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Autor: Rasche, G. / Woolcock, W.S.

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The Effect of Radiative Capture on Threshold $\pi^- p$ Scattering and the Theory of the Panofsky Ratio

by **G. Rasche**

Institut für Theoretische Physik der Universität, Schönberggasse 9, CH-8001 Zürich, Switzerland

and **W. S. Woolcock**

Research School of Physical Sciences, The Australian National University, Canberra, Australia

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Abstract. The effect of the (γn) channel on threshold $\pi^- p$ scattering is considered. The symmetric 3×3 matrix of s-wave scattering amplitudes is written in terms of the components of a real symmetric 3×3 matrix, which can be expanded in a power series in q^2. The behaviour near the $\pi^- p$ threshold of the cross-sections for the processes $\pi^- p \to \pi^0 n$ and $\pi^- p \to \gamma n$ is obtained and the Panofsky ratio in flight is calculated. The 2×2 matrix of s-wave scattering amplitudes for the open channels $(\pi^0 n)$, (γn) below the $\pi^- p$ threshold is also obtained and shown to be unitary. The theory of the Panofsky ratio for the decay of the $\pi^- p$ $1s$ state is developed.

1. Introduction

In previous papers [1–2] we reviewed work on electromagnetic corrections in pion–nucleon scattering below 300 MeV and obtained convenient parametric forms for the s-wave amplitudes for the processes $\pi^\pm p \to \pi^\pm p$, $\pi^- p \to \pi^0 n$ near the $\pi^\pm p$ threshold (that is, $W = M + \mu$ or $q^2 = 0$).[1]) In the latter case we neglected the effect of the (γn) channel on the $(\pi^- p)$, $(\pi^0 n)$ system and considered the two channel problem only. It was emphasized by us previously [3] that, if electromagnetic effects in pion–nucleon scattering are to be considered consistently, it is necessary to take account of the (γn) channel, as well as Coulomb and mass difference effects, in the analysis of the results of $\pi^- p$ experiments. The purpose of this paper is to consider the effect of the (γn) channel on $\pi^- p$ scattering at very low energies.

When the (γn) channel is taken into account we have a three channel problem for the s-wave amplitudes. We shall not be considering partial waves with $l > 0$ in this paper; for each such partial wave there are in fact four channels, as discussed in [3]. In Section 2 we write the symmetric 3×3 matrix $\mathscr{F}$ of s-wave scattering amplitudes in terms of the components of a real symmetric 3×3 matrix $\mathbf{A}$ which generalizes the 2×2 matrix $\mathbf{A}$ of [2]. The components of $\mathbf{A}$ are real for $q^2 > 0$ and there are good reasons for believing that $\mathbf{A}(q^2)$ can be continued to a complex neighbourhood of $q^2 = 0$ to give a function analytic at $q^2 = 0$. Assuming that this is the case, $\mathbf{A}(q^2)$

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[1]) We use the notation of references [1–2] and label equations from these references with the prefixes 1-, 2- respectively.

has a power series expansion in q^2 which is valid in a neighbourhood of $q^2 = 0$; one would then expect to be able to analyse data from very low energy $\pi^- p$ experiments in terms of s-wave scattering lengths and curvatures. From the expressions for $\mathscr{F}_{0-}$ and $\mathscr{F}_{\gamma-}$ (where the subscript γ labels the (γn) channel) we shall calculate the cross-sections for the processes $\pi^- p \to \pi^0 n$ and $\pi^- p \to \gamma n$, respectively, and thus evaluate the Panofsky ratio (the ratio of these cross-sections) in flight at very low energies.

In Section 3 we shall obtain from the 3×3 matrix $\mathscr{F}$ of Section 2 the 2×2 matrix $\mathscr{F}^{(2)}$ for the open channels $(\pi^0 n)$, (γn) when one goes below the $\pi^- p$ threshold. It will be checked explicitly that this matrix derives from a unitary 2×2 S-matrix. The matrix $\mathscr{F}^{(2)}$, considered as a function of total centre-of-momentum frame energy W, has a simple pole with factorizable residue just below the real axis, very close to the energy of the $1s$ state of a $\pi^- p$ atom with only the Coulomb potential acting. From this fact we obtain the expression for the Panofsky ratio for the decay of the $\pi^- p$ $1s$ state (which is a 'very nearly bound state' in the terminology of Taylor [4]). This is the ratio which has been measured experimentally with considerable accuracy [5]. Section 4 is devoted to some numerical considerations, including the relationship between the measured value of the Panofsky ratio and our present knowledge of very low energy $\pi^- p$ charge exchange scattering and π^- photoproduction from neutrons.

