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# On bound states of the infinite harmonic crystal

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*Abstract.* We study spectral properties of the infinite harmonic crystal with impurities. In particular, we consider the threshold behavior of the bound state in one dimension. We show that infinitely many bound states may occur. The proof of this fact relies on corresponding results for Schrödinger operators. A final remark is concerned with random masses.

## Introduction

Recently the spectral properties of the infinite harmonic crystal with a finite number of impurities have been studied by V. Hemmen [2] who extended earlier results of Romerio and Wreszinski [1]. Inspired by these two articles we got the impression that it might be of some interest to see how one can apply methods from the theory of Schrödinger operators to study spectral properties of the harmonic crystal with impurities. The outline of this paper is as follows. First we recover the results of [1] and [2] by looking at the problem from a somewhat different angle. Then, restricting attention to one dimension, we consider the weak coupling limit of a harmonic chain with impurities and study the threshold behavior. For related literature about Schrödinger operators we refer to [8, 9]. In Section 3, we discover the borderline at which the number of bound states may become infinite. A question about this problem was contained in [1]. We heavily rely on the corresponding results about the Schrödinger operator with potential $-c/(1 + x^2)$, $c > 0$. Finally we show that some of our results hold with probability one for random masses where 'random' is understood in a restricted sense.

Let us add a few remarks about the analogy between the equation studied in this paper and Schrödinger operators. We are going to investigate the spectral properties of an equation of the type

$$\alpha_{k+1}x_{k+1} + \alpha_{k-1}x_{k-1} + \beta_k x_k = \lambda x_k$$

where $\{x_k\} \in l^2(\mathbb{Z})$ and $\alpha_k$, $\beta_k$ will be subject to certain conditions. If $\alpha_{k+1} = \alpha_{k-1} = -1$ we have the case of a lattice Laplacin plus potential. Then the analogy with Schrödinger operators is complete and this case seems to have attracted most attention. As a reference we quote [14], where, if the existence of bound states is considered, the perturbation is kept *diagonal* although the $\alpha_k$ might be more general than in the pure

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Laplacian. However, in our problem $\alpha_k = -J/M_k$, $\beta_k = 2J/M_k (J > 0)$ where $M_k = M + \Delta_k$ and $\Delta_k$ is small in some sense, so the perturbation is *non diagonal.* $M_k$ are the masses in the crystal. We didn't feel encouraged to treat our problem in position space along the traditional lines [14] for even the diagonal case looks very complicated. We prefer to work in momentum space where we can use analytical tools. In Section 3, the connection with Schrödinger operators is established by exploiting the fact that the lattice Laplacian in momentum space looks parabolic at the band edge. A scaling argument will be essential. This work led us to a reconsideration of previous works about weakly coupled Schrödinger operators [8, 9]. This will be done in a companion paper [12].

## 1. Known results reviewed

We consider an infinite harmonic crystal in $v$ dimensions. The masses $M_n$ sit at the lattice sites $n \in \mathbb{Z}^v$. The Cartesian coordinates $x_\alpha(n)$ $(\alpha = 1, \ldots, v)$ denote the deviations in $\mathbb{R}^v$ of the particle at site $n$. The equation of motion read

$$M_n \ddot{x}_\alpha(n) = -\sum_{m,\beta} \Phi_{\alpha,\beta}(n-m) x_\beta(m) \tag{1.1}$$

where $\Phi_{\alpha,\beta}(\cdot)$ $(\alpha, \beta = 1, \ldots, v)$ is the translation invariant interaction matrix. We assume $\Phi_{\alpha,\beta}(n-m) = 0$, $|n-m| > N_0$ and comment on generalizations in the remarks. For stability reasons $\Phi \geqslant 0$. We look for solutions of (1.1) which oscillate in time like $\exp(\pm i\sqrt{\mu}\, t)(\mu > 0)$. Moreover we discuss systems which can be considered as the perturbations of a basic crystal with masses $M$. In [2] the following eigenvalue problem was derived

$$M^{-1}\Phi x + \sum_{n \in \Lambda_1} \lambda_n \Phi^{1/2} P_n \Phi^{1/2} x + \sum_{n \in \Lambda_2} \lambda_n \Phi^{1/2} P_n \Phi^{1/2} x = \mu x. \tag{1.2}$$

