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Physical justification for using the tensor product to describe two quantum systems as one joint system

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Abstract. We require the following three conditions to hold on two systems being described as a joint system: (1°) the structure of the two systems is preserved; (2°) a measurement on one of the systems does not disturb the other one; (3°) maximal information obtained on both systems separately gives maximal information on the joint system. With these conditions we show, within the framework of the propositional system formalism, that if the systems are classical the joint system is described by the cartesian product of the corresponding phase spaces, and if the systems are quantal the joint system is described by the tensor product of the corresponding Hilbert spaces.

Introduction

As we know the states of a physical system are described by the points in a phase space in classical mechanics and by the unit vectors of a complex Hilbert space in quantum mechanics. If we consider two classical systems S_1 and S_2 with corresponding phase spaces Ω_1 and Ω_2, then the joint system S is naturally described by means of the phase space Ω which is the cartesian product $\Omega_1 \times \Omega_2$. For two quantum systems S_1 and S_2 with corresponding Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ it is one of the axioms of quantum mechanics that the states of the joint system S are described by the unit vectors of the tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$. This not obvious procedure to construct the states of the joint system in quantum mechanics originated in the wave-mechanics formalism. Indeed if we describe both S_1 and S_2 by means of the Hilbert space $L^2(\mathbb{R}^3)$ of all square integrable complex functions of three variables, and a state of S_1 resp. S_2 is represented by a function $\psi_1(\mathbf{x})$ resp. $\psi_2(\mathbf{y})$ such that $\int |\psi_1(\mathbf{x})|^2 \, d\mathbf{x} = 1$ and $\int |\psi_2(\mathbf{y})|^2 \, d\mathbf{y} = 1$, then it is natural to represent a state of the joint system S by a square integrable complex function of six variables $\psi(\mathbf{x}, \mathbf{y})$ such that $\int |\psi(\mathbf{x}, \mathbf{y})|^2 \, d\mathbf{x} \, d\mathbf{y} = 1$. This amounts to saying that we take the Hilbert space $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ to describe the joint system S.

Since not every complex function of six variables can be written as a product of two functions of three variables, $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ contains functions that are not in the cartesian product $L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and in fact one can prove that $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ is

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isomorphic to $L^2(\mathbb{R}^3) \otimes L^2(\mathbb{R}^3)$ ([1], p. 51). This is the reason why in general, when we consider S_1 and S_2 to be described by abstract Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, we take $\mathscr{H}_1 \otimes \mathscr{H}_2$ to describe the joint system S.

Our aim in this article is to try to understand the physics behind these coupling procedures for two systems. We shall see that some very general and physically intuitive requirements on two systems S_1 and S_2 lead us in a natural way to describe the joint system by means of the tensor product of the corresponding Hilbert spaces if S_1 and S_2 are quantum systems. The same requirements for classical systems force us to describe the joint system by the cartesian product of the corresponding phase spaces.

To carry out this program, we used the propositional system formalism of Piron [3] because we think it gives a clearer insight into the physical notions behind quantum mechanics which are often not so easy to grasp in the ordinary Hilbert space formalism. Moreover it is a formalism capable of describing both quantum systems and classical systems by means of the same mathematical artillery.

The origins of the propositional system formalism go back to [2]. The physical interpretation exposed here is given in [3]. It was realized long ago [4], [5] that the observables of a system are more fundamental physical notions than the states of the system, which is the opposite of what is done in the Hilbert space formalism, where one defines first the states and afterwards the observables. This idea gave birth to the C^*-algebra approach of quantum mechanics. But still, an observable being represented by an operator is not a physically very clear idea. One can remark however that it is always possible to split an experiment into different *yes-no experiments* (observations which permit only the two alternatives yes or no as an answer).[2]) Indeed, in general the values of an observable cover a subset of the real line. A determination of the observable is obtained by dividing the real line into small intervals and then deciding for each one of these intervals whether it contains the measured value. The collection of all the yes-no experiments obtained in this way determines completely the observable. In phase space formalism of classical mechanics this collection is the set of all subsets of phase space of the form $f^{-1}(A)$, where A is a subset of the measuring range of the observable represented by the real function f, and f is completely determined by the knowledge of this collection (see [3] theorem 1.21). In the Hilbert space formalism of quantum mechanics this collection is the set of all projection operators of the spectral decomposition of the corresponding selfadjoint operator. So the idea grew to make a quantum formalism by considering the collection of all the possible yes-no experiments on a physical system and to study its structure.

Taking into account some very basic physical facts Piron shows in his excellent book [3] that one is led in a very natural way to define operations on this set, which turn it into a complete, orthocomplemented, weakly modular lattice, a well-known mathematical object. The structure of this object is now a direct consequence of physical arguments, which is not the case for, e.g. linearity in the C^*-algebra approach. Work on such lattice formalisms has been done by Mackey [6], Jauch [7], Piron [3], and many others. In [3] it is proven that with some additional suppositions (for instance the possibility to define states for the system), this approach yields a structure which is isomorphic to the structure of ordinary quantum mechanics in the Hilbert

[2]) A *yes-no experiment* is represented in the phase space formalism of classical mechanics by a function f from the phase space to the set {yes, no} (or, which is the same, by the subset f^{-1} (yes) of the phase space). In the Hilbert space formalism of quantum mechanics it is represented by a projection operator P (or, which is the same, by the closed subspace $P\mathscr{H}$ of the Hilbert space $\mathscr{H}$).

space formalism. Assuming a different additional property, named distributivity, the formalism leads to a structure isomorphic to the one used in classical mechanics. We will give a short review of the basic notions and definitions involved in the propositional system formalism in the following section.

1. The propositional formalism

If S is a physical system we will denote by $\mathscr{L}$ the collection of all the yes-no experiments on S. Let us remark that in fact $\mathscr{L}$ is just the collection of all the properties of S because with every property corresponds a yes-no experiment and vice-versa. This to point out the great generality of this formalism which starts by studying the structure of the set of properties of a physical system. If S is a classical system, described in phase space formalism, $\mathscr{L}$ is the collection of all the subsets of the phase space Ω. We'll denote this collection $\mathscr{P}(\Omega)$. If S is a quantum system, described in the Hilbert space formalism, $\mathscr{L}$ is the collection of all the closed subspaces of the Hilbert space $\mathscr{H}$. We'll denote this set likewise $\mathscr{P}(\mathscr{H})$.

We will say that a yes-no experiment is '*true*' whenever we are *certain* that the answer yes will be obtained. For a classical system described by $\mathscr{P}(\Omega)$ this will be the case when the point of Ω representing the state lies in the corresponding subset A of Ω, otherwise we get with certainty the answer no. For a quantum system described by $\mathscr{P}(\mathscr{H})$ this will be the case when the state vector x belongs to the corresponding closed subspace G of $\mathscr{H}$. Remark that only when $x \in G^{\perp}$ we get with certainty the answer no. In both classical and quantum cases we'll say '*A is true*' and '*G is true*'.