2. The Three-Channel Scattering Matrix

We recall the result of [2] that, for the two-channel $(\pi^- p)$, $(\pi^0 n)$ case, the matrix $\mathscr{F}_2$ of s-wave amplitudes is given by the equation

$$\mathscr{F}_2^{-1} = \mathbf{C}_2^{-1}\mathbf{A}_2^{-1}\mathbf{C}_2^{-1} + \mathbf{H}_2 - i\mathbf{Q}_2, \tag{1}$$

where

$$\mathbf{C}_2 = \begin{pmatrix} C_0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{H}_2 = \begin{pmatrix} C_0^{-2}\beta h & 0 \\ 0 & 0 \end{pmatrix},$$
$$\mathbf{Q}_2 = \begin{pmatrix} q & 0 \\ 0 & q_0 \end{pmatrix}, \quad \mathbf{A}_2 = \begin{pmatrix} A_{--} & A_{0-} \\ A_{0-} & A_{00} \end{pmatrix}. \tag{2}$$

This result comes from (2-65) and (2-72). We have suppressed the energy dependence of the quantities appearing in (1) and (2) and will continue to do so most of the time through this paper. It will be convenient to think of all the quantities as functions of the variable W. In the model developed in [2], the matrix $\mathbf{A}_2$ is analytic at $q^2 = 0$ provided the inner potential matrix $\mathbf{W}(q^2; r)$ is such that $\mathbf{d}(q^2)$ is analytic at $q^2 = 0$. Since $q^2(W)$ is analytic at $W = M + \mu$, we can equally well consider $\mathbf{A}_2$ to be analytic at $W = M + \mu$.

When the (γn) channel is taken into account it is natural to assume that there exists a 3×3 matrix $\mathbf{A}$ which is connected to the 3×3 matrix $\mathscr{F}$ of s-wave scattering amplitudes by an equation exactly like (1), namely

$$\mathscr{F}^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{C}^{-1} + \mathbf{H} - i\mathbf{Q}, \tag{3}$$

where

$$\mathbf{C} = \begin{pmatrix} C_0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} C_0^{-2}\beta h & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathbf{Q} = \begin{pmatrix} q & 0 & 0 \\ 0 & q_0 & 0 \\ 0 & 0 & k \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} A_{--} & A_{0-} & A_{\gamma-} \\ A_{0-} & A_{00} & A_{\gamma 0} \\ A_{\gamma-} & A_{\gamma 0} & A_{\gamma\gamma} \end{pmatrix}, \tag{4}$$

with k the magnitude of the momentum of either γ or n in the centre-of-momentum frame. The assumption expressed in (3) and (4) is the natural extension of the result for the two-channel case and is in accord with intuitive expectations resulting from Gamow factor arguments. Further, it leads to results in accord with empirical observation, such as the finiteness of the Panofsky ratio. One would also expect (though a rigorous proof does not exist) that the matrix $\mathbf{A}$ is analytic at $q^2 = 0$, so that its elements have power series expansions in q^2 valid in a (complex) neighbourhood of $q^2 = 0$. As in [2], we shall write

$$\mathbf{A}(W) = \mathbf{a} + \boldsymbol{\alpha} q^2(W) + \cdots. \tag{5}$$

The next steps are just complicated algebra. From (4) we have

$$\det \mathbf{A}(\mathbf{C}^{-1}\mathbf{A}^{-1}\mathbf{C}^{-1} + \mathbf{H} - i\mathbf{Q})$$
$$= \begin{pmatrix} C_0^{-2}(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + C_0^{-2}\beta h \det \mathbf{A} - iq \det \mathbf{A} & C_0^{-1}(A_{\gamma-}A_{\gamma 0} - A_{0-}A_{\gamma\gamma}) & C_0^{-1}(A_{0-}A_{\gamma 0} - A_{00}A_{\gamma-}) \\ C_0^{-1}(A_{\gamma-}A_{\gamma 0} - A_{0-}A_{\gamma\gamma}) & (A_{--}A_{\gamma\gamma} - A_{\gamma-}^2) - iq_0 \det \mathbf{A} & (A_{0-}A_{\gamma-} - A_{--}A_{\gamma 0}) \\ C_0^{-1}(A_{0-}A_{\gamma 0} - A_{00}A_{\gamma-}) & (A_{0-}A_{\gamma-} - A_{--}A_{\gamma 0}) & (A_{--}A_{00} - A_{0-}^2) - ik \det \mathbf{A} \end{pmatrix}, \tag{6}$$

where

$$\det \mathbf{A} = A_{--}A_{00}A_{\gamma\gamma} + 2A_{0-}A_{\gamma-}A_{\gamma 0} - A_{--}A_{\gamma 0}^2 - A_{00}A_{\gamma-}^2 - A_{\gamma\gamma}A_{0-}^2. \tag{7}$$

