}_i)/\bar{m}_i. \tag{3.19}$$

4. Relativistic Modifications

It has been pointed out in the introduction, the question of taking into account relativistic corrections is very delicate. In Ref. [16] the modification of the additive electromagnetic amplitude is described; if applied correctly it gives this amplitude non-relativistically up to all orders and relativistically up to first order in the fine structure constant.

How to make relativistic corrections in order to go from the nuclear to the charge-independent quantities in a potential model is much more doubtful. This question is considered in [17] and in the appendix of [6]. Without further justification, we simply state that we try to take these relativistic effects into account by substituting the relativistic energy in the lab system of each particle for its non-relativistic mass. This means that the reduced mass m_i is replaced by the reduced energy in the lab system. The somewhat different procedure proposed in [6] would result only in minute changes (of the order $(q^{\text{lab}})^2/s$) in our numerical applications to the πN system.

5. Perturbation Expressions for πN Scattering

5.1. $\pi^+ p$ Scattering

At low energies the $\pi^+ p$ problem is a one-channel case, the physical state being identical with the isospin state with total isospin 3/2. Obviously $\tau = \bar{\tau} = 1$. We label the scattering phases by the subscript 3 to indicate the isospin quantum number. From (2.19), (3.3), (3.7) and (3.8) we have

$$\bar{R} = \bar{h} = \bar{F}\cos\bar{\delta}_3 + \bar{G}\sin\bar{\delta}_3 \quad r \geqslant r_0 \tag{5.1}$$

$$R = h\cos c_3' + (\hat{G}\cos\bar{\delta}_3 - \hat{F}\sin\delta_3)\sin c_3' \quad r \geqslant r_0 \tag{5.2}$$

where

$$h = \hat{F}\cos\bar{\delta}_3 + \hat{G}\sin\bar{\delta}_3$$

$$\delta_3 = \bar{\delta}_3 + c_3'. \tag{5.3}$$

There is no need to label the functions, since everything is self-evident. The prime on c_3' distinguishes it from the corresponding correction in the $\pi^- p$ case.

To get the perturbation result for c_3' to first order in the Coulomb parameter we use (3.13). Obviously Δ is of first order and so we can replace R by $\bar{R}$ in the integral on the right side of (3.13). Furthermore, we can neglect all terms of higher order in the Wronskian determinant of (3.13). This means that in (5.2) we can replace

$$\cos(c_3') \text{ by } 1$$

$$(\hat{G}\cos(\bar{\delta}_3) - \hat{F}\sin(\bar{\delta}_3))\sin(c_3') \text{ by } (\bar{G}\cos(\bar{\delta}_3) - \bar{F}\sin(\bar{\delta}_3))c_3'.$$

Inserting all this into (3.13) and using

$$W_{r_0}[\bar{G}, \bar{F}] = \bar{m} = m \tag{5.4}$$

where m is the reduced mass of the $\pi^+ p$ system, we arrive at the result

$$c_3' = m^{-1} W_{r_0}[\bar{h}, h] - X \tag{5.5}$$

where

$$X = 2\int_0^{r_0} V(r)\bar{R}^2(r)\,dr. \tag{5.6}$$

Equations (5.5) and (5.6) give explicitly the perturbation result for c_3'. They contain the Coulomb potential via V and the special choice of the additive electromagnetic amplitude via h.

5.2. $\pi^- p$ Scattering

Neglecting the γn state we have two channels for low energies. We label the physical states by

$$|-\rangle = |\pi^- p\rangle \tag{5.7}$$

$$|0\rangle = |\pi^0 n\rangle. \tag{5.8}$$

The isospin eigenstates for total isospin $\frac{1}{2}$ and 3/2 we label

$$|1\rangle \quad \text{and} \quad |3\rangle \tag{5.9}$$

respectively. We always assume that the hadronic interaction conserves isospin, so that $\bar{K}$ is diagonal in the isospin basis. Consequently $\bar{\tau}$ is the transformation matrix between the physical basis and the isospin basis:

$$\bar{\tau}_{-3} = \sqrt{1/3} \quad \bar{\tau}_{-1} = -\sqrt{2/3}$$

$$\bar{\tau}_{03} = \sqrt{2/3} \quad \bar{\tau}_{01} = \sqrt{1/3}. \tag{5.10}$$

The corresponding phases are denoted by $\bar{\delta}_3$ and $\bar{\delta}_1$. Isospin conservation implies that $\bar{\delta}_3$ is the same as in $\pi^+ p$ scattering.

We now have to make a choice for $\bar{M}$. In the $\pi^+ p$ system this problem does not arise, because it is natural to take the $\pi^+ p$ reduced mass m equal to m in the Schrödinger

equation. For the two-channel system we define

$$\overline{M} = m \cdot \mathbb{1}. \tag{5.11}$$

Again this choice seems natural, since the $\pi^- p$ reduced mass is equal to m. We also define

$$\overline{Q} = q \cdot \mathbb{1}. \tag{5.12}$$

Two further useful definitions are

$$\bar{h}_\alpha = \bar{F} \cos \bar{\delta}_\alpha + \bar{G} \sin \bar{\delta}_\alpha \quad r \geqslant r_0 \tag{5.13}$$

$$h_{i\alpha} = F_i \cos \bar{\delta}_\alpha + G_i \sin \bar{\delta}_\alpha \quad r \geqslant r_0 \tag{5.14}$$

where $i = 0, -$ and $\alpha = 3,1$. One should note that because of (3.2) and $\bar{m}_- = \bar{m}_0 = m$, in our two-channel case $\bar{F}_i$ and $\bar{G}_i$ do not depend on the index i. Due to charge-independence, $\bar{h}_3 = \bar{h}$ as defined in (5.1). Using (2.19), (3.3), (3.7), (3.8), (3.14) and (3.15) we then write

$$\bar{R}_{i\alpha} = \bar{\tau}_{i\alpha} \bar{h}_\alpha \quad r \geqslant r_0 \tag{5.15}$$

$$R_{i\alpha} = (\bar{\tau}_{i\alpha} + (\bar{\tau}C)_{i\alpha})(h_{i\alpha} \cos c_\alpha + (\hat{G}_i \cos \bar{\delta}_\alpha - \hat{F}_i \sin \bar{\delta}_\alpha) \sin c_\alpha) \quad r \geqslant r_0. \tag{5.16}$$