1.1. Structure of $\mathscr{L}$

The above defined notion of truth for a yes-no experiment gives rise to the basic structure of $\mathscr{L}$: we'll say that b '*implies*' c (notation $b < c$) iff c is true whenever b is true. It is easy to see that this defines a partial order relation, i.e. a reflexive, antisymmetric and transitive relation on $\mathscr{L}$. This partial order relation is the inclusion on $\mathscr{P}(\Omega)$ and on $\mathscr{P}(\mathscr{H})$.

(*1°*) *$\mathscr{L}$, $<$ is a complete lattice. This means that for every family $(b_i)_{i \in I}$ of elements in $\mathscr{L}$ there exists a greatest lower bound $\wedge_{i \in I}$ and a least upper bound $\vee_{i \in I} b_i$.*

In $\mathscr{P}(\Omega)$, the greatest lower bound of a family $(A_i)_{i \in I}$ is $\cap_{i \in I} A_i$, the least upper bound is $\cup_{i \in I} A_i$.

For a family of closed subspaces $(F_i)_{i \in I}$ in $\mathscr{P}(\mathscr{H})$ we have a greatest lower bound $\cap_{i \in I} F_i$, and a least upper bound $\overline{\text{span}_{i \in I}(F_i)}$.

Physically speaking, $\wedge_{i \in I} b_i$ will be true iff all the b_i's are true. The physical interpretation of the least upper bound is less obvious: it may happen that $\vee_{i \in I} b_i$ is true while none of the b_i's is true. For a classical system, this last possibility is excluded.

(*2°*) *$\mathscr{L}$, $<$ is an orthocomplemented lattice, with orthocomplementation $'$. This means that $'$ is a bijection mapping $\mathscr{L}$ to $\mathscr{L}$ such that for $b, c \in \mathscr{L}$*

$$(b')' = b$$
$$b < c \Rightarrow c' < b'$$
$$b \vee b' = I, \qquad b \wedge b' = 0 \quad \textit{where } I = \vee_{b \in \mathscr{L}} b,\ 0 = \wedge_{b \in \mathscr{L}} b.$$

For $A \in \mathscr{P}(\Omega)$ we have $A' = \Omega \backslash A$ and for $G \in \mathscr{P}(\mathscr{H})$ we have $G' = G^{\perp}$.

The physical interpretation is easy: if b is true, then we'll get for certain the answer no for b'. This interpretation holds in Hilbert space formalism as well as in phase space formalism.

(3°) *A lattice* $\mathscr{L}, <$ *is said to be* distributive *if*:

$$\forall b, c, d: b \vee (c \wedge d) = (b \vee c) \wedge (b \vee d)$$

It is easy to see that $\mathscr{P}(\Omega)$ is distributive, whereas $\mathscr{P}(\mathscr{H})$ is not if $\dim \mathscr{H} > 1$. This distributivity gives us the fundamental difference between quantum mechanics and classical mechanics. Although we see that not any $\mathscr{L}$ is distributive, there exists a weaker condition which is always fulfilled:

(4°) *$\mathscr{L}$ is weakly modular. This means that whenever $b < c$, the sublattice generated by $\{b, b', c, c'\}$ is distributive.* This condition is verified in $\mathscr{P}(\mathscr{H})$.

Pairs which generate distributive sublattices are interesting in themselves because they show classical features. This motivates the following definition:

(5°) *Definition*: *$b, c \in \mathscr{L}$ are said to be compatible iff the sublattice generated by $\{b, b', c, c'\}$ is distributive.*

We shall denote this by $b \leftrightarrow c$.

In a $\mathscr{P}(\mathscr{H})$ the relation $F \leftrightarrow G$ holds iff the corresponding projection operators commute. In $\mathscr{P}(\Omega)$ any two elements are compatible.

Other pairs of elements can also be interesting:

(6°) *Definition*: *$b, c \in \mathscr{L}$ are said to be a modular pair iff*

$$\forall x < c \qquad (b \vee x) \wedge c = (b \wedge c) \vee x$$

We shall denote this by $(b, c)M$.

One easily sees that $b \leftrightarrow c$ implies $(b, c)M$.

In the following we shall always assume that the lattice $\mathscr{L}$ of yes-no experiments of a system satisfies axioms (1°), (2°), (4°). For a more complete physical justification of this assumption the reader is referred to [3].

1.2. The states of a physical system

Let us consider a system S, described by the complete, ortho-complemented, weakly modular lattice $\mathscr{L}$ of its yes-no experiments. The next step in the study of the system is to investigate which yes-no experiments are true at a certain moment. These are indeed the properties which are elements of reality for the system at that moment: they represent the state of the system. This collection of 'true' yes-no experiments is completely characterized by its greatest lower bound which is again a yes-no experiment (see 1.1 (1°)). One usually represents the state by this yes-no experiment.

Moreover we want the system to be totally described by this collection. This compels its greatest lower bound to be a minimal element of $\mathscr{L}$: no additional yes-no experiment can give us more information. In mathematical language, this means the following for the proposition p corresponding to a certain state: $x \in \mathscr{L}, x < p \Rightarrow x = 0$ or $x = p$.

An element p which has this property is called an *atom* of $\mathscr{L}$. Since on the other hand for every property of S there has to be a state in which the system possesses this property, we have the following axiom:

(1°) *$\mathscr{L}$ is an atomic lattice. This means that for every element b of $\mathscr{L}$ there exists an atom p such that $p < b$.*

In $\mathscr{P}(\Omega)$ the atoms are the points of Ω and in $\mathscr{P}(\mathscr{H})$ they are the one-dimensional subspaces of $\mathscr{H}$. There is one more axiom to be fulfilled by $\mathscr{L}$. For its physical understanding which requires a deep analysis of the measuring process we refer the reader to [3] ch. 4.

(2°) *$\mathscr{L}$ satisfies the covering law. This means that if p is an atom of $\mathscr{L}$ and $a, x \in \mathscr{L}$ such that $a \wedge p = 0$ and $a < x < a \vee p$ then $x = a$ or $x = a \vee p$.*

A collection $\mathscr{L}$ of yes-no experiments on a physical system which satisfies 1.1 (1°), (2°), (4°) and 1.2 (1°), (2°) will be called a propositional system.

As we said before a classical system is described by a distributive propositional system which implies that any two propositions are compatible. When the opposite is true, i.e. no element of $\mathscr{L}$ except 0 and I is compatible with all the others ($\mathscr{L}$ is irreducible), we say that the system is a pure quantum system. In general a system is intermediate between those two extremes. It is in fact a quantum system with superselection rules and can be considered as a combination of pure quantum systems (see [3] chapter 2). $\mathscr{P}(\mathscr{H})$ is a pure quantum system.

The following two representation theorems show us that although we took $\mathscr{P}(\Omega)$ and $\mathscr{P}(\mathscr{H})$ as examples for a classical system and a pure quantum system, they are already the most general realizations.

(3°) If $\mathscr{L}$ is the propositional system of a classical system, and so a complete orthocomplemented distributive atomic lattice, then $\mathscr{L}$ is isomorphic to $\mathscr{P}(\Omega)$ where Ω is the set of all the atoms of $\mathscr{L}$ (for proof see [3] p. 9).