$x$ is short-hand for $x_\alpha(n)$. $P_n$ denotes the projection onto site $n$

$$(P_n x)_\alpha(n) = x_\alpha(n).$$

We put

$$\lambda_n = \frac{1}{M_n} - \frac{1}{M}, \qquad -\frac{1}{M} < \lambda_n < \infty \tag{1.3}$$

and define

$$\Lambda_1 = \{n \in \mathbb{Z}^v \mid \lambda_n > 0\}$$
$$\Lambda_2 = \{n \in \mathbb{Z}^v \mid \lambda_n < 0\}.$$

We denote by $N_i \leq \infty (i = 1, 2)$ the number of points in $\Lambda_i$. We assume that

$$\lambda_n \to 0 \quad \text{as } |n| \to \infty. \tag{1.4}$$

The operators act in the Hilbert space $\mathscr{H} = \bigoplus_1^v l^2(Z^v)$ or its Fourier transformed version $\hat{\mathscr{H}} = L^2(BZ)$, where $BZ = [-\pi, \pi]^v$ is the Brillouin zone. That means we keep the energy in the system finite. Then $\hat{P}_n$ is the projection onto the function $(2\pi)^{-1/2} \exp(inp)$, $p \in BZ$, $v = 1$.

As a consequence of (1.4) the two sums on the left side of (1.2) are self-adjoint, compact perturbations of the self-adjoint *dynamical matrix* $M^{-1}\phi$. The spectrum of $M^{-1}\phi$ is absolutely continuous and covers a closed interval $[0, \mu_0]$ ([2], [3]). We are interested in eigenvalues (bound states) $\mu > \mu_0$ of (1.2). References [1] and [2] deal with eigenvalue problem (1.2). We go one step further noticing that for $\mu > \mu_0$ eigenvalue problem (1.2) is equivalent to

$$A_\mu y \equiv (\mu - M^{-1}\Phi)^{-1/2}\left(\sum_{\Lambda_1} \lambda_n \Phi^{1/2} P_n \Phi^{1/2} + \sum_{\Lambda_2} \lambda_n \Phi^{1/2} P_n \Phi^{1/2}\right)\cdot(\mu - M^{-1}\Phi)^{-1/2} y = y$$

$$y \equiv (\mu - M^{-1}\Phi)^{1/2} x \tag{1.5}$$

so that the compact self-adjoint operator $A_\mu$ must have eigenvalue 1. We make the following observations:

(1) $A_\mu$ is composed of a positive $(\Sigma_{\Lambda_1})$ and a negative $(\Sigma_{\Lambda_2})$ part.
(2) The multiplicity of the eigenvalue $\mu$ in (1.2) is equal to the multiplicity of eigenvalue 1 in (1.5).
(3) $A_\mu$ depends analytically on $\mu$ if $\mu \in \mathbb{C}\backslash[0, \mu_0]$.
(4) $\|A_\mu\| \to 0$ monotonically as $\mu \to \infty$, for $\|(\mu - M^{-1/2}\Phi)^{1/2}(\mu' - M^{-1/2}\Phi)^{-1/2}\| \leqslant 1$ when $\mu' > \mu > \mu_0$.
(5) The positive eigenvalues of $A_\mu$ decrease monotonically as $\mu \to \infty$. This follows, e.g. from [4].