From (3) and (6) it follows that

$$C_0^{-2}\Delta\mathscr{F}_{--} = A_{--} - iq_0(A_{--}A_{00} - A_{0-}^2) - ik(A_{--}A_{\gamma\gamma} - A_{\gamma-}^2) - q_0 k \det \mathbf{A}, \tag{8}$$

$$C_0^{-1}\Delta\mathscr{F}_{0-} = A_{0-} - ik(A_{0-}A_{\gamma\gamma} - A_{\gamma-}A_{\gamma 0}), \tag{9}$$

$$C_0^{-1}\Delta\mathscr{F}_{\gamma-} = A_{\gamma-} - iq_0(A_{00}A_{\gamma-} - A_{0-}A_{\gamma 0}), \tag{10}$$

$$\Delta\mathscr{F}_{00} = A_{00} - ik(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + (\beta h - iqC_0^2)(A_{--}A_{00} - A_{0-}^2 - ik \det \mathbf{A}), \tag{11}$$

$$\Delta\mathscr{F}_{\gamma 0} = A_{\gamma 0} + (\beta h - iqC_0^2)(A_{--}A_{\gamma 0} - A_{0-}A_{\gamma-}), \tag{12}$$

$$\Delta\mathscr{F}_{\gamma\gamma} = A_{\gamma\gamma} - iq_0(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + (\beta h - iqC_0^2)(A_{--}A_{\gamma\gamma} - A_{\gamma-}^2 - iq_0 \det \mathbf{A}), \tag{13}$$

where

$$\Delta = 1 - iq_0 A_{00} - ikA_{\gamma\gamma} - q_0 k(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + (\beta h - iqC_0^2)[A_{--} - iq_0(A_{--}A_{00} - A_{0-}^2) - ik(A_{--}A_{\gamma\gamma} - A_{\gamma -}^2) - q_0 k \det \mathbf{A}] \tag{14}$$

and $\det \mathbf{A}$ is given in (7). These results should be compared with those in (2-73)–(2-76) to see the modifications produced by the inclusion of the (γn) channel. Note that from (3) it is clear that $\mathscr{F}$ satisfies the unitarity relation

$$\bar{\mathscr{F}}^{-1} - \mathscr{F}^{-1} = 2i\mathbf{Q}.$$

It follows from (1-1), (1-17), (9) and (10) that, taking the contribution of the s-wave only into account, we have

$$\sigma_{0-} = 4\pi q_0 q^{-1} C_0^2 |\Delta|^{-2} |A_{0-} - ik(A_{0-}A_{\gamma\gamma} - A_{\gamma -}A_{\gamma 0})|^2,$$
$$\sigma_{\gamma -} = 4\pi k q^{-1} C_0^2 |\Delta|^{-2} |A_{\gamma -} - iq_0(A_{00}A_{\gamma -} - A_{0-}A_{\gamma 0})|^2. \tag{15}$$

Thus the Panofsky ratio in flight, $\sigma_{0-}/\sigma_{\gamma -}$, has a finite limit as $W \downarrow (M + \mu)$, namely

$$\lim_{W \downarrow (M+\mu)} \frac{\sigma_{0-}}{\sigma_{\gamma -}} = \frac{q_0 |a_{0-} - ik(a_{0-}a_{\gamma\gamma} - a_{\gamma -}a_{\gamma 0})|^2}{k|a_{\gamma -} - iq_0(a_{00}a_{\gamma -} - a_{0-}a_{\gamma 0})|^2}; \tag{16}$$

it is understood in (16) that q_0, k are to be evaluated at $W = M + \mu$. It is expected that measurements of the Panofsky ratio in flight will eventually be made at very low energies. To analyse the results, it will however be necessary to take account of the variation of q_0, k and $\mathbf{A}$ with energy (using for $\mathbf{A}$ the first two terms in the expansion (5)) and also to include the contribution of the p-waves to σ_{0-}, $\sigma_{\gamma -}$.

3. The $(\pi^0 n)$, (γn) Scattering Matrix Below the $\pi^- p$ Threshold

To make the necessary analytic continuation to obtain the 2×2 scattering matrix

$$\mathscr{F}^{(2)} = \begin{pmatrix} \mathscr{F}_{00} & \mathscr{F}_{\gamma 0} \\ \mathscr{F}_{\gamma 0} & \mathscr{F}_{\gamma\gamma} \end{pmatrix}$$

for $M_n + \mu_0 < W < M + \mu$, we note that, *above* the $\pi^- p$ threshold, one may write

$$\beta h(\eta_-) - iqC_0^2(\eta_-) = \beta g(\eta_- + i.0) + i\pi\beta \coth \pi\eta_-, \tag{17}$$

where

$$g(\eta_- + i.0) = \lim_{\epsilon \downarrow 0} g(\eta_- + i\epsilon),$$

$$g(z) = -2 \int_0^\infty dt \frac{t}{(t^2 - z^2)(e^{2\pi t} - 1)}, \qquad z \notin \mathbb{R}. \tag{18}$$