To first order we have

$$R_{i\alpha} \doteqdot \bar{\tau}_{i\alpha} h_{i\alpha} + \bar{\tau}_{i\alpha} c_\alpha (\bar{G} \cos \bar{\delta}_\alpha - \bar{F} \sin \bar{\delta}_\alpha) + (\bar{\tau}C)_{i\alpha} \bar{h}_\alpha. \tag{5.17}$$

We replace $|R\rangle_\alpha$ by $|\bar{R}\rangle_\alpha$ in the right-hand side of (3.13) and define

$$X_{\beta\alpha} = \int_0^{r_0} dr\, {}_\beta\langle \bar{R} | \overline{M}^{-1} \Delta | \bar{R} \rangle_\alpha. \tag{5.18}$$

Inserting (5.17) for $R_{i\alpha}$ on the left-hand side of (3.13) and using (5.4) as well as the orthogonality of $\bar{\tau}$ we arrive at

$$m^{-1} \sum_i \bar{\tau}_{i\beta} \bar{\tau}_{i\alpha} W_{r_0}[\bar{h}_\beta, h_{i\alpha}] - c_\alpha \langle \beta | \mathbb{1} | \alpha \rangle - C_{\beta\alpha} \sin(\bar{\delta}_\alpha - \bar{\delta}_\beta) \doteqdot X_{\beta\alpha}. \tag{5.19}$$

Taking $\alpha = \beta$ in (5.19) and using (5.10) we get

$$c_3 \doteqdot -X_{33} + (1/3m)\, W_{r_0}[\bar{h}_3, h_{-3}] + (2/3m)\, W_{r_0}[\bar{h}_3, h_{03}] \tag{5.20}$$

$$c_1 \doteqdot -X_{11} + (2/3m)\, W_{r_0}[\bar{h}_1, h_{-1}] + (1/3m)\, W_{r_0}[\bar{h}_1, h_{01}]. \tag{5.21}$$

Taking $\alpha = 1$ and $\beta = 3$ we get

$$C_{31} \sin(\bar{\delta}_3 - \bar{\delta}_1) \doteqdot X_{31} + (\sqrt{2}/3m)\, W_{r_0}[\bar{h}_3, h_{-1}] - (\sqrt{2}/3m)\, W_{r_0}[\bar{h}_3, h_{01}]. \tag{5.22}$$

From (3.17) we know that

$$C_{31} = -C_{13}. \tag{5.23}$$

Equations (5.20)–(5.23) give the electromagnetic corrections in perturbation theory.

We now calculate the quantities X_{33}, X_{11} and X_{31} more explicitly. To this end we put

$$|\bar{R}\rangle_\alpha = \bar{h}_\alpha(r) |\alpha\rangle \quad \alpha = 3, 1 \quad r \leqslant r_0. \tag{5.24}$$

It is seen that $\bar{h}_3$ and $\bar{h}_1$ are the regular solutions of (3.1) in the isospin basis. Due to charge-independence the system of equations (3.1) decouples in this basis. The $\bar{h}_\alpha$ as defined in (5.24) have to fit smoothly at $r = r_0$ to the $\bar{h}_\alpha$ as defined in (5.13). We put

$$\Delta m = m_0 - m \quad \Delta q^2 = q_0^2 - q^2 \tag{5.25}$$

$$V_- = \langle -|V|- \rangle. \tag{5.26}$$

One should note that V_- is minus the potential going into (5.6). Collecting all the definitions and using (5.10) we get

$$X_{33} = 2/3m \int_0^{r_0} dr(\bar{h}_3)^2 (mV_- + 2\Delta m U_{33} - \Delta q^2) \tag{5.27}$$

$$X_{11} = 1/3m \int_0^{r_0} dr(\bar{h}_1)^2 (4mV_- + 2\Delta m U_{11} - \Delta q^2) \tag{5.28}$$

$$X_{31} = \sqrt{2}/3m \int_0^{r_0} dr\bar{h}_1 \bar{h}_3(-2mV_- + 2\Delta m U_{11} - \Delta q^2). \tag{5.29}$$

In order to give the nuclear scattering amplitudes a_{--} and a_{0-} in terms of the charge-independent quantities $\bar{\delta}_\alpha$ and the corrections c_α, $C_{\alpha\beta}$ we define

$$a_\alpha = \sin(\bar{\delta}_\alpha + c_\alpha) \exp(i\bar{\delta}_\alpha + ic_\alpha). \tag{5.30}$$

Using (5.10) we get from (A10) and (3.5) in perturbation theory

$$a_{--} = (3q)^{-1} \exp(2i\hat{v}) [a_3 + 2a_1 + 2\sqrt{2}\, C_{31}(a_3 - a_1)] \tag{5.31}$$

$$a_{0-} = (9qq_0)^{-1/2} \exp(i\hat{v}) [(a_3 - a_1)(\sqrt{2} + C_{31})] \tag{5.32}$$

where $\hat{v} = \hat{v}_-$. Note that the exponential functions in (5.31) and (5.32) corresponding to the additive electromagnetic amplitude have not been approximated by an expansion.