We denote by $P_\Omega(A)$ the spectral projection of a self-adjoint operator $A$ associated with the Borel set $\Omega$. The above statements imply $(\mu > \mu_0)$

$$\begin{aligned}&\dim(\mathrm{Ran}\, P_{[1,\infty)}(A_\mu)) = \dim(\mathrm{Ran}\, P_{[\mu,\infty)}(\text{l.h.s. of } (1.2))\\ &= \text{number of bound states} \geq \mu \text{ (including multiplicities).}\end{aligned} \tag{1.6}$$

If $N_1 < \infty$ it follows from (1) and (2) above and the fact that $\Sigma_{\Lambda_1}$ has finite rank $N_1 \nu$ that the number of bound states is bounded by $N_1 \nu$. This result was also obtained in Reference [2] and partially in [1] (if $N_2 = 0$). When $\Lambda_2 = \emptyset$ a necessary and sufficient condition for the existence of at least one bound state is that $\|A_\mu\| \geq 1$ for $\mu$ close to $\mu_0$. It is a dimensional question whether $\|A_\mu\|$ blows up or not as $\mu \downarrow \mu_0$. For illustration we consider the special case of a nearest neighbour interaction

$$\hat{\Phi}_{\alpha,\beta}(p) = J(\nu - \sum_{i=1}^{\nu} \cos p_i)\, \delta_{\alpha,\beta}, \quad J > 0 \tag{1.7}$$

in $\hat{\mathscr{H}} = L^2(BZ)$. Then $\mu_0 = 2\nu J/M$.

Near a corner $p_0$ of the cube $BZ = [-\pi, \pi]^\nu$ we have

$$\hat{\Phi}_{\alpha,\alpha}(p) \sim \hat{\Phi}_{\alpha,\alpha}(p_0) - \frac{J}{2}(p - p_0)^2 \tag{1.8}$$

with $\hat{\Phi}_{\alpha,\alpha}(p_0) = M \cdot \mu_0$.

From this it follows that if $N_1 = 1$, $N_2 = 0$ and

(1) $\nu = 1, 2$, we find a unique (simple) bound state for arbitrary values of $\lambda_{n_0}$ $(n_0 \in \Lambda_1)$.

(2) $\nu \geqslant 3$, we find a *threshold* $\lambda^*$ given by (calculate $\|A_{\mu_0}\|$ = non zero eigenvalue of $A_{\mu_0}$ and put it equal to 1)

$$\frac{1}{\lambda^*} = \frac{M}{(2\pi)^\nu} \int_{BZ} \frac{\nu - \Sigma_{i=1}^{\nu} \cos p_i}{\nu + \Sigma_{i=1}^{\nu} \cos p_i} d^\nu p. \tag{1.9}$$

This means that a bound state occurs if and only if $\lambda_{n_0} > \lambda^*(n_0 \in \Lambda_1)$. The value of the mass ratio $\alpha^* = M^*/M = 1/(1 + \lambda^* M)$ resulting from (1.9) for $\nu = 3$ is by construction larger than that obtained in [1].

*Remarks.* (1) Our arguments in the above example go through for any short range interaction. If the interaction is no longer of short range the analytic properties of $\hat{\Phi}$ change. In one dimension we may get $\hat{\Phi}(p) \sim \hat{\Phi}(p_0) - c|p - p_0|^\delta$ with $\delta < 1, c > 0$, instead of (1.8). That means a threshold.

(2) The operator corresponding to $A$ in the theory of Schrödinger operators would be $(-\Delta - E)^{-1/2} V(-\Delta - E)^{-1/2}, E < 0$, where $\Delta$ is the Laplacian. This operator is discussed in [12]. However, in [8, 9], $|V|^{1/2}(-\Delta - E)^{-1} V^{1/2}(V^{1/2} = |V|^{1/2} \operatorname{sgn} V)$ was considered. This will also be done in Section 3.

(3) We have recovered the results of references [1] and [2]. The 'dotted line' argument in [2] corresponds to our 'threshold story'.