The next step is to define new variables κ, ξ for $M_n + \mu_0 < W < M + \mu$ by

$$\kappa^2 = (4W^2)^{-1}[W^2 - (M - \mu)^2][(M + \mu)^2 - W^2], \qquad \kappa > 0,$$
$$\xi = \beta/2\kappa, \tag{19}$$

and to recognize that when W is in this interval the correct forms of $\mathscr{F}_{00}$, $\mathscr{F}_{\gamma 0}$, $\mathscr{F}_{\gamma\gamma}$ are obtained from the expressions in (11)–(14) by replacing $(\beta h - iqC_0^2)$ by the right side of (17) and then replacing η_- by $i\xi$. This gives

$$\beta h(\eta_-) - iqC_0^2(\eta_-) \rightarrow \beta[f(\xi) + \pi \cot \pi\xi], \tag{20}$$

a real valued function of W, where, from (18),

$$f(\xi) = -2\int_0^\infty dt \frac{t}{(t^2+\xi^2)(e^{2\pi t}-1)}. \tag{21}$$

With the replacement (20) we then have for the elements of $\mathscr{F}^{(2)}$ the expressions

$$\Delta\mathscr{F}_{00} = A_{00} - ik(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + x(A_{--}A_{00} - A_{0-}^2 - ik \det \mathbf{A}), \tag{22}$$

$$\Delta\mathscr{F}_{\gamma 0} = A_{\gamma 0} + x(A_{--}A_{\gamma 0} - A_{0-}A_{\gamma -}), \tag{23}$$

$$\Delta\mathscr{F}_{\gamma\gamma} = A_{\gamma\gamma} - iq_0(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + x(A_{--}A_{\gamma\gamma} - A_{\gamma -}^2 - iq_0 \det \mathbf{A}), \tag{24}$$

where

$$\begin{aligned}\Delta = 1 &- iq_0A_{00} - ikA_{\gamma\gamma} - q_0k(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2)\\ &+ x[A_{--} - iq_0(A_{--}A_{00} - A_{0-}^2) - ik(A_{--}A_{\gamma\gamma} - A_{\gamma -}^2) - q_0 k \det \mathbf{A}],\end{aligned} \tag{25}$$

$$x = \beta[f(\xi) + \pi \cot \pi\xi], \tag{26}$$

and $f(\xi)$ is given by (21). To check explicitly that $\mathscr{F}^{(2)}$ as given by (22)–(25) satisfies the usual unitarity relation it is simplest to invert it. Some tedious algebra shows that

$$\Delta \det \mathscr{F}^{(2)} = (A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + x \det \mathbf{A}, \tag{27}$$

so that

$$\left[\mathscr{F}^{(2)-1} + i\begin{pmatrix} q_0 & 0\\ 0 & k\end{pmatrix}\right][(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) + x \det \mathbf{A}]$$
$$= \begin{pmatrix} A_{\gamma\gamma} + x(A_{--}A_{\gamma\gamma} - A_{\gamma -}^2) & -A_{\gamma 0} + x(A_{0-}A_{\gamma -} - A_{--}A_{\gamma 0})\\ -A_{\gamma 0} + x(A_{0-}A_{\gamma -} - A_{--}A_{\gamma 0}) & A_{00} + x(A_{--}A_{00} - A_{0-}^2)\end{pmatrix}.$$

Thus

$$\bar{\mathscr{F}}^{(2)-1} - \mathscr{F}^{(2)-1} = 2i\begin{pmatrix} q_0 & 0\\ 0 & k\end{pmatrix},$$

as required.

This result can also be obtained more transparently as follows. Note that, because of the Coulomb interaction in the $\pi^- p$ channel, it is the matrix $\mathbf{C}^{-1}\mathscr{F}C^{-1}$ which is to be analytically continued below the $\pi^- p$ threshold (cf. Ref. [6]). Using the replacement (20), we have from (3) that the continuation of $\mathbf{C}\mathscr{F}^{-1}\mathbf{C}$ for $M_n + \mu_0 < W < M + \mu$ is

$$\mathbf{B} - i\begin{pmatrix} 0 & 0 & 0\\ 0 & q_0 & 0\\ 0 & 0 & k\end{pmatrix}.$$

where

$$\mathbf{B} = \mathbf{A}^{-1} + \begin{pmatrix} x & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

a real matrix. From this it follows that

$$\mathscr{F}^{(2)-1} = \begin{pmatrix} B_{00} & B_{\gamma 0}\\ B_{\gamma 0} & B_{\gamma\gamma}\end{pmatrix} - B_{--}^{-1}\begin{pmatrix} B_{0-}^2 & B_{0-}B_{\gamma -}\\ B_{0-}B_{\gamma -} & B_{\gamma -}^2\end{pmatrix} - i\begin{pmatrix} q_0 & 0\\ 0 & k\end{pmatrix},$$

so that

$$\operatorname{Im} \mathscr{F}^{(2)-1} = -\begin{pmatrix} q_0 & 0 \\ 0 & k \end{pmatrix}.$$