(4°) If $\mathscr{L}$ is the propositional system of a pure quantum system, and so a complete orthocomplemented, irreducible, weakly modular, atomic lattice satisfying the covering law, then $\mathscr{L}$ is isomorphic to the set $\mathscr{P}(V)$ of all biorthogonal subspaces of a vectorspace V over some field K. The orthocomplementation defines on K an involutive anti-automorphism and on V a nondegenerate sesquilinear form; the weak modularity ensures that the whole space is linearly generated by any element and the corresponding orthogonal subspace (for proof see [8] and [3]). If we take the field in this representation theorem of Piron to be $\mathbb{C}$ and the involutive anti-automorphism of $\mathbb{C}$ to be the conjugation then the vectorspace V becomes a Hilbertspace over $\mathbb{C}$ and so $\mathscr{L}$ is isomorphic to our example $\mathscr{P}(\mathscr{H})$. In the following we shall always restrict ourselves to the case where $K = \mathbb{C}$.

2. The description of two systems as a joint system

Let us consider two physical systems S_1 and S_2 with their corresponding propositional systems $\mathscr{L}_1$ and $\mathscr{L}_2$. We want to describe the two systems as being one joint system S with corresponding propositional system $\mathscr{L}$. Which will be the physical requirements to ask on $\mathscr{L}_1$, $\mathscr{L}_2$ and $\mathscr{L}$?

2.1. — First of all we do not want S_1 and S_2 to lose their identity by coupling them. So every property of S_1 and every property of S_2 has to be a property of S. The mathematical translation of this requirement is the existence of a map h_1 from $\mathscr{L}_1$ to $\mathscr{L}$ and a map h_2 from $\mathscr{L}_2$ to $\mathscr{L}$.

— We also require the physical structure of S_1 and S_2 to be conserved when they are considered as parts of S. So h_1 and h_2 have to conserve the structure of $\mathscr{L}_1$ and $\mathscr{L}_2$. It is sufficient to ask the following:

(i) for $a_1, b_1 \in \mathscr{L}_1$ and $a_1 \leftrightarrow b_1$ we have $h_1(a_1) \leftrightarrow h_1(b_1)$

(ii) for $(a_i)_{i \in I} a_i \in \mathscr{L}_1$ we have $h_1(\vee_{i \in I} a_i) = \vee_{i \in I} h_1(a_i)$.

From these two requirements follows immediately that

(iii) $h_1(0_1) = 0$

(iv) $h_1(a_1') = h_1(a_1)' \wedge h_1(I_1)$

(v) $h_1(\wedge_i a_i) = \wedge_{i \in I} h_1(a_i)$.

So (i) and (ii) make h_1 to conserve the structure of $\mathscr{L}_1$. We ask the same properties for h_2.[3])

— $h_1(I_1)$ is a yes-no experiment on S which is true whenever S_1 exists, so it is always true. From this we conclude that $h_1(I_1) = I$ and in an analogous way $h_2(I_2) = I$.

A c-morphism such that the identity is mapped to the identity is said to be unitary.

2.2 We also want to couple S_1 and S_2 in such a way that it remains always possible to consider only one of them without disturbing the other. No measurement on S_1 may disturb S_2 and vice versa. The mathematical translation of this requirement is the following:

$$\text{If } a_1 \in \mathscr{L}_1, \quad a_2 \in \mathscr{L}_2 \quad \text{then } h_1(a_1) \leftrightarrow h_2(a_2).$$

2.3. We do not lose information in coupling S_1 with S_2. So we will ask that when we perform a measurement on S_1 which gives us maximal information together with a measurement on S_2 which gives us maximal information this will give us a maximal information measurement on S.

The mathematical translation of this requirement is the following (see 1.2):

If p_1 is an atom of $\mathscr{L}_1$ and p_2 an atom of $\mathscr{L}_2$, then: $h_1(p_1) \wedge h_2(p_2)$ is an atom of $\mathscr{L}$.

These three rather weak requirements will be sufficient to prove the following two theorems.

2.4. If S_1 and S_2 are two classical systems described by $\mathscr{P}(\Omega_1)$ and $\mathscr{P}(\Omega_2)$ then the joint system S is described by $\mathscr{P}(\Omega_1 \times \Omega_2)$.

2.5. If S_1 and S_2 are two pure quantum systems by $\mathscr{P}(\mathscr{H}_1)$ and $\mathscr{P}(\mathscr{H}_2)$ then the joint system S is described by $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ or $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2^*)$ where $\mathscr{H}_2^*$ is the dual of $\mathscr{H}_1$.

In the next sections we will prove those two theorems.

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3) A map from a propositional system $\mathscr{L}$ to a propositional system $\mathscr{L}'$ with these properties is called a c-morphism. An isomorphism of two propositional systems is a c-morphism such that the inverse map is also a c-morphism (i.e. a bijective c-morphism).

3. The classical case

Let us consider two classical systems S_1 and S_2 described by $\mathscr{P}(\Omega_1)$ and $\mathscr{P}(\Omega_2)$ and the joint system S described by $\mathscr{P}(\Omega)$. First of all we remark that requirement 2.2 is always fulfilled since we want S to be a classical system. So we will prove 2.4 by using only 2.1 and 2.3.

Theorem. *If there exist two unitary c-morphisms*

$$h_1: \mathscr{P}(\Omega_1) \rightarrow \mathscr{P}(\Omega)$$
$$h_2: \mathscr{P}(\Omega_2) \rightarrow \mathscr{P}(\Omega)$$

such that $h_1(p_1) \wedge h_2(p_2)$ is an atom of $\mathscr{P}(\Omega)$ whenever p_1 is an atom of $\mathscr{P}(\Omega_1)$ and p_2 is an atom of $\mathscr{P}(\Omega_2)$, then $\mathscr{P}(\Omega)$ is isomorphic to $\mathscr{P}(\Omega_1 \times \Omega_2)$.

Proof. Consider the following map:

$$I: \Omega_1 \times \Omega_2 \rightarrow \Omega$$
$$(p_1, p_2) \rightarrow h_1(p_1) \wedge h_2(p_2).$$

(i) *I is injective*
Indeed suppose

$$\begin{aligned} & h_1(p_1) \wedge h_2(p_2) = h_1(q_1) \wedge h_2(q_2) \\ \Rightarrow\ & h_1(p_1) \wedge h_2(p_2) = [h_1(p_1) \wedge h_2(p_2)] \wedge [h_1(q_1) \wedge h_2(q_2)] \\ & \qquad = h_1(p_1 \wedge q_1) \wedge h_2(p_2 \wedge q_2) \end{aligned}$$

if $p_1 \neq q_1 \Rightarrow p_1 \wedge q_1 = \phi \Rightarrow h_1(p_1 \wedge q_1) = \phi \Rightarrow h_1(p_1) \wedge h_2(p_2) = \phi.$

(ii) *I is surjective*
Indeed

$$\begin{aligned} \Omega &= h_1(\Omega_1) \wedge h_2(\Omega_2) = h_1(\vee_{p_1 \in \Omega_1} p_1) \wedge h_2(\vee_{p_2 \in \Omega_2} p_2) \\ &= \{\vee_{p_1 \in \Omega_1} h_1(p_1)\} \wedge \{\vee_{p_2 \in \Omega_2} h_2(p_2)\} \\ &= \vee_{p_1 \in \Omega_1} \vee_{p_2 \in \Omega_2} (h_1(p_1) \wedge h_2(p_2)) \\ &= \{h_1(p_1) \wedge h_2(p_2) \mid (p_1, p_2) \in \Omega_1 \times \Omega_2\}. \end{aligned}$$

If we consider now the map i: $\mathscr{P}(\Omega) \rightarrow \mathscr{P}(\Omega_1, \Omega_2)$ such that $i(A) = \{(p_1, p_2) \mid I[(p_1, p_2)] \in A\}$ then it is easy to check that i is an isomorphism.