6. Special Potentials for πN Scattering

We will see in the next section that the numerical calculation of the phase shift corrections at low energies is not very sensitive to the shape of the potentials. One can therefore use very simple expressions for U and V.

The electromagnetic potential should take account of the extended charge distribution of the particles. We take for V the potential of a uniformly charged sphere with radius r_C and a point charge,

$$V_\pm(r) = \begin{cases} \pm e^2(1.5 - 0.5r^2/r_C^2)/r_C & r \leqslant r_C. \\ \pm e^2/r & r \geqslant r_C. \end{cases} \tag{6.1}$$

The hadronic potentials must reproduce the charge-independent phase shifts which are the free parameters in a phase shift analysis. But the complexity of the connection between a potential and the corresponding phase shift makes it hard to choose directly a potential with the right property. It is much easier to choose an ansatz for the solutions

of the purely hadronic equation (3.1) and to determine the parameters of this ansatz in terms of the charge-independent phase shifts. To test the dependence of the electromagnetic corrections on the ansatz, two different expressions have been used in the numerical calculations. We refer to them by I and II and use the following abbreviations

$$x = qr, \quad x_N = qr_N \tag{6.2}$$

$$\xi_{\alpha;lj}(x) = j_l(x)\cos\bar{\delta}_{\alpha;lj} - n_l(x)\sin\bar{\delta}_{\alpha;lj}$$

$$\eta_{\alpha;lj}(x) = j_{l+1}(x)\cos\bar{\delta}_{\alpha;lj} - n_{l+1}(x)\sin\bar{\delta}_{\alpha;lj}. \tag{6.3}$$

In (6.3) we have made the angular momentum dependence explicit in an obvious way.

Ansatz I consists in taking the simplest polynomial for $\bar{h}_{\alpha;lj}(x)$ in the range $x \leqslant x_N$ which has the behaviour x^{l+1} for small x to take account of the centrifugal term in the Schrödinger equation:

$$\bar{h}_{\alpha;lj}(x) = \begin{cases} (m/q)^{1/2}\, x^{l+1}(A_{\alpha;lj} + B_{\alpha;lj}\, x) & x \leqslant x_N \\ (m/q)^{1/2}\, x\xi_{\alpha;lj} & x \geqslant x_N. \end{cases} \tag{6.4}$$

The constants A and B are fixed by the requirement of continuity of the function $\bar{h}$ and its derivative at $x = x_N$. The result is

$$A_{\alpha;lj} = x_N^{-l}(\xi_{\alpha;lj}(x_N) + x_N \eta_{\alpha;lj}(x_N))$$

$$B_{\alpha;lj} = -x_N^{-l}\, \eta_{\alpha;lj}(x_N). \tag{6.5}$$

A typical form of the potential corresponding to ansatz I is shown in Figure 1. In fact the quantities $\bar{h}$, A, B in (6.4) should carry an index I to indicate their relation to ansatz I; for reasons of typographical simplicity we suppress this index and use the same letters in the following for ansatz II.

Ansatz II consists in taking for $\bar{h}_{\alpha;lj}(x)$ in the range $x \leqslant x_N$ a more complicated polynomial corresponding to a continuous potential at $x = 0$ and $x = x_N$:

$$\bar{h}_{\alpha;lj}(x) = \begin{cases} (m/q)^{1/2}\, x^{l+1}(A_{\alpha;lj} + B_{\alpha;lj}\, x^2 + C_{\alpha;lj}\, x^4) & x \leqslant x_N \\ (m/q)^{1/2}\, x\xi_{\alpha;lj} & x \geqslant x_N. \end{cases} \tag{6.6}$$

The constants A, B and C are fixed by the continuity requirements of the function $\bar{h}$, of its derivative and of the corresponding potential at $x = x_N$. The result is

$$A_{\alpha;lj} = (x_N^{-l}/8)\,[(8 - x_N^2)\,\xi_{\alpha;lj}(x_N) + x_N(2l+7)\,\eta_{\alpha;lj}(x_N)]$$

$$B_{\alpha;lj} = (x_N^{-l-1}/8)\,[2x_N\,\xi_{\alpha;lj}(x_N) - (4l+10)\,\eta_{\alpha;lj}(x_N)]$$

$$C_{\alpha;lj} = -(x_N^{-l-3}/8)\,[x_N\,\xi_{\alpha;lj}(x_N) - (2l+3)\,\eta_{\alpha;lj}(x_N)]. \tag{6.7}$$

A typical form of the potential corresponding to ansatz II is shown in Figure 1; it is finite at $r = 0$.

At low energies the potentials are only weakly energy dependent. From the definitions of r_N, r_C and r_0 it follows that the range r_0 in the formulae of the preceding sections is equal to the greater of r_N and r_C. It is not possible to determine both parameters r_N