We look now for the position of the pole in $\mathscr{F}^{(2)}$ which lies very close to the position of the $1s$ bound state of the $\pi^- p$ atom with only the Coulomb potential acting between π^- and p. This bound state is at $\kappa = \beta/2$ or $\xi = 1$. Suppose then that the pole in $\mathscr{F}^{(2)}$ is at the value of W corresponding to

$$\xi_0 = 1 + \delta_0, \tag{28}$$

where δ_0 is complex and $|\delta_0| \ll 1$. In fact, using (34)–(36) and numbers given in Section 4, $|\delta_0| \approx 10^{-3}$. Since the pole position is so close to $\xi = 1$ we shall take q_0, k and the elements of $\mathbf{A}$ to be fixed at their values for $\xi = 1$, which are real. The value of W corresponding to $\xi = 1$ is given by $M + \mu - W_1 = \beta^2/8m$. We are obviously justified in using nonrelativistic kinematics and therefore replace the first equation in (19) by

$$M + \mu - W = \kappa^2/2m. \tag{29}$$

In finding the pole position we are continuing in W away from the real axis; κ becomes complex and so then do ξ and x (see (19) and (26)). Since we are working in the neighbourhood of $\xi = 1$, we shall generalize (28) and use the variable δ defined by

$$\delta = \xi - 1. \tag{30}$$

From (19), (29) and (30)

$$M + \mu - W = (\beta^2/8m)(1 + \delta)^{-2}. \tag{31}$$

Also, from [7],

$$f(\xi) = -\ln \xi + 1/2\xi + \psi(\xi),$$

so that

$$f(1) = -\gamma + \tfrac{1}{2} = -0.0772\cdots.$$

Since we are considering values of δ for which $|\delta| \ll 1$, we also have

$$\pi \cot \pi\xi \approx 1/\delta.$$

It is now clear that $f(\xi)$ may be neglected in (26) and that, to a very accurate approximation (around 1 part in 10^4),

$$x = \beta/\delta, \quad x_0 = \beta/\delta_0. \tag{32}$$

We now write

$$\Delta\mathscr{F}^{(2)} = \mathbf{N}_1 + x\mathbf{N}_2,$$
$$\Delta = D_1 + xD_2,$$

where

$$\mathbf{N}_1 = \begin{pmatrix} A_{00} - ik(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) & A_{\gamma 0} \\ A_{\gamma 0} & A_{\gamma\gamma} - iq_0(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2) \end{pmatrix}, \tag{33}$$

$$\mathbf{N}_2 = \begin{pmatrix} A_{--}A_{00} - A_{0-}^2 - ik \det \mathbf{A} & A_{--}A_{\gamma 0} - A_{0-}A_{\gamma -} \\ A_{--}A_{\gamma 0} - A_{0-}A_{\gamma -} & A_{--}A_{\gamma\gamma} - A_{\gamma -}^2 - iq_0 \det \mathbf{A} \end{pmatrix}, \tag{34}$$

$$D_1 = 1 - iq_0 A_{00} - ikA_{\gamma\gamma} - q_0 k(A_{00}A_{\gamma\gamma} - A_{\gamma 0}^2), \tag{35}$$

$$D_2 = A_{--} - iq_0(A_{--}A_{00} - A_{0-}^2) - ik(A_{--}A_{\gamma\gamma} - A_{\gamma-}^2) - q_0 k \det \mathbf{A}. \tag{36}$$

Thus the pole in $\mathscr{F}^{(2)}$ is, to an excellent approximation, at the position where

$$x_0 = -D_1/D_2, \quad \delta_0 = -\beta D_2/D_1, \tag{37}$$

the values of D_1, D_2 being calculated at $W = W_1$. Using (31), (32) and (37), we may write for $\mathscr{F}^{(2)}$ the following approximate expression which is valid very close to the pole:

$$\mathscr{F}^{(2)} \approx \frac{\mathbf{N}_1 + x_0\mathbf{N}_2}{D_2(x - x_0)} \approx -\frac{(D_2\mathbf{N}_1 - D_1\mathbf{N}_2)}{D_2^2}\left(\frac{\beta^3}{4m}\right)(1 + \delta_0)^{-3}x_0^{-2}(W - W_0)^{-1}. \tag{38}$$

From (33)–(36),

$$D_2\mathbf{N}_1 - D_1\mathbf{N}_2 = \begin{pmatrix} \alpha_0^2 & \alpha_0\alpha_\gamma \\ \alpha_0\alpha_\gamma & \alpha_\gamma^2 \end{pmatrix},$$

where

$$\begin{aligned} \alpha_0 &= A_{0-} - ik(A_{0-}A_{\gamma\gamma} - A_{\gamma-}A_{\gamma 0}), \\ \alpha_\gamma &= A_{\gamma-} - iq_0(A_{00}A_{\gamma-} - A_{0-}A_{\gamma 0}). \end{aligned} \tag{39}$$

Thus the residue of $\mathscr{F}^{(2)}$ at the pole factorizes, a result which follows from (27) without direct calculation. The expression (38) for $\mathscr{F}^{(2)}$ then becomes

$$\mathscr{F}^{(2)} \approx -\frac{1}{2}\begin{pmatrix} q_0 & 0 \\ 0 & k \end{pmatrix}^{-1/2}\begin{pmatrix} c_0^2 & c_0c_\gamma \\ c_0c_\gamma & c_\gamma^2 \end{pmatrix}\begin{pmatrix} q_0 & 0 \\ 0 & k \end{pmatrix}^{-1/2}(W - W_0)^{-1}, \tag{40}$$

where

$$\begin{aligned} c_0 &= (2q_0)^{1/2}(\beta^3/4m)^{1/2}D_1^{1/2}(D_1 - \beta D_2)^{-3/2}\alpha_0, \\ c_\gamma &= (2k)^{1/2}(\beta^3/4m)^{1/2}D_1^{1/2}(D_1 - \beta D_2)^{-3/2}\alpha_\gamma. \end{aligned} \tag{41}$$

Since D_1 and $(D_1 - \beta D_2)$ are close to 1, it is clear which square roots are to be taken in (41). Since it is an approximation (though a very good one) the expression (40) is not exactly unitary when W is real and close to Re W_0.