4. The quantum case

To prove 2.5 we will make an intensive use of a study of structure preserving maps of propositional systems made in [9]. We already defined a c-morphism to be a map preserving the physical structure represented by the axioms 1.1 (1°), (2°), (4°). In 1.1 (6°) we defined modular pairs. It is easy to see that not every c-morphism will map a modular pair into a modular pair. If it does we will call it an m-morphism [9] [10].

In [9] it is proven that every unitary m-morphism from $\mathscr{P}(\mathscr{H})$ to $\mathscr{P}(\mathscr{H}')$ can be generated by a family of unitary and antiunitary maps from $\mathscr{H}$ to $\mathscr{H}'$. If all these maps are unitary the m-morphism is said to be linear and if they are antiunitary the m-morphism is said to be antilinear. Otherwise it will be called mixed.

We will also need the following results of [9].

4.1. Proposition (see [9] proposition 2.5)[4])

Let $\mathscr{H}$ and $\mathscr{H}'$ be two complex Hilbertspaces and f a c-morphism from $\mathscr{P}(\mathscr{H})$ to $\mathscr{P}(\mathscr{H}')$. Then f is an m-morphism iff $\forall x, y$ non zero vectors in $\mathscr{H}: f(\overline{x-y}) \subset f(\bar{x}) + f(\bar{y})$.

4.2. **Theorem** (see [9], theorem 3.1)

Let $\mathscr{H}$ and $\mathscr{H}'$ be two complex Hilbert spaces, with dimension greater than or equal to 3. *Let f be an m-morphism mapping $\mathscr{P}(\mathscr{H})$ into $\mathscr{P}(\mathscr{H}')$. Then for every couple (x, y) of non zero vectors of $\mathscr{H}$, there exists a bijective bounded linear map F_{yx} mapping $f(\bar{x})$ into $f(\bar{y})$, such that the set of maps $\{F_{xy};\ x, y \in \mathscr{H}\}$ has the following properties:*

$$F_{xx} = \mathbb{1}_{f(\bar{x})} \tag{1}$$

$$F_{xy} = (F_{yx})^{-1} \tag{2}$$

$$F_{zy}F_{yx} = F_{zx} \tag{3}$$

$$F_{y+z,x} = F_{y,x} + F_{z,x} \tag{4}$$

$$F_{\lambda x, \lambda y} = F_{x,y} \qquad \lambda \in \mathbb{C}, \quad \lambda \neq 0 \tag{5}$$

$$F_{y,x} \text{ is an isomorphism if } \|x\| = \|y\|. \tag{6}$$

For every non zero x in $\mathscr{H}$, there exists moreover two orthogonal projections $P_1^{\bar{x}}$ and $P_2^{\bar{x}}$, elements of $\mathscr{P}(f(\bar{x}))$, *such that*

$$P_1^{\bar{x}} \cdot P_2^{\bar{x}} = 0 \tag{7}$$

$$P_1^{\bar{x}} + P_2^{\bar{x}} = \mathbb{1}_{f(\bar{x})} \tag{8}$$

$$P_i^{\bar{x}} = F_{yx} P_i^{\bar{x}} F_{xy} \quad i = 1, 2 \tag{9}$$

and

$$F_{\lambda x, x} = \lambda P_1^{\bar{x}} + \bar{\lambda} P_2^{\bar{x}}. \tag{10}$$

If for one non zero x in $\mathscr{H}$ the projection $P_2^{\bar{x}}(P_1^{\bar{x}})$ is zero, then all the $P_2^{\bar{y}}(P_1^{\bar{y}})$ are zero and f is a linear (anti-linear) m-morphism. (11)

We will also use the following result of [9].

4.3. **Theorem** (see [9], corollary 4.2)
Let $\mathscr{H}$, $\mathscr{H}'$ be complex Hilbert spaces with dimension greater than two. Let f be a unitary c-morphism mapping $\mathscr{P}(\mathscr{H})$ into $\mathscr{P}(\mathscr{H}')$ such that there exists one atom of p of $\mathscr{P}(\mathscr{H})$ which is mapped to an atom of $f(p)$ of $\mathscr{P}(\mathscr{H}')$. Then f is an isomorphism.

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[4]) If x is a vector of a Hilbert space $\mathscr{H}$, we will write $\bar{x}$ to denote the one dimensional subspace generated by x.

4.4. We will also need a few results from lattice theory which we will use frequently in the calculations.

(1°) In a weakly modular, complete orthocomplemented lattice $\mathscr{L}$ if $b \in \mathscr{L}$ and $a_i \in \mathscr{L}$ $\forall i \in \mathscr{L}$ and $b \leftrightarrow a_i$ $\forall i \in I$, then:

$$\vee_{i \in I}(b \wedge a_i) = b \wedge (\vee_{i \in I} a_i) \tag{12}$$

(see [3], theorem 2.21)

(2°) In a weakly modular, orthocomplemented lattice $\mathscr{L}$ we have two criteria which enable us to see whether two elements are compatible:

$$\text{If } b, c \in \mathscr{L}: b \leftrightarrow c \Leftrightarrow (b \wedge c) \wedge (b' \wedge c) = c \tag{13}$$

$$\Leftrightarrow (b \vee c') \wedge c = b \wedge c \tag{14}$$

(see [3] Section 2.2)

(3°) In a weakly modular, orthocomplemented lattice $\mathscr{L}$, the triplet (b, c, d) is distributive whenever one of the three elements is compatible with each of the two others.