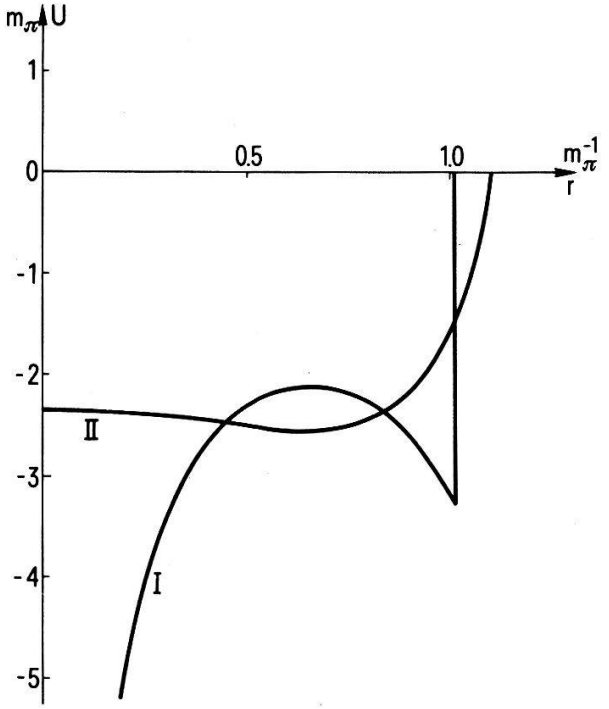


Figure 1
The hadronic $p33$ potential U for ansatz I and ansatz II at $T_{\text{kin}}^{\text{lab}} = 191.9$ MeV.

and r_C together with the charge-independence phase shifts because the data are not accurate enough. If one chooses a value of the order m_π^{-1} [1]) for r_C, then the combined $\pi^+ p$ and $\pi^- p$ data determine well a value for r_N. For $r_C = 1.42 m_\pi^{-1}$ the phase shift analysis at the first resonance described in Section 7 gives the values

$$r_N = \begin{cases} (1.01 \pm 0.14)\, m_\pi^{-1} & \text{(ansatz I)} \\ (1.10 \pm 0.17)\, m_\pi^{-1} & \text{(ansatz II).} \end{cases} \tag{6.8}$$

7. Phase Shift Analysis Below 250 MeV

We now describe the numerical procedure for performing the phase shift analysis, including the corrections described in the previous sections. As mentioned in the introduction we can fit the data in a charge-independent way by including the corrections

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[1]) m_π is the mass of the charged pion. One fermi corresponds to $0.71 m_\pi^{-1}$.

due to the electromagnetic mass differences. This means in particular that we can fit the π^+p and π^-p data with the same charge-independent $p33$ phase shift.

The data analysed here are the same as in Ref. [1], consisting of measurements of $\sigma_{tot}(\pi^+p)$, $\sigma_{tot}(\pi^-p)$, $\sigma(\pi^-p \to \pi^0 n)$, $d\sigma/d\Omega(\pi^+p \to \pi^+p)$ and $d\sigma/d\Omega(\pi^-p \to \pi^-p)$. A strictly energy-independent analysis of these data is not possible because they are measured at different energies. Therefore we describe now our procedure for the analysis in detail.

7.1. Free and fixed parameters

As shown in the first two columns of Table I we divide all the data into sets, each of which contains the results of either the π^+p or the π^-p experiments in a small energy interval. Each data set is fitted separately by minimizing χ^2, varying the charge-independent s- and p-wave phases $\bar{\delta}_\alpha$ at two energies T_1 and T_2. These are the energies of the differential cross-section measurements. Starting from these phases and including the corrections we obtain the nuclear s- and p-wave phase shifts δ_α and the mixing parameter C_{31} in the coupled-channel case at the energies T_1 and T_2.

The same nuclear quantities at other energies within the energy interval are then obtained according to

$$q^{2l+1}\operatorname{ctg}\delta_\alpha - q_1^{2l+1}\operatorname{ctg}\delta_{\alpha;1} = (q_2^{2l+1}\operatorname{ctg}\delta_{\alpha;2} - q_1^{2l+1}\operatorname{ctg}\delta_{\alpha;1})\frac{q^2-q_1^2}{q_2^2-q_1^2} \tag{7.1}$$

$$C_{31} - C_{31;1} = (C_{31;2} - C_{31;1})\frac{q^2-q_1^2}{q_2^2-q_1^2}\,. \tag{7.2}$$

Here the additional indices 1 and 2 refer to the first and the second energy of column 4 in Table I. Column 3 lists explicitly the free parameters for each least-squares fit. Some small phases near the resonance at 190 MeV, which are omitted from the list, cannot be determined well by the data and are therefore kept fixed in the corresponding fit.

Table I

Minimized χ^2 for each least-squares fit in small energy intervals; a, b and c refer to the three analyses a), b) and c)

Experiments	T_{kin}^{lab} intervals (MeV)	Free parameters: Charge-independent phases	Free parameters: At T (MeV)	Number of data points	Degrees of freedom	χ^2 at minimum: a	b	c
π^+p	(71.6/118.9)	$s31, p31, p33$ $s31, p31, p33$	94.5 114.1	11	5	7.7	7.7	7.7
π^+p	(120.4/155.8)	$s31, p31, p33$ $s31, p31, p33$	142.9 124.8	21	15	26.4	26.4	26.4
π^+p	(161.2/194.3)	$p33$ $p33$	166.0 194.3	32	30	24.7	24.6	24.2
π^+p	(205.3/236.3)	$s31, p31, p33$ $s31, p31, p33$	214.6 236.3	30	24	22.3	22.3	22.3
π^-p	(76.7/96.0)	$s11, p11, p13$	88.5 119.3	6	3	6.2	6.4	9.1
π^-p	(114.4/144.1)	$s11, p11, p13$ $s11, p11, p13$	119.3 144.1	18	12	24.8	24.8	19.3
π^-p	(159.6/192.3)	$s11, p11, p13$ $s11, \quad p13$	161.9 191.9	26	21	29.0	29.0	29.5
π^-p	(208.9/237.9)	$s11, p11, p13$ $s11, p11, p13$	219.6 237.9	24	18	19.8	20.0	22.2

In the fifth data set of Table I only the phases at T_1 are varied, while the phases at T_2 are taken from the fit in the sixth set.