In accordance with the standard theory for narrow multi-channel resonances (see, for example, Ref. [4]), the total width of the resonance is

$$\begin{aligned} \Gamma = |c_0|^2 + |c_\gamma|^2 &= (\beta^3/2m)|D_1||D_1 - \beta D_2|^{-3} \\ &\times [q_0A_{0-}^2 + kA_{\gamma-}^2 + q_0k^2(A_{0-}A_{\gamma\gamma} - A_{\gamma-}A_{\gamma 0})^2 \\ &\quad + q_0^2k(A_{00}A_{\gamma-} - A_{0-}A_{\gamma 0})^2], \end{aligned} \tag{42}$$

while the partial widths Γ_0, Γ_γ for decay of the resonance into $(\pi^0 n)$, (γn) respectively are

$$\Gamma_0 = |c_0|^2, \quad \Gamma_\gamma = |c_\gamma|^2. \tag{43}$$

For this interpretation in terms of the exponential decay of a nearly stable state to be consistent, it is necessary that

$$-2\,\mathrm{Im}\,W_0 = \Gamma. \tag{44}$$

Since the expression (40) for $\mathscr{F}^{(2)}$ is only approximately unitary, (44) is not exactly true. It is, however, true to a very high level of accuracy. For, from (31) and (37),

$$-2\,\mathrm{Im}\,W_0 = (\beta^3/2m)\,\mathrm{Im}(\bar{D}_1 D_2)|D_1 - \beta D_2|^{-4}[|D_1|^2 - \beta\,\mathrm{Re}(\bar{D}_1 D_2)].$$

Now from (35), (36) and (39),

$$\mathrm{Im}(\bar{D}_1 D_2) = q_0|\alpha_0|^2 + k|\alpha_\gamma|^2,$$

whilst it is also true that

$$|D_1|^2 - \beta\,\mathrm{Re}(\bar{D}_1 D_2) \approx |D_1||D_1 - \beta D_2|;$$

since $\beta\,\mathrm{Im}(\bar{D}_1 D_2) \approx 6\cdot 10^{-5}$, this equality is accurate to 2 parts in 10^9. Thus (44) holds to the same accuracy.

We conclude that the Panofsky ratio as measured experimentally for the decay of the 1s state of the $\pi^- p$ atom to $(\pi^0 n)$ and (γn) is, from (39), (41) and (43),

$$P_{\mathrm{exp}} = \frac{\Gamma_0}{\Gamma_\gamma} = \frac{q_0|\alpha_0|^2}{k|\alpha_\gamma|^2} = \frac{q_0|A_{0-} - ik(A_{0-}A_{\gamma\gamma} - A_{\gamma-}A_{\gamma 0})|^2}{k|A_{\gamma-} - iq_0(A_{00}A_{\gamma-} - A_{0-}A_{\gamma 0})|^2}. \tag{45}$$

The values of q_0, k and the elements of $\mathbf{A}$ are to be taken at $W = W_1$. These values hardly differ at all from those at $W = M + \mu$, as we shall see in Section 4, so that, comparing (16) with (45), the Panofsky ratio measured in flight at very low energies will differ very little from the value measured for the decay of the 1s state of the $\pi^- p$ atom. In the next section we shall assess the present situation concerning the indirect calculation of the Panofsky ratio from the right side of (45).

4. Numerical Considerations

In [2] we considered Coulomb and mass difference effects which shift a_{--}, a_{0-}, a_{00} and α_{--}, α_{0-}, α_{00} from their strictly nuclear (charge independent) values. When the (γn) channel is taken into account, as we have done in this paper, there may be further shifts in these six quantities. The very low energy experimental data on $\pi^- p \to \pi^- p$ and $\pi^- p \to \pi^0 n$ should be analysed using for the s-wave amplitudes the equations (8), (9) and (14) of this paper rather than (2-73), (2-74) and (2-76). But, as we pointed out in [2], the analysis will not be possible unless the six $\pi^- p$ parameters required can be reduced to four charge independent ones, about two of which (namely a_3, α_3), there is information available independently by analysis of $\pi^+ p$ experiments. Since we can see no way of estimating the shifts in the six parameters which may arise from the presence of the (γn) channel, the only course open is to assume that these shifts can be neglected and to use only the corrections which we calculated in [2] neglecting the influence of the (γn) channel. It is worth emphasizing that there is no point in making more refined theories for the two-channel electromagnetic corrections as long as the three-channel problem cannot be treated in a better way.