## 2. Weak coupling limit

By 'weak coupling limit' we think of $\lambda_n$ being multiplied by a coupling constant $\varepsilon$ ($\varepsilon > 0$). We investigate the limit $\varepsilon \to 0$ and study how the trade-off between the light and heavy masses influences the spectrum. Throughout this section we consider the harmonic chain with nearest neighbour interaction (1.7). The function

$$f(p) \equiv \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \lambda_n e^{inp} \qquad p \in [-\pi, \pi] \tag{2.1}$$

will play a crucial role in the sequel. We can express $\hat{A}_\mu$ (1.5) in $\hat{\mathscr{H}}$ by means of $f$ as

$$\hat{A}_\mu(p, p') = \frac{\hat{\Phi}(p)^{1/2} f(p - p') \hat{\Phi}(p')^{1/2}}{(\mu - M^{-1}\hat{\Phi}(p))^{1/2}(\mu - M^{-1}\hat{\Phi}(p'))^{1/2}} \tag{2.2}$$

where $\hat{\Phi}(p) = J(1 - \cos p)$, $J > 0$ and $\mu > \mu_0 = 2J/M$.

The key idea in the proof of our next theorem will be the decomposition of $\hat{A}_\mu$ as

$$\hat{A}_\mu = \hat{A}_\mu^{(1)} + \hat{A}_\mu^{(2)} \tag{2.3}$$

according to a corresponding decomposition of $f$, namely

$$f(p - p') = f_1(p, p') + f_2(p, p') \tag{2.4}$$

where

$$f_1(p, p') \equiv \frac{f(p - \pi) f(\pi - p')}{f(0)}. \tag{2.5}$$

We assume for the moment that $2\pi f(0) = \Sigma\, \lambda_n \neq 0$. Suppose that

$$f' \in \Lambda_\delta, \tag{2.6}$$

where $\Lambda_\delta$ denotes the class of Lipschitz continuous functions of order $\delta(0 < \delta < 1)$. We need some information about $f_2$, in particular an upper bound. From (2.4), (2.5), and the periodicity of $f$ and $f'$ we get

$$f_2(p, \pm\pi) = f_2(\pm\pi, p') = 0 \tag{2.7}$$

$$\frac{\partial}{\partial p} f_2(p, \pm\pi) = \frac{\partial f_2}{\partial p'}(\pm\pi, p') = 0. \tag{2.8}$$

In the first quadrant of $BZ$ (where $0 \le p \le \pi$, $0 \le p' \le \pi$) we have on account of (2.4), (2.5) and (2.7)

$$\begin{aligned} f_2(p, p') &= \int_\pi^p \frac{\partial f_2}{\partial \tilde{p}}(\tilde{p}, p')\, d\tilde{p} \\ &= \int_\pi^p (f'(\tilde{p} - p') - f'(\tilde{p} - \pi))\, d\tilde{p} - \int_\pi^p f'(\tilde{p} - \pi)(f(\pi - p') \\ &\quad - f(0))f(0)^{-1}\, d\tilde{p}. \end{aligned} \tag{2.9}$$

We conclude from (2.9) and (2.6) that

$$\begin{aligned} |f_2(p, p')| &\le \text{const}\, |\pi - p|(|\pi - p'|^\delta + |\pi - p'|) \\ &\le \text{const}\, |\pi - \mathrm{p}||\pi - \mathrm{p}'|^\delta. \end{aligned}$$

The other quadrants give analogous bounds and we can write the net result in a concise form as

$$|f_2(p, p')| \le \text{const}\, (1 + \cos p)^{1/2}(1 + \cos p')^{\delta/2}. \tag{2.10}$$

Since the $\lambda_n$ are real (2.1) gives $f(-p) = \overline{f(p)}$, hence $f_2(p, p') = \overline{f_2(p', p)}$. Therefore we might also interchange $p$ and $p'$ on the right-hand side of (2.10). So far we worked under the assumption that $f(0) \neq 0$. If $f(0) = 0$ we replace $f_1(p, p')$ in (2.4), (2.5) by

$$f_1(p, p') \equiv f(p - \pi) + f(\pi - p'). \tag{2.11}$$

The estimate (2.10) still holds true.