The inelasticities η_α and the d- and f-wave phase shifts are fixed parameters included in the fits. Their values are taken from Ref. [1]. The partial-wave isospin amplitudes in equation (5.30) are now modified by the inelasticities in the usual way,

$$a_\alpha = \eta_\alpha \sin(\bar{\delta}_\alpha + c_\alpha)\exp(i\bar{\delta}_\alpha + ic_\alpha) - (1-\eta_\alpha)/(2i). \tag{7.3}$$

The ranges r_N and r_C described in Section 6 are two further parameters. They are needed only for the calculation of the corrections and are therefore kept fixed at some reasonable values. Fixing r_C at typical values of 1 or 2 fm, r_N was determined by independent least-squares fits with different r_N to the data in the third and the seventh sets of Table I. This has to be done because the mass difference corrections are more dependent on r_N than was the case earlier for the Coulomb corrections. After this preliminary procedure to obtain the right range parameters, all fits in the different energy intervals were done with fixed values for r_C and r_N.

7.2. Scattering amplitudes from the parameters

To calculate the nuclear quantities from the charge-independent phase shifts $\bar{\delta}_\alpha$ we use the electromagnetic phase shift corrections described in the previous sections. In all the numerical calculations of this section we have done this only for the $s1$ and $p3$ partial waves. For the other phases the corrections are much smaller than the accuracy to which these phases can be determined from the existing data.

The inclusion of the corrections by means of our computer program does not follow equations (5.5) and (5.20–5.22). The nuclear quantities are calculated from the potentials of Section 6 by integrating the Schrödinger equation (2.3) from 0 to $r_0 = \max(r_C, r_N)$, followed by expanding the solutions at r_0 according to equation (2.19). We have checked that the perturbation expressions of Section 5 give the same phase shift corrections within 1 %.

The functions $\hat{F}_i$ and $\hat{G}_i$ and the corresponding Coulomb phases $\hat{v}_i$ depend on the choice of the additive electromagnetic amplitudes, which are those of Ref. [16] multiplied by form factors $(1 - t/m_\rho^2)^{-4}$, where $m_\rho \approx 5m_\pi$. For the phase shift analysis of the existing data it is well justified to use first-order expressions in the Coulomb parameter for $\hat{v}_i$, $\hat{F}_i$ and $\hat{G}_i$. Therefore the Coulomb phases to be used in equation (A10) are (see Ref. [12])

$$\hat{v}_{i;l} = \arg\Gamma(l+1+i\gamma_i) + \Delta v_{i;l} \tag{7.4}$$

where

$$\Delta v_{i;0} = \gamma_i/2[\ln(1+\varepsilon) + \varepsilon/6\,(1+\varepsilon)^{-3}(18+27\varepsilon+11\varepsilon^2)]$$

$$\Delta v_{i;1} = \gamma_i/2[\ln(1+\varepsilon) - \varepsilon/6(1+\varepsilon)^{-3}\,(6+5\varepsilon+\varepsilon^2)]$$

$$\varepsilon = 4q_i^2/m_\rho^2 \approx 0.16 q_i^2 m_\pi^{-2}.$$

Since we had a computer program for the usual Coulomb wave functions F_l and G_l, the functions $\hat{F}_{i;l}$ and $\hat{G}_{i;l}$ used for the expansion (2.19) at r_0 are approximated by (see Appendix III)

$$\hat{F}_{i;l}(r_0) \doteq F_l(q_i r_0, \gamma_i) + G_l(q_i r_0, \gamma_i)\,\Delta v_{i;l}$$

$$\hat{G}_{i;l}(r_0) \doteq G_l(q_i r_0, \gamma_i) - F_l(q_i r_0, \gamma_i)\,\Delta v_{i;l}. \tag{7.5}$$

7.3. *Results*

To determine the charge-independent $I = 3/2$ s- and p-wave phase shifts, a least-squares fit was made for each of the four $\pi^+ p$ data sets defined in Table I. The corresponding $I = 3/2$ phases at energies needed for analysing the $\pi^- p$ data sets were obtained by using equation (7.1) for the charge-independent phase shifts. With these fixed values of the $I = 3/2$ phases, a fit was made for each of the four $\pi^- p$ data sets to determine the charge-independent $I = \frac{1}{2}$ s- and p-wave phase shifts.

Three independent analyses were done, labelled by a), b) and c). They are characterized by the following features:

a) ansatz II (see Section 6) $r_C = 1.42 m_\pi^{-1}$

b) ansatz I $r_C = 1.42 m_\pi^{-1}$

c) ansatz II $r_C = 0.71 m_\pi^{-1}$.

As described above, each analysis was started by fixing a value for r_N. From the preliminary fits to the third and seventh data set, we obtained

$$r_N = \begin{cases} 1.10 \pm 0.17 m_\pi^{-1} & \text{a)} \\ 1.01 \pm 0.14 & \text{b)} \\ 1.34 \pm 0.13 & \text{c)} \end{cases}$$

Changing the given values by the standard errors quoted results in an increase of 1 in the χ^2 for the seventh data set. The minimized χ^2 for all the fits with the above range parameters is presented in the last three columns of Table I. It can be seen that a reasonably good fit is possible with charge-independent phase shifts if one includes not only the Coulomb corrections, but also the mass difference corrections according to the method described in this work.

The higher χ^2 values for the fits to the $\pi^- p$ data sets is not surprising if one recalls the fact that the $I = 3/2$ phases must first be provided by equation (7.1) from the values obtained from the $\pi^+ p$ fits. It would be desirable to have the $\pi^+ p$ and $\pi^- p$ measurements at the same energy.

The differences between the corrections using ansatz I and ansatz II are unimportant (analysis a) and b)). This is illustrated in Table II for the most important $p3$ corrections.