(see [3], theorem 2.25) (15)

4.5. We will now consider two pure quantum systems S_1 and S_2 described by $\mathscr{P}(\mathscr{H}_1)$ and $\mathscr{P}(\mathscr{H}_2)$ and the joint system S described by $\mathscr{P}(\mathscr{H})$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are complex Hilbert spaces with dimension greater than two and $\mathscr{H}$ is a complex Hilbert space. We will also suppose that 2.1, 2.2 and 2.3 are fulfilled; so

(i) there exist two unitary c-morphisms

$$h_1: \mathscr{P}(\mathscr{H}_2) \to \mathscr{P}(\mathscr{H}) \qquad h_2: \mathscr{P}(\mathscr{H}_2) \to \mathscr{P}(\mathscr{H}). \tag{16}$$

(ii) such that for every $G_1 \in \mathscr{P}(\mathscr{H}_1)$ and $G_2 \in \mathscr{P}(\mathscr{H}_2)$ we have:

$$h_1(G_1) \leftrightarrow h_2(G_2). \tag{17}$$

(iii) and such that if p_1 is an atom of $\mathscr{P}(\mathscr{H}_1)$, p_2 an atom of $\mathscr{P}(\mathscr{H}_2)$, then

$$h_1(p_1) \wedge h_2(p_2) \text{ is an atom of } \mathscr{P}(\mathscr{H}). \tag{18}$$

With these different assumptions we will first prove some lemmas in preparation of the proof of 2.5.

Lemma 1. *If we define the following maps:*

$$u_{\bar{x}_2}: \mathscr{P}(\mathscr{H}_1) \to \mathscr{P}(h_2(\bar{x}_2)) \qquad G_1 \mapsto h_1(G_1) \wedge h_2(\bar{x}_2) \tag{19}$$

$$v_{\bar{x}_1}: \mathscr{P}(\mathscr{H}_2) \to \mathscr{P}(h_1(\bar{x}_1)) \qquad G_2 \mapsto h_2(G_2) \wedge h_1(\bar{x}_1) \tag{20}$$

then $u_{\bar{x}_2}$ *and* $v_{\bar{x}_1}$ *are isomorphisms.*

Proof.

(i) Take $(G_1^i)_{i \in I}$ so that $G_1^i \in \mathscr{P}(\mathscr{H}_1)$ $\forall_{i \in I}$

$$u_{\bar{x}_2}(\vee_{i \in I} G_1^i) = h_1(\vee_{i \in I} G_1^i) \wedge h_2(\bar{x}_2) = (\vee_{i \in I} h_1(G_1^i)) \wedge h_2(\bar{x}_2)$$

because $h_1(G_1^i) \leftrightarrow h_2(\bar{x}_2) \forall i \in I$ and (12) we have:

$$u_{\bar{x}_2}(\vee_{i\in I} G_1^i) = \vee_{i\in I}(h_1(G_1^i) \wedge h_2(\bar{x}_2)) = \vee_{i\in I} u_{\bar{x}_2}(G_1^i).$$

(ii) Take $G_1, H_1 \in \mathscr{P}(\mathscr{H}_1)$ such that $G_1 \leftrightarrow H_1$. Let us first calculate $u_{\bar{x}_2}(H_1')$

$$\begin{aligned} u_{\bar{x}_2}(H_1') &= h_1(H_1') \wedge h_2(\bar{x}_2) = h_1(H_1)' \wedge h_2(\bar{x}_2) = (h_1(H_1)' \vee h_2(\bar{x}_2)') \wedge h_2(\bar{x}_2) \\ &= (h_1(H_1) \wedge h_2(\bar{x}_2))' \wedge h_2(\bar{x}_2) = u_{\bar{x}_2}(H_1)' \wedge h_2(\bar{x}_2). \end{aligned}$$

It is easy to see that if $a \in \mathscr{P}(h_2(\bar{x}_2))$, then $a' \wedge h_2(\bar{x}_2)$ is in fact the orthocomplement of a in $\mathscr{P}(h_2(\bar{x}_2))$. We can see that:

$$\begin{aligned} (u_{\bar{x}_2}(G_1) \vee (u_{\bar{x}_2}(H_1)' \wedge h_2(\bar{x}_2))) \wedge u_{\bar{x}_2}(H_1) &= (u_{\bar{x}_2}(G_1) \vee u_{x_2}(H_1')) \wedge u_{\bar{x}_2}(H_1) \\ &= u_{\bar{x}_2}((G_1 \vee H_1') \wedge H_1) = u_{\bar{x}_2}(G_1 \wedge H_1) \\ &= u_{\bar{x}_2}(G_1) \wedge u_{\bar{x}_2}(H_1). \end{aligned}$$

This proves (14) that $u_{\bar{x}_2}(G_1) \leftrightarrow u_{\bar{x}_2}(H_1)$.

(iii) If p_1 is an atom of $\mathscr{P}(\mathscr{H}_1)$, then $u_{\bar{x}_2}(p_1) = h_1(p_1) \wedge h_2(\bar{x}_2)$ is an atom of $\mathscr{P}(\mathscr{H})$ (18), so an atom of $\mathscr{P}(h_2(\bar{x}_2))$.

(iv) $u_{\bar{x}_2}(\mathscr{H}_1) = h_1(\mathscr{H}_1) \wedge h_2(\bar{x}_2) = h_2(\bar{x}_2)$

So we proved that $u_{\bar{x}_2}$ is a unitary c-morphism which maps atoms onto atoms. Using theorem 4.3 we conclude that $u_{\bar{x}_2}$ is an isomorphism. In an analogous way we prove that $v_{\bar{x}_1}$ is an isomorphism.

Lemma 2. *h_1 and h_2 are m-morphisms.*

Proof. Take two non zero vectors $x_1, y_1 \in \mathscr{H}_1$ and a vector $x \in h_1(\overline{x_1 - y_1})$. From lemma 1 we know that $v_{\overline{x_1 - y_1}}$ is an isomorphism of $\mathscr{P}(\mathscr{H}_2)$ to $\mathscr{P}(h_1(\overline{x_1 - y_1}))$. Hence there exists an atom $\bar{x}_2 \in \mathscr{P}(\mathscr{H}_2)$ such that $v_{\overline{x_1 - y_1}}(\bar{x}_2) = \bar{x}$.

So

$$\bar{x} = h_2(\bar{x}_2) \wedge h_1(\overline{x_1 - y_1}). \tag{21}$$

Since $\overline{x_1 - y_1} < \bar{x}_1 \vee \bar{y}_1$ we have

$$h_1(\overline{x_1 - y_1}) < h_1(\bar{x}_1) \vee h_1(\bar{y}_1) \Rightarrow h_2(\bar{x}_2) \wedge h_1(\overline{x_1 - y_1}) < h_2(\bar{x}_2) \wedge (h_1(\bar{x}_1) \vee h_1(\bar{y}_1))$$

from (15) $= [h_2(\bar{x}_2) \wedge h_1(\bar{x}_1)] \vee [h_2(\bar{x}_2) \wedge h_1(\bar{y}_1)]$

from (18) $= [h_2(\bar{x}_2) \wedge h_1(\bar{x}_1)] + [h_2(\bar{x}_2) \wedge h_1(\bar{y}_1)]$.

From (21) we see that $x = y + z$ where $y \in h_1(\bar{x}_1)$ and $z \in h_1(\bar{y}_1)$.
Since x was chosen at random, we have:

$$h_1(\overline{x_1 - y_1}) \subset h_1(\bar{x}_1) + h_1(\bar{y}_1).$$

Using proposition 4.1 we conclude that h_1 is an m-morphism. In an analogous way, one proves that h_2 is an m-morphism.