For the scattering length parameters we thus take the charge-independent scattering lengths of (2-88) and modify them by the corrections suggested in Table 3 of [2], giving

$$\begin{aligned} a_{--} &= 0.0783\,\mu^{-1}, \\ a_{0-} &= -0.1192\,\mu^{-1}, \\ a_{00} &= -0.0040\,\mu^{-1}. \end{aligned} \tag{46}$$

With the scattering lengths of (2-88) the curvatures given in (2-87) will be modified in order to fit the present (rather poor) very low energy s-wave data. Using the corrections suggested in Table 3 the curvatures become

$$\alpha_{--} = -0.0153\,\mu^{-3},$$
$$\alpha_{0-} = -0.0412\,\mu^{-3},$$
$$\alpha_{00} = -0.0139\,\mu^{-3}. \tag{47}$$

Using (46) and (47) we see that the change in A_{--}, A_{0-} and A_{00} in going from $W = M + \mu$ to $W = W_1$ is far beyond the last significant figure.

The experimental value of the Panofsky ratio [5] is

$$P_{\text{exp}} = 1.533 \pm 0.021.$$

This gives, using (45) and the values

$$q_0(W_1) = 0.20113, \quad k(W_1) = 0.92716,$$

the result

$$|A_{\gamma-}(W_1)| = (0.0448 \pm 0.0011)\,\mu^{-1}.$$

The error comes from the error on P_{exp} and an uncertainty of 0.0021 on a_{0-} obtained from the third and fifth columns of Table 3 of [2]. This last uncertainty may be a substantial underestimate. Note that, as we shall see later when rough numbers are given for $a_{\gamma 0}$ and $a_{\gamma\gamma}$,

$$|\alpha_0(W_1)| = |A_{0-}(W_1)|, \quad |\alpha_\gamma(W_1)| = |A_{\gamma-}(W_1)|,$$

to an extremely high degree of accuracy. To sum up, since W_1 is so close to $(M + \mu)$, we have

$$|a_{\gamma-}| = (0.0448 \pm 0.0011)\,\mu^{-1}. \tag{48}$$

The sign of $a_{\gamma-}$ cannot be determined; we take it to be positive. Once this is fixed, the approximate charge independence of the pion–nucleon interaction resolves the sign ambiguity for $\alpha_{\gamma 0}$.

We now consider the input leading to (48). Time-reversal invariance has been assumed and there are no grounds for questioning this assumption. The theory of the Panofsky ratio has now been formulated in a much more satisfactory way than previously and the value of the Panofsky ratio is a firm experimental number. The value of a_{0-} and the uncertainty to be attributed to it are much more doubtful pieces of input. The calculation of $(a_1 - a_3)$ by Bugg et al. [8] has defects connected with the unsatisfactory nature of the electromagnetic corrections they apply (see Section 4 of [1]). Moreover, their quoted error does not include systematic uncertainties which may be substantial. There is uncertainty in our calculation in [2] of the difference between a_{0-} and $(\sqrt{2/3})(a_3 - a_1)$ and there is a further correction arising from the presence of the (γn) channel which cannot be estimated. We are therefore somewhat skeptical about the value of a_{0-} and its error, which we have used as input to (48).

We now review the evidence on $a_{\gamma-}$ from results on the photoproduction reaction $\gamma n \to \pi^- p$. Such results are obtained from experiments with deuterons which look at the reaction $\gamma d \to \pi^- pp$. The analysis of results on such experiments involves making a 'spectator approximation', which takes account of the presence of the spectator proton in the deuteron. There are also several approximations made in all analyses of data on pion photoproduction from free nucleons; the final state theorem is used, the pion–nucleon interaction is taken to be charge independent and mass differences are neglected. This last approximation introduces a substantial uncertainty into the extrapolation to threshold of the multipole amplitude E_{0+} since for each of the four photoproduction processes this amplitude is rapidly varying near threshold and the extrapolation has to be made from some 15 MeV (in laboratory energy of the photon) above threshold. We hope to consider these approximations in the analysis of pion photoproduction data in a further paper.

For the present, we have attempted to extrapolate to threshold the real part of E_{0+} for the processes $\gamma n \to \pi^- p$, $\gamma n \to \pi^0 n$ and $\gamma p \to \pi^+ n$, using values taken from Figures 1, 5 and 11 in the paper of Berends and Donnachie [9]. From Appendix B of [3] we have[2]), for the partial wave with $J = \frac{1}{2}$, $P = -1$,

$$\mathscr{F}_{\gamma -} = -\sqrt{2}\, E_{0+}^{\gamma n \to \pi^- p},$$

$$\mathscr{F}_{\gamma 0} = -\sqrt{2}\, E_{0+}^{\gamma n \to \pi^0 n}.$$

At the energies from which the extrapolation is made, $C_0 \approx 1$ and so, using (10) and (12) and noting that the second terms on the right sides can be neglected, we have as a result of our extrapolation

$$a_{\gamma -} \approx -\sqrt{2}\,(-0.0318)\,\mu^{-1} = 0.0450\,\mu^{-1},$$

$$a_{\gamma 0} \approx -\sqrt{2}\,(0.0020)\,\mu^{-1} = -0.0028\,\mu^{-1}. \tag{49}$$