Table II

$p3$ corrections in degrees for $\pi^+ p(c')$ and $\pi^- p(c_3, c_1, C_{31})$ scattering for the three different analyses a), b) and c).

T_{kin}^{lab}	$c'_{;1+}$			$c_{3;1+}$			$c_{1;1+}$			$C_{31;1+}$		
(MeV)	a	b	c	a	b	c	a	b	c	a	b	c
88.5	−0.38	−0.38	−0.38	0.42	0.42	0.40	−0.02	−0.02	−0.02	−0.17	−0.17	−0.12
119.3	−0.56	−0.55	−0.55	0.35	0.36	0.32	−0.03	−0.03	−0.02	−0.05	−0.05	−0.01
144.1	−0.56	−0.55	−0.52	0.14	0.14	0.01	−0.02	−0.02	−0.01	0.01	0.01	0.05
161.9	−0.39	−0.39	−0.35	−0.09	−0.10	−0.14	−0.02	−0.02	0	0.04	0.04	0.07
191.9	−0.01	−0.01	0.09	−0.36	−0.35	−0.38	−0.03	−0.03	0	0.06	0.06	0.09
219.6	0.24	0.24	0.29	−0.40	−0.39	−0.42	−0.03	−0.03	0	0.08	0.08	0.10
237.9	0.31	0.32	0.35	−0.36	−0.36	−0.36	−0.03	−0.03	0.01	0.09	0.09	0.10

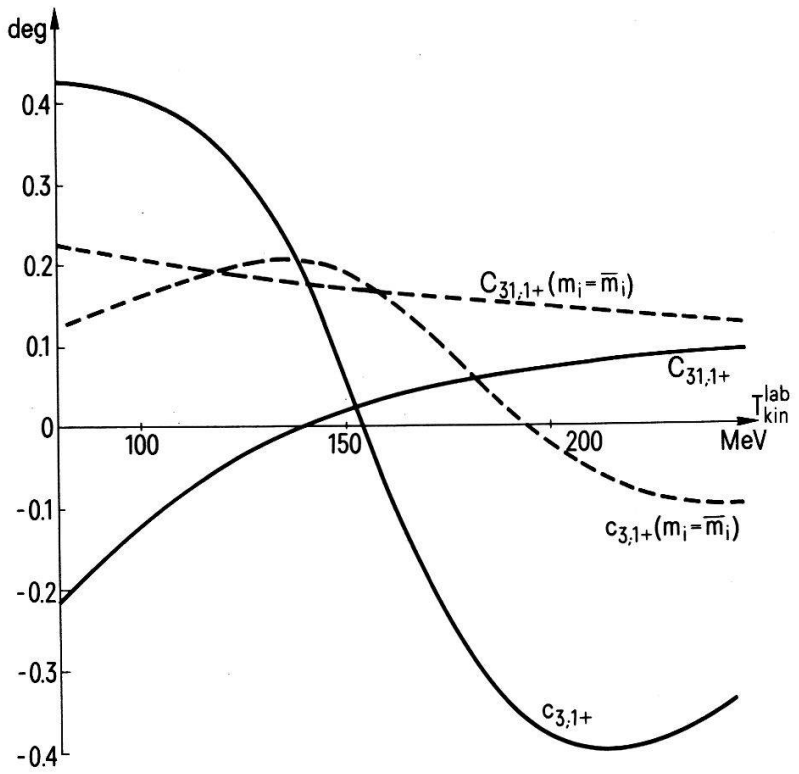

Figure 2
$p33$ phase shift corrections $c_{3;1+}$ and mixing parameters $C_{31;1+}$ (solid lines); the dashed lines are the corresponding corrections without any mass differences ($m_i = \bar{m}_i$).

This is again a confirmation that the corrections are only weakly dependent on the shape of the hadronic potentials, provided that they have a suitable range. Comparison of columns a and c shows that the corrections are more dependent on the choice of r_C. The fits below 100 MeV prefer for r_C a value about $1.4m_\pi^{-1}$ rather than about 0.7. This fact has no physical meaning because the chosen electromagnetic potential is not expected to describe exactly the actual physical situation within the nuclear range.

Figure 2 shows the decomposition of the corrections $c_{3;1+}$ and $C_{31;1+}$ into Coulomb and mass difference corrections. From this illustration it is clear that in analysing πN data at low energies mass difference corrections and Coulomb corrections play equally important roles.

Tables III and IV contain the charge-independent phase shifts and the corrections for the s- and p-waves resulting from the phase shift analysis a).

Table III
Charge-independent πN phase shifts in degrees

T_{kin}^{lab} (MeV)	$\delta_{3;0+}$	$\delta_{1;0+}$	$\delta_{3;1-}$	$\delta_{1;1-}$	$\delta_{3;1+}$	$\delta_{1;1+}$
88.5	-8.70 ± 0.12	8.59 ± 0.14	-1.74 ± 0.22	-1.68 ± 0.08	16.97 ± 0.06	-0.99 ± 0.13
119.3	-8.15 ± 0.19	9.77 ± 0.16	-3.41 ± 0.24	-1.05 ± 0.28	32.94 ± 0.06	-1.71 ± 0.12
144.1	-12.94 ± 0.28	9.64 ± 0.17	-3.66 ± 0.34	-0.55 ± 0.22	51.35 ± 0.07	-1.44 ± 0.12
161.9	-12.50	10.19 ± 0.23	-4.36	0.72 ± 0.42	67.33 ± 0.16	-1.23 ± 0.25
191.9	-14.78	10.62 ± 0.50	-5.18	1.96	91.68 ± 0.55	-2.79 ± 0.29
219.6	-17.30 ± 0.29	13.13 ± 0.26	-5.97 ± 0.54	3.20 ± 0.37	109.78 ± 0.22	-2.96 ± 0.13
237.9	-18.40 ± 0.32	13.11 ± 0.34	-6.22 ± 0.48	5.13 ± 0.30	117.20 ± 0.21	-3.22 ± 0.11