Since h_1 and h_2 are m-morphisms we can consider the mappings

$$F_{y_1, x_1} : h_1(\bar{x}_1) \to h_1(\bar{y}_1)$$

$$K_{y_2, x_2} : h_2(\bar{x}_2) \to h_2(\bar{y}_2)$$

defined in theorem 4.2, with their corresponding properties proved in [9]. We can prove then the following:

Lemma 3. *If* $x \in h_1(\overline{x_1}) \wedge h_2(\overline{x_2})$ *and* $y_1 \in \mathscr{H}_1$, $y_2 \in \mathscr{H}_2$, *then*:

$$F_{y_1,x_1}(x) \in h_1(\overline{y_1}) \wedge h_2(\overline{x_2}) \tag{22}$$

$$K_{y_2,x_2}(x) \in h_1(\overline{x_1}) \wedge h_2(\overline{y_2}) \tag{23}$$

and

$$F_{y_1,x_1}K_{y_2,x_2}(x) = K_{y_2,x_2}F_{y_1,x_1}(x). \tag{24}$$

Proof. Take $x \in h_1(\overline{x_1}) \wedge h_2(\overline{x_2})$ and $y_2 \in \mathscr{H}_2$, then:

$x = K_{y_2,x_2}(x) + u$ where $u \in h_2(\overline{x_2 - y_2})$, $K_{y_2,x_2}(x) \in h_2(\overline{y_2})$.

We have to show that $K_{y_2,x_2}(x) \in h_1(\overline{x_1})$.

Since $h_1(\overline{x_1}) \leftrightarrow h_2(\overline{y_2})$ and $h_1(\overline{x_1}) \leftrightarrow h_2(\overline{x_2 - y_2})$ we have (13)

$$h_2(\overline{y_2}) = [h_2(\overline{y_2}) \wedge h_1(\overline{x_1})] \vee [h_2(\overline{y_2}) \wedge h_1(\overline{x_1})']$$

$$h_2(\overline{x_2 - y_2}) = [h_2(\overline{x_2 - y_2}) \wedge h_1(\overline{x_1})] \vee [h_2(\overline{x_2 - y_2}) \wedge h_1(\overline{x_1})'].$$

So $K_{y_2,x_2}(x) = v + w$ where $v \in h_2(\overline{y_2}) \wedge h_1(\overline{x_1})$ and $w \in h_2(\overline{y_2}) \wedge h_1(\overline{x_1})'$ and $u = z + t$ where $z \in h_2(\overline{x_2 - y_2}) \wedge h_1(\overline{x_1})$ and $t \in h_2(\overline{x_2 - y_2}) \wedge h_1(\overline{x_1})'$.

So: $x - v - z = w + t$, hence $w + t \in h_1(\overline{x_1}) \wedge h_1(\overline{x_1})'$ which implies $w + t = 0$ and since $h_2(\overline{x_2 - y_2}) \wedge h_2(\overline{y_2}) = 0$ we have $w = t = 0$. So we see that $K_{y_2,x_2}(x) = v \in h_1(\overline{x_1}) \wedge h_2(\overline{y_2})$.

In an analogous way we prove that $F_{y_1,x_1}(x) \in h_1(\overline{y_1}) \wedge h_2(\overline{x_2})$.

To prove (24) we proceed as follows:

$$x = K_{y_2,x_2}(x) + u$$

$$F_{y_1,x_1}(x) = F_{y_1,x_1}K_{y_2,x_2}(x) + F_{y_1,x_1}(u)$$

and

$$F_{y_1,x_1}(x) \in h_1(\overline{y_1}) \wedge h_2(\overline{x_2})$$

$$F_{y_1,x_1}K_{y_2,x_2}(x) \in h_1(\overline{y_1}) \wedge h_2(\overline{y_2})$$

$$F_{y_1,x_1}(u) \in h_1(\overline{y_1}) \wedge h_2(\overline{x_2 - y_2})$$

So:

$$K_{y_2,x_2}F_{y_1,x_1}(x) = F_{y_1,x_1}K_{y_2,x_2}(x).$$

Lemma 4. h_1 *and* h_2 *are both non-mixed m-morphisms.*

Proof. Suppose h_1 to be mixed. Then $P_1^{\bar{x}}[h_1(\overline{x_1})] \neq 0$ and $P_2^{\bar{x}}[h_1(\overline{x_1})] \neq 0$. Take

$z \in P_1^{\bar{x}}[h_1(\overline{x_1})]$ and $t \in P_2^{\bar{x}}[h_1(\overline{x_1})]$.

Since from lemma 1 the map $v_{\bar{x}_2}: \mathscr{P}(\mathscr{H}_2) \to \mathscr{P}(h_1(\overline{x_1}))$ is surjective, there exist elements z_2 and $t_2 \in \mathscr{H}_2$ such that

$$v_{\bar{x}_1}(\overline{z_2}) = \bar{z} \quad \text{and} \quad v_{\bar{x}_1}(\overline{t_2}) = \bar{t} \quad \text{or} \quad z \in h_1(\overline{x_1}) \wedge h_2(\overline{z_2}) \quad \text{and} \quad t \in h_1(\overline{x_1}) \wedge h_2(\overline{t_2}).$$

Take $K_{t_2,z_2}(z) = t' \in \bar{t}$

$K_{t_2,z_2}F_{\lambda x_1,x_1}(z) = K_{t_2,z_2}(\lambda z) = \lambda t'.$

On the other hand, because of (24), we must obtain the same result if we calculate

$F_{\lambda x_1,x_1}K_{t_2,z_2}(z) = F_{\lambda x_1,x_1}(t') = \bar{\lambda} t'.$

But $\lambda t' \neq \bar{\lambda} t'$ if $\mathrm{Im}(\lambda) \neq 0$. So h_1 cannot be mixed. In an analogous way we prove that h_2 cannot be mixed.

Lemma 5. *Take $z_1 \in \mathscr{H}_1$, $z_2 \in \mathscr{H}_2$, and $z \in h_1(\bar{z}_1) \wedge h_2(\bar{z}_2)$ such that z_1, z_2, z are non zero vectors. Put*

$$\alpha = \frac{\|z_1\| \, \|z_2\|}{\|z\|} \tag{25}$$

and define the following maps:

$$U_{\bar{x}_2}: \mathscr{H}_1 \to h_2(\bar{x}_2) \qquad x_1 \mapsto \frac{\alpha}{\|x_2\|} F_{x_1,z_1}K_{x_2,z_2}(z) \tag{26}$$

$$V_{\bar{x}_1}: \mathscr{H}_2 \to h_1(\bar{x}_1) \qquad x_2 \mapsto \frac{\alpha}{\|x_1\|} K_{x_2,z_2}F_{x_1,z_1}(z) \tag{27}$$

then the $U_{\bar{x}_2}$ are all unitary maps or antiunitary maps according to whether h_1 is linear or antilinear (see (11)) and the $v_{\bar{x}_1}$ are all unitary maps or antiunitary maps according to whether h_2 is linear or antilinear.