Another way to obtain $a_{\gamma -}$ is to use the threshold value of $E_{0+}^{\gamma p \to \pi^+ n}$, namely $0.0274\,\mu^{-1}$, and the threshold value of the ratio $(d\sigma/d\Omega)(\gamma n \to \pi^- p)/(d\sigma/d\Omega)(\gamma p \to \pi^+ n)$ given by Adamovich et al. [10], namely (1.336 ± 0.017), to obtain

$$a_{\gamma -} \approx \sqrt{2}\,(0.0274)(1.336)^{1/2}\,\mu^{-1} = 0.0448\,\mu^{-1}. \tag{50}$$

We have not quoted errors on these determinations of $a_{\gamma -}$ since the values of Re E_{0+} which we used have no errors assigned to them. There is no doubt, however, that the error on either of the above determinations of $a_{\gamma -}$ is at least 0.0015. On comparing (49) and (50) with (48) and noting that the error in (48) is also an underestimate, it is clear that there is satisfactory agreement between the experimental value of the Panofsky ratio and the value deduced indirectly from data on very low energy pion–nucleon scattering and pion photoproduction from nucleons near threshold. The agreement is obviously much better than we have any right to expect, and it looks as though there will have to be substantial improvements in our knowledge of low energy pion phenomena before there is any danger of this agreement being upset.

Note that the almost perfect agreement between (48) and (49) or (50) depends very much on using a_{0-} and not $\sqrt{2}/3(a_3 - a_1)$ in calculating $|a_{\gamma -}|$ in (48). The importance of this was pointed out in [2], in the remarks around (2-89). The use of $\sqrt{2}/3(a_3 - a_1)$ would give $|a_{\gamma -}| = 0.0459$ in (48). The agreement with (49) or (50) is still satisfactory, but the effect of the difference between a_{0-} and $\sqrt{2}/3(a_3 - a_1)$ on the calculation of $|a_{\gamma -}|$ from the Panofsky ratio will become more important as data from experiments involving very low energy pions improves.

We mentioned at the beginning of this section that to analyse data on the s-wave

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[2]) This result differs from that given in Appendix B of [3] by a factor i. This change is necessary if we use the pion photoproduction amplitudes given in the literature and keep to the convention for time-reversal invariance used in the helicity formalism.

amplitudes for $\pi^- p \rightarrow \pi^- p$ and $\pi^- p \rightarrow \pi^0 n$ at low energies, one should use equations (8), (9) and (14). To do this one seems to require values of $a_{\gamma -}$, $a_{\gamma 0}$ and $a_{\gamma\gamma}$ and also of $\alpha_{\gamma -}$, $\alpha_{\gamma 0}$ and $\alpha_{\gamma\gamma}$. From data on Re E_{0+} for pion photoproduction already discussed, we have

$$\begin{aligned} a_{\gamma -} &\approx 0.0448\,\mu^{-1}, \quad \alpha_{\gamma -} \approx -0.0192\,\mu^{-3}, \\ a_{\gamma 0} &\approx -0.0028\,\mu^{-1}, \quad \alpha_{\gamma 0} \approx +0.0024\,\mu^{-3}. \end{aligned} \tag{51}$$

For $a_{\gamma\gamma}$, $\alpha_{\gamma\gamma}$ we rely on the result of Pfeil et al. [11] for Compton scattering off protons that, at energies near the pion photoproduction threshold, f_{EE}^{1-} is almost exactly equal to the sum of the Born term and the integral over the physical region; other terms in the dispersion relation are negligible. If the same is true for Compton scattering off neutrons, we deduce that

$$a_{\gamma\gamma} \approx +0.0035\,\mu^{-1}, \quad \alpha_{\gamma\gamma} \approx -0.0009\,\mu^{-3}. \tag{52}$$

Even if the values in (52) and (53) are only very rough estimates, it is true that at very low energies, to an extremely good approximation,

$$C_0^2 \mathscr{F}_{--} \approx A_{--}, \quad C_0 \mathscr{F}_{0-} \approx A_{0-}. \tag{53}$$

For example, at $W = M + \mu$, using (46), (51) and (52),

$$C_0^2 \mathscr{F}_{--} = (0.0783 + i0.0050)\,\mu^{-1}, \quad C_0 \mathscr{F}_{0-} = (-0.1192 + i0.0004)\,\mu^{-1}.$$

Since the experiments at very low energies will give only $|\mathscr{F}_{--}|^2$ and $|\mathscr{F}_{0-}|^2$ it is clear that the approximations (53) are extremely good and that the numbers given in (51) and (52) are not needed for the analysis of $\pi^- p$ elastic and charge exchange scattering experiments at very low energies.

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