Table IV
Electromagnetic corrections to πN phase shifts (c' for π^+p, c_3 and c_1 for π^-p scattering) and mixing parameter C_{31} in degrees for partial waves $s1$ and $p3$ corresponding to a form factor $(1 - t/m_\rho^2)^{-4}$ in the electromagnetic amplitude

$T_{\text{kin}}^{\text{lab}}$ (MeV)	$c'_{;0+}$	$c_{3;0+}$	$c_{1;0+}$	$C_{31;0+}$	$c'_{;1+}$	$c_{3;1+}$	$c_{1;1+}$	$C_{31;1+}$
88.5	0.12	−0.07	0	0.10	−0.38	0.42	−0.02	−0.17
119.3	0.11	−0.04	−0.03	0.10	−0.56	0.35	−0.03	−0.05
144.1	0.13	−0.04	−0.05	0.10	−0.56	0.14	−0.02	0.01
161.9	0.12	−0.03	−0.07	0.10	−0.39	−0.09	−0.02	0.04
191.9	0.12	−0.02	−0.09	0.10	−0.01	−0.36	−0.03	0.06
219.6	0.12	−0.01	−0.11	0.08	0.24	−0.40	−0.03	0.08
237.9	0.12	−0.01	−0.12	0.08	0.31	−0.36	−0.03	0.09

7.4. Conclusion

It should be mentioned that the phase shift corrections and with them the phase shift analysis are influenced by the choice of the hypothetical charge-independent masses. If for example we choose $\bar{m}_1 \neq m_1$, we have mass difference corrections in the single-channel case also. But with the present accuracy of the experimental data there is no hope of determining a value for $\bar{m}$ by fitting the data.

We do not claim that the model-dependent calculation of the electromagnetic corrections presented here predicts exact numbers. Nevertheless it shows two things. Firstly, it gives for the first time a rough estimate of the mass difference corrections to the phase shifts. These corrections cannot be neglected any longer in a phase shift analysis of precise low energy πN experiments. Secondly, a charge-independent analysis of the present low energy πN scattering data is possible with the inclusion of mass difference corrections using even a very simple potential model.

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APPENDIX I

We show that the matrix K defined by (2.20) is real and symmetric. The arguments given here apply equally well to the matrices $\mathring{K}$, $\hat{K}$ and $\bar{K}$ defined in Sections 2 and 3. The reality of K is a simple consequence of the fact that all the matrices in (2.3) are real and so we can choose the $R_{i\alpha}$ to be real functions.

The symmetry of K is a consequence of the fact that $M^{-1}Q^2$, U and V are symmetric in the physical basis. We therefore have from (2.3) the symbolic relation

$$_\beta\langle R|M^{-1}D|R\rangle_\alpha - {}_\alpha\langle R|M^{-1}D|R\rangle_\beta = 0.$$

This equation reads explicitly

$$\frac{d}{dr}\sum_{i=1}^{n} m_i^{-1} W_r[R_{i\beta}, R_{i\alpha}] = 0 \tag{A1}$$

where

$$W_r[R_{i\beta}, R_{i\alpha}] = R_{i\beta}\frac{d}{dr}R_{i\alpha} - R_{i\alpha}\frac{d}{dr}R_{i\beta}. \tag{A2}$$

Because $W_{r=0}[R_{i\beta}, R_{i\alpha}] = 0$ for two regular solutions, we have from (A1)

$$\sum_{i=1}^{n} m_i^{-1} W_r[R_{i\beta}, R_{i\alpha}] = 0 \qquad r > 0. \tag{A3}$$

Inserting the asymptotic expansion (2.19) and using the relation

$$W_r[\hat{G}_i, \hat{F}_i] = m_i \tag{A4}$$

we have

$$\sum_{i=1}^{n} (\kappa_{i\beta}\sigma_{i\alpha} - \sigma_{i\beta}\kappa_{i\alpha}) = 0$$

or, in matrix form,

$$\kappa^t \sigma = \sigma^t \kappa.$$

We therefore have

$$K^t = (\sigma\kappa^{-1})^t = (\kappa^{-1})^t \sigma^t = (\kappa^{-1})^t \sigma^t \kappa\kappa^{-1} = (\kappa^{-1})^t \kappa^t \sigma\kappa^{-1} = \sigma\kappa^{-1} = K. \tag{A5}$$

Thus the transpose K^t of K coincides with K and we have established that K is symmetric

APPENDIX II

The total scattering amplitudes f and g in the c.m. system for the process $i \to f$ are given by

$$\mathring{f}_{fi} = \sum_{l,j} (j + \tfrac{1}{2}) P_l(\cos\theta)\, \mathring{a}_{fi;lj} \tag{A6}$$

$$\mathring{g}_{fi} = \sum_{l,j} (-1)^{j-l-1/2} P_l^1(\cos\theta)\, \mathring{a}_{fi;lj}. \tag{A7}$$

Here we have introduced explicitly the partial-wave indices (lj). It is clear how the formulae of Sections 2 and 3 have to be modified to apply to arbitrary l. It should be particularly noted that in practical calculations one might need a different potential $U^{(lj)}$ for each partial wave. The differential cross-section for the process $i \to f$ is

$$\frac{d\sigma_{fi}}{d\Omega} = q_f q_i^{-1}(|\mathring{f}_{fi}|^2 + |\mathring{g}_{fi}|^2). \tag{A8}$$