Estimate (2.10) provides us with a bound on $\hat{A}_\mu^{(2)}(p, p')$, namely

$$\begin{aligned} |\hat{A}_\mu^{(2)}(p, p')| &\le \frac{\hat{\Phi}(p)^{1/2}|f_2(p - p')|\hat{\Phi}(p')^{1/2}}{(\mu - M^{-1}\hat{\Phi}(p))^{1/2}(\mu - M^{-1}\hat{\Phi}(p'))^{1/2}} \\ &\le \text{const}\, \frac{(1 + \cos p)^{1/2}(1 + \cos p')^{\delta/2}}{(\mu - \mu_0 + (J/M)(1 + \cos p))^{1/2}(\mu - \mu_0 + (J/M)(1 + \cos p'))^{1/2}} \\ &\le \text{const}\, (1 + \cos p')^{(\delta - 1)/2} \end{aligned} \tag{2.12}$$

uniformly for $\mu \in (\mu_0, \infty)$, $\delta \in (0, 1)$.

By symmetry

$$|\hat{A}^{(2)}(p, p')| \le \text{const}\, (1 + \cos p)^{(\delta - 1)/2}. \tag{2.13}$$

Therefore

$$\sup_p \int_{-\pi}^{\pi} |\hat{A}_\mu^{(2)}(p, p')|\, dp' = \sup_{p'} \int_{-\pi}^{\pi} |\hat{A}_\mu^{(2)}(p, p')|\, dp \le C < \infty \tag{2.14}$$

when $C$ is independent of $\mu \in (\mu_0, \infty)$.

This shows that $\hat{A}_\mu^{(2)}$ is a bounded operator and $\|\hat{A}_\mu^{(2)}\| \le C(\mu \in (\mu_0, \infty))$, for (2.14) implies that $\hat{A}_\mu^{(2)}$ is bounded from $L^1$ to $L^1$ and from $L^\infty$ to $L^\infty$, and one can apply the Riesz–Thorin interpolation theorem [5].

*Remarks.* (1) One can easily show that $A_\mu^{(2)}$ converges in operator norm to $\hat{A}_{\mu_0}^{(2)}$ as $\mu \downarrow \mu_0$. Since $\hat{A}_\mu^{(2)}(\mu > \mu_0)$ is Hilbert–Schmidt $A_{\mu_0}^{(2)}$ is compact.

(2) By a more detailed analysis one can deal with the border case $\delta = 0$, that means $f \in C^1$ only. It turns out that $\hat{A}_\mu^{(2)} \to \hat{A}_{\mu_0}^{(2)}$ strongly as $\mu \downarrow \mu_0$ and the limiting operator is still bounded (but no longer compact). (2.10) holds true with $\delta = 0$ but (2.14) is violated. The ensuing integral kernels are, as far as the singularities are concerned, of the following type

$$(Tf)(x) = \frac{1}{x}\int_0^x f(x')\,dx', \qquad (T^*f)(x) = \int_x^a \frac{f(x')}{x'}\,dx', \quad x \in [0, a], 0 < a \le \infty.$$

The singularity at $x = x' = 0$ in the kernel of $T$ is intended to mirror the singularities at $\pm\pi$ of the actual kernel $\hat{A}_{\mu_0}^{(2)}(p, p')$. $T$ is known (by a simple but somewhat tricky proof) to be bounded on any $L^p[0, a]$, $1 < p < \infty, 0 < a \le \infty$ [7, p. 229]. We are now prepared to prove

**Theorem 2.1.** *Suppose*

$$\sum_{n=-\infty}^{\infty} |n|^{1+\delta}|\lambda_n| < \infty, \qquad -\frac{1}{M} < \lambda_n < \infty, \quad \delta > 0.$$