$U_{\bar{x}_2}$ generates $u_{\bar{x}_2}$ and $V_{\bar{x}_1}$ generates $v_{\bar{x}_1}$. (28)

Proof. It follows from (22) and (23) that $U_{\bar{x}_2}$ and $V_{\bar{x}_1}$ are well defined. It follows from the definition and (10) that $U_{\bar{x}_2}$ is linear or antilinear according to whether h_1 is linear or antilinear.

Take now $\|t_1\| = \|z_1\|$ and $\|t_2\| = \|z_2\|$ with $t_1 \in \bar{x}_1$ and $t_2 \in \bar{x}_2$

$$\|U_{\bar{x}_2}(x_1)\| = \frac{\alpha}{\|x_2\|} \|F_{x_1,z_1}K_{x_1,z_1}(z)\|$$

$$= \frac{\alpha}{\|x_2\|} \frac{\|x_1\|}{\|z_1\|} \frac{\|x_2\|}{\|z_2\|} \|F_{t_1,z_1}K_{t_2,z_2}(z)\|.$$

Using (6) and (25) we see that:

$$\|U_{\bar{x}_2}(x_1)\| = \frac{\|z_1\| \, \|z_2\|}{\|z\| \, \|x_2\|} \frac{\|x_1\| \, \|x_2\|}{\|z_1\| \, \|z_2\|} \|z\| = \|x_1\|.$$

So $U_{\bar{x}_2}$ is an isometry (or anti-isometry).

In an analogous way we prove that $V_{\bar{x}_1}$ is an isometry (or anti-isometry).

From (22) and (23) we see that:

$\overline{U_{\bar{x}_2}(x_1)} = h_1(\bar{x}_1) \wedge h_2(\bar{x}_2) = \overline{V_{\bar{x}_1}(x_2)}$ for $x_1 \in \mathscr{H}_1$ and $x_2 \in \mathscr{H}_2$.

If we define:

$\tilde{u}_{\bar{x}_2}: \mathscr{P}(\mathscr{H}_1) \to \mathscr{P}(h_2(\bar{x}_2))$

$G_1 \mapsto \{U_{\bar{x}_2}(x_1); x_1 \in G_1\}$

$\tilde{v}_{x_1}: \mathscr{P}(\mathscr{H}_2) \to \mathscr{P}(h_1(\bar{x}_1))$

$G_2 \mapsto \{V_{\bar{x}_1}(x_2); x_2 \in G_2\}$

then it is easy to check that $\tilde{u}_{\bar{x}_2}$ and $\tilde{v}_{\bar{x}_1}$ are unitary c-morphisms different from the zero morphism, mapping atoms onto atoms. Take now an orthonormal base $\{e_i\}_{i \in I}$ in G_1. Then $\{U_{x_2}(e_i)\}_{i \in I}$ is an orthonormal base in $\tilde{u}_{\bar{x}_2}(G_1)$, and:

$$u_{\bar{x}_2}(G_1) = u_{\bar{x}_2}(\vee_{i \in I} \bar{e}_1) = \vee_{i \in I} u_{\bar{x}_2}(\bar{e}_i)$$
$$= \vee_{i \in I} (h_1(\bar{e}_i) \wedge h_2(\bar{x}_2)) = \vee_{i \in I} \overline{U_{\bar{x}_2}(e_i)} = \tilde{u}_{\bar{x}_2}(G_1).$$

So $\tilde{u}_{\bar{x}_2} = u_{\bar{x}_2}$ and analogously $\tilde{v}_{\bar{x}_1} = v_{\bar{x}_1}$. Since $u_{\bar{x}_2}$ and $v_{\bar{x}_1}$ are surjective, $U_{\bar{x}_2}$ and $V_{\bar{x}_1}$ will be unitary or antiunitary.

Lemma 6.

If $\|x_1\| = \|x_2\|$ *for* $x_1 \in \mathscr{H}_1$ *and* $x_2 \in \mathscr{H}_2$, *then* $U_{\bar{x}_2}(x_1) = V_{\bar{x}_1}(x_2)$. (29)

If $\{e_i)_{i \in I}$ *is an orthonormal base of* $\mathscr{H}_1$ *and* $\{f_j\}_{j \in J}$ *an orthonormal base of* $\mathscr{H}_2$, *then* $\{U_{\bar{f}_j}(e_i)\}_{i \in I, j \in J}$ *is an orthonormal base of* $\mathscr{H}$. (30)

Proof. (29) is a direct consequence of the definitions (26), (27) and of (24).

Let us prove (30). Suppose $j \neq j'$, then, since $\overline{f_j} \perp \overline{f_{j'}}$, we have $h_2(\overline{f_j}) \perp h_2(\overline{f_{j'}})$. This makes $U_{\bar{f}_j}(x_1) \perp U_{\bar{f}_{j'}}(x_1)\ \forall x_1 \in \mathscr{H}_1$.

Since $U_{\bar{f}_j}$ is a unitary or antiunitary operator we have also

$U_{\bar{f}_j}(e_i) \perp U_{\bar{f}_j}(e_{i'})$ for $i \neq i'$

and

$\|U_{\bar{f}_j}(e_i)\| = 1 \quad \forall i, j$

So $\{U_{\bar{f}_j}(e_i)\}$ is an orthonormal set in $\mathscr{H}$.

Take

$$\vee_{i \in I, j \in J} \overline{U_{\bar{f}_j}(e_i)} = \vee_{i \in I, j \in J}(h_1(\bar{e}_i) \wedge h_2(\overline{f_j})) = h_1(\mathscr{H}_1) \wedge h_2(\mathscr{H}_2) = \mathscr{H}$$

Hence $\{U_{\bar{f}_j}(e_i)\}$ is an orthonormal base in $\mathscr{H}$.

Let us consider now the dual space $\mathscr{H}_1^*$ of $\mathscr{H}_1$ and the canonical antiisomorphism

$*: \mathscr{H}_1 \to \mathscr{H}_1^* \qquad x_1 \mapsto x_1^*$ where $x_1^*(y_1) = \langle x_1, y_1 \rangle\ \forall y_1 \in \mathscr{H}_1$.

We know that if $\{e_i\}_{i \in I}$ is an orthonormal base of $\mathscr{H}_1$ and $\{f_j\}_{j \in J}$ is an orthonormal base of $\mathscr{H}_2$, then $\{e_i^*\}_{i \in I}$ is an orthonormal base of $\mathscr{H}_1^*$, $\{e_i \otimes f_j\}_{i \in I, j \in J}$ is an orthonormal base of $\mathscr{H}_1 \otimes \mathscr{H}_2$.