The explicit form of (2.23) is, for general (lj),

$$\mathring{a}_{fi;lj} = q_i^{-1} \sin\hat{v}_{i;l} \exp(i\hat{v}_{i;l}) \langle f|\mathbb{1}|i\rangle + a_{fi;lj} \tag{A9}$$

where

$$a_{fi;lj} = (q_i q_f)^{-1/2} \exp(i\hat{v}_{i;l} + i\hat{v}_{f;l}) \langle f|K^{(lj)}(\mathbb{1} - iK^{(lj)})^{-1}|i\rangle. \tag{A10}$$

Insertion of (A9) and (A10) into (A6) and (A7) shows the influence of the choice of $\hat{V}$ on the nuclear quantities $K^{(lj)}$. Defining the additive electromagnetic amplitude $\hat{f}_{ii}$ as

$$\hat{f}_{ii} = \sum_{l,j} (j+\tfrac{1}{2}) P_l(\cos\theta) q_i^{-1} \sin\hat{v}_{i;l} \exp(i\hat{v}_{i;l}) \tag{A11}$$

we have, more explicitly,

$$\mathring{f}_{fi} = \langle f|\mathbb{1}|i\rangle \hat{f}_{ii} + \sum_{l,j} (j+\tfrac{1}{2})\, P_l(\cos\theta)\, a_{fi;lj} \tag{A12}$$

$$\mathring{g}_{fi} = \sum_{l,j} (-1)^{j-l-1/2} P_l^1(\cos\theta)\, a_{fi;lj}. \tag{A13}$$

APPENDIX III

We consider the two equations for $x \geqslant 0$

$$\left(\frac{d^2}{dx^2} + 1 - 2\gamma/x\right) R(x) = 0$$

$$\left(\frac{d^2}{dx^2} + 1 - 2\gamma/x + 2\gamma D(x)\right) \hat{R}(x) = 0 \tag{A14}$$

where $D(x)$ is continuous

$$|xD(x)| < c_1 \qquad 0 \leqslant x \leqslant 1$$

$$|x^2 D(x)| < c_2 \qquad x > 1$$

c_1 and c_2 are two constants. We show the connection between their solutions, neglecting terms of order γ^2 and higher.

F and G are the regular and irregular solutions respectively of the first equation, $\hat{F}$ and $\hat{G}$ the corresponding solutions of the second. The Wronskians are, by definition of the normalization,

$$W_x[F, G] = W_x[\hat{F}, \hat{G}] = -1. \tag{A15}$$

According to equation (2.11) we have asymptotically

$$F(x) \underset{x\to\infty}{\sim} F(x)\cos\Delta v + G(x)\sin\Delta v$$

$$\hat{G}(x) \underset{x\to\infty}{\sim} G(x)\cos\Delta v - F(x)\sin\Delta v \tag{A16}$$

where Δv is the difference between the phases of $\hat{F}$ and F.

From the equation (A14) we get the derivative of the Wronskian of any two solutions R and $\hat{R}$,

$$\frac{d}{dx} W_x[R, \hat{R}] = -2\gamma D(x)\, R(x)\, \hat{R}(x) \tag{A17}$$

and, by integration, we have

$$W_x[R, \hat{R}] = W_\infty[R, \hat{R}] + 2\gamma \int_x^\infty dy D(y)\, R(y)\, \hat{R}(y). \tag{A18}$$

We now set $\hat{R} = \hat{F}$ and obtain two integral equations by replacing R by either F or G.

$$\tilde{W}_x[F, \hat{F}] = -\sin \Delta v + 2\gamma \int_x^\infty dy D(y) F(y) \hat{F}(y)$$

$$W_x[G, \hat{F}] = \cos \Delta v + 2\gamma \int_x^\infty dy D(y) G(y) \hat{F}(y). \tag{A19}$$

The terms involving Δv result from the Wronskians for $x \to \infty$, using equation (A16). Replacing the functions in the integrand by the corresponding functions for $\gamma = 0$, (A19) becomes a linear system to determine $\hat{F}$ and the derivative $\hat{F}'$ up to the first order in γ. We immediately arrive at the first order results

$$\hat{F}(x) \doteq F(x) + G(x) \Delta v + \sin x \int_x^\infty dy 2\gamma D(y) \sin y \cos y - \cos x \int_x^\infty dy\, 2\gamma D(y) \sin^2 y \tag{A20}$$

$$\hat{F}'(x) \doteq F'(x) + G'(x) \Delta v + \cos x \int_x^\infty dy 2\gamma D(y) \sin y \cos y + \sin x \int_x^\infty dy 2\gamma D(y) \sin^2 y. \tag{A21}$$

Because Δv is, in lowest order, proportional to γ we have replaced $\cos \Delta v$ by 1 and $\sin \Delta v$ by Δv. We obtain similar expressions for $\hat{G}$ and $\hat{G}'$.

To save computer time we have neglected the integral terms in the equations (A20) and (A21) in our actual calculations, where x is equal qr_0, F and G are the usual Coulomb wave functions, and $\hat{F}$ and $\hat{G}$ are the wave functions at r_0. In our calculations q is within the range m_π–$2m_\pi$ and $r_0 \approx 1.4 m_\pi^{-1}$. The neglected integrals are about 5% of the phase difference Δv, which is equal to the integral

$$\Delta v = \int_0^\infty dy 2\gamma D(y) \sin^2 y.$$

As we have pointed out in Ref. [12], the effect of Δv itself on the phase shift corrections is small for $l = 0$ and negligible for $l > 0$. Therefore the approximation (7.5) is well justified.

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