We will consider now the following maps:

(i) If h_1 and h_2 are linear, we define

$$\phi_{e,f}: \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H} \qquad \Sigma_{ij} x_{ij} e_i \otimes f_j \mapsto \Sigma_{ij} x_{ij} U_{\bar{f}_j}(e_i)$$

(ii) If h_1 and h_2 are antilinear, we define

$$\psi_{e,f}: \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H} \qquad \Sigma_{ij} x_{ij} e_i \otimes f_j \mapsto \Sigma_{ij} x_{ij} U_{\bar{f}_j}(e_i)$$

(iii) If h_1 is antilinear and h_2 is linear, we define

$$\mu_{e,f}: \mathscr{H}_1^* \otimes \mathscr{H}_2 \to \mathscr{H} \qquad \Sigma_{ij} x_{ij} e_i^* \otimes f_j \mapsto \Sigma_{ij} x_{ij} U_{\bar{f}_j}(e_i)$$

(iv) If h_1 is linear and h_2 antilinear, we define

$$\nu_{e,f}: \mathscr{H}_1^* \otimes \mathscr{H}_2 \to \mathscr{H} \qquad \Sigma_{ij} x_{ij} e_i^* \otimes f_j \mapsto \Sigma_{ij} x_{ij} U_{\bar{f}_j}(e_i).$$

It is easy to see that $\phi_{e,f}$ and $\mu_{e,f}$ are unitary maps and $\psi_{e,f}$ and $\nu_{e,f}$ are antiunitary maps. Since we constructed these maps starting from two bases they need not be canonical maps. If we want however $\mathscr{H}$ to be the tensor product, we have to prove that $\phi_{e,f}$, $\psi_{e,f}$, $\mu_{e,f}$, $\nu_{e,f}$ are independent of the chosen bases $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$.

Lemma 7. *$\phi_{e,f}$ and $\mu_{e,f}$ are canonical unitary maps and $\psi_{e,f}$ and $\nu_{e,f}$ are canonical antiunitary maps.*

Proof. We will prove the lemma only for one of the four maps. The three other proofs are completely analogous.

Take another orthonormal base $\{p_k\}_{k\in I}$ of $\mathscr{H}_1$ and another orthonormal base $\{q_l\}_{l\in J}$ of $\mathscr{H}_2$. We have to prove that $\mu_{e,f} = \mu_{p,q}$.

Take $x \in \mathscr{H}_1^* \otimes \mathscr{H}_2$, so

$$x = \Sigma_{i,j} x_{ij} e_i^* \otimes f_j = \Sigma_{k,l} y_{kl} p_k^* \otimes q_l.$$

So

$$x_{ij} = \Sigma_{k,l} \langle p_k, e_i \rangle \langle f_j, q_l \rangle y_{kl}$$

$$\begin{aligned}
\mu_{pq}(x) &= \Sigma_{k,l} y_{kl} U_{\bar{q}l}(p_k) \\
&= \Sigma_{k,l} y_{kl} U_{\bar{q}_l}(\Sigma_i \langle e_i, p_k \rangle e_i) \\
&= \Sigma_{k,l} y_{kl} \Sigma_i \langle p_k, e_i \rangle U_{\bar{q}_l}(e_i) \\
&= \Sigma_{k,l,i} y_{kl} \langle p_k, e_i \rangle V_{\bar{e}_i}(q_l) \\
&= \Sigma_{k,l,i} y_{kl} \langle p_k, e_i \rangle V_{\bar{e}_i}(\Sigma_j \langle f_j, q_l \rangle f_j) \\
&= \Sigma_{k,l,i,j} y_{kl} \langle p_k, e_i \rangle \langle f_j, q_l \rangle V_{\bar{e}_i}(f_j) \\
&= \Sigma_{k,l,i,j} y_{kl} \langle p_k, e_i \rangle \langle f_j, q_l \rangle U_{\bar{f}_j}(e_i) \\
&= \Sigma_{i,j} x_{ij} U_{\bar{f}_j}(e_i) \\
&= \mu_{e,f}(x).
\end{aligned}$$

In the following theorem we will prove 2.5.

Theorem. *Consider $\mathscr{P}(\mathscr{H}_1)$, $\mathscr{P}(\mathscr{H}_2)$, $\mathscr{P}(\mathscr{H})$ where $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}$ are complex Hilbert spaces with* $\dim \mathscr{H}_1 > 2$ *and* $\dim \mathscr{H}_2 > 2$. *If h_1, h_2 fulfill the following conditions:*

(i) *h_1, h_2 are unitary c-morphisms*

$h_1: \mathscr{P}(\mathscr{H}_1) \to \mathscr{P}(\mathscr{H})$

$h_2: \mathscr{P}(\mathscr{H}_2) \to \mathscr{P}(\mathscr{H})$

(ii) $\forall G_1 \in \mathscr{P}(\mathscr{H}_1), \forall G_2 \in \mathscr{P}(\mathscr{H}_2)$ *we have* $h_1(G_1) \leftrightarrow h_2(G_2)$
(iii) $\forall p_1$ *atom in* $\mathscr{P}(\mathscr{H}_1)$, $\forall p_2$ *atom in* $\mathscr{P}(\mathscr{H}_2)$:
$h_1(p_1) \wedge h_2(p_2)$ *is an atom of* $\mathscr{P}(\mathscr{H})$

then $\mathscr{P}(\mathscr{H})$ *is canonically isomorphic to* $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ *or to* $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2^*)$.

Proof. From lemma 2 and lemma 4 we know that h_1 and h_2 are non mixed m-morphisms. From lemma 7 we conclude that:

(i) When h_1 and h_2 are linear, a canonical isomorphism from $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ to $\mathscr{P}(\mathscr{H})$ is generated by $\phi_{e,f}$ as follows:

$$\phi: \mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2) \to \mathscr{P}(\mathscr{H}) \qquad G \mapsto \{\phi_{e,f}(x): x \in G\}.$$

(ii) When h_1 and h_2 are antilinear, a canonical isomorphism from $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ to $\mathscr{P}(\mathscr{H})$ is generated by $\psi_{e,f}$.
(iii) When h_1 is linear and h_2 antilinear, a canonical isomorphism from $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2^*)$ to $\mathscr{P}(\mathscr{H})$ is generated by $\mu_{e,f}$.
(iv) When h_1 is antilinear and h_2 is linear, a canonical isomorphism from $\mathscr{P}(\mathscr{H}_1 \otimes \mathscr{H}_2^*)$ to $\mathscr{P}(\mathscr{H})$ is generated by $\nu_{e,f}$.

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REFERENCES

[1] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Part I: Functional Analysis* (Academic Press, New York 1972).
[2] G. Birkhoff and J. von Neumann, *The Logic of Quantum Mechanics*, Ann. Math. *37*, 823 (1936).
[3] C. Piron, *Foundations of Quantum Physics* (W.A. Benjamin Inc. 1976).
[4] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton 1955).
[5] I. E. Segal, Ann. Math. *48*, 930 (1947).
[6] G. W. Mackey, *Mathematical Foundations of Quantum Mechanics* (W. A. Benjamin, New York 1963).
[7] J. M. Jauch, *Foundations of Quantum Mechanics* (Addison-Wesley Inc. 1968).
[8] C. Piron, Helv. Phys. Acta *37*, 440 (1964).
[9] D. Aerts and I. Daubechies, *About the Structure Preserving Maps of a Quantum Mechanical Propositional System*; to be published in Helv. Phys. Acta.
[10] D. Aerts and C. Piron: In preparation.