*In the weak coupling limit that is for sufficiently small* $\varepsilon$ *(replacing* $\lambda_n$ *by* $\varepsilon\lambda_n$, $\varepsilon > 0$*) we have the alternatives*

(i) *if* $\Sigma\lambda_n \ge 0$, *there exists a unique, simple bound state,*

(ii) *if* $\Sigma\lambda_n < 0$, *there exists no bound state.*

*Proof.* The condition on the $\lambda_n$ implies $f' \in \Lambda_\delta$ (2.6). Suppose first that $(2\pi)^{-1}\Sigma\lambda_n = f(0) \ne 0$. $\hat{A}_\mu^{(1)}$ (2.5) is a rank-one projection with a positive or negative non-zero eigenvalue depending on whether $f(0) > 0$ or $f(0) < 0$. As $\mu \downarrow \mu_0$ this eigenvalue tends to $+\infty$, resp $-\infty$, while for $\mu \to \infty$ it tends to 0. $\hat{A}_\mu^{(2)}$ stays uniformly bounded as $\mu \downarrow \mu_0$ and the norm can be made arbitrarily small by choosing $\varepsilon$ small enough. Hence $\hat{A}_\mu^{(2)}$ is a small perturbation of $\hat{A}_\mu^{(1)}$. Therefore $\hat{A}_\mu$ has a simple 'large' eigenvalue which is equal to $\pm\|\hat{A}_\mu\|$. But $\|\hat{A}_\mu\|$ decreases monotonically as $\mu \to \infty$ (see Section 1), moreover, $\|\hat{A}_\mu\| \to \infty$ as $\mu \downarrow \mu_0$ and $\|\hat{A}_\mu\| \to 0$ as $\mu \to \infty$ so that $1 \in \sigma(\hat{A}_\mu)$ for a unique value $\mu \in (\mu_0, \infty)$ if and only if $f(0) > 0$. This proves (i) and (ii) of Theorem 2.1 except when $f(0) = 0$.

Suppose now that $f(0) = 0$. By (2.11) $\hat{A}_\mu^{(1)}$ is a sum of two oblique rank-one projections. The eigenvalues are given by

$$\operatorname{Re}(F, G) \pm \sqrt{(\operatorname{Re}(F, G))^2 + \|F\|^2\|G\|^2 - |(F, G)|^2}$$

where

$$G(p) = (\mu - M^{-1}\hat{\Phi}(p))^{-1/2}\hat{\Phi}(p)^{1/2}$$

$$F(p) = G(p)f(p - \pi), \quad p \in [-\pi, \pi].$$

By the Schwarz inequality one eigenvalue is always strictly positive. Since $F(0) = 0$ and $f' \in \Lambda_\delta$ we find in the limit $\mu \downarrow \mu_0$ that $(F, G) \to$ const, $\|F\| \to$ const, however $\|G\| \to \infty$.

Therefore $\hat{A}_\mu^{(1)}$ exhibits a large positive eigenvalue at $\|G\|\|F\|$ as $\mu$ approaches $\mu_0$. Again $\hat{A}_\mu^{(2)}$ is a small perturbation if $\varepsilon$ is small and one can proceed as in the first part of the proof.

*Remarks*

(1) Theorem 2.1 also holds true for $\delta = 0$.

(2) In the theory of Schrödinger operators the quantity that corresponds to $\Sigma\lambda_n$ is $\int V(x)\,dx$.

(3) One can derive an asymptotic formula for the bound state in the weak coupling limit (see [9] for the Schrödinger case). We only give the result in terms of position space quantities. $\mu$ denotes the bound state

$$(2MJ)^{-1/2}(\mu - \mu_0)^{1/2} = \varepsilon \sum_{n=-\infty}^{\infty} \lambda_n - \varepsilon^2 M \cdot \left(\sum_{n=-\infty}^{\infty} \lambda_n^2 + 2 \sum_{n,m=-\infty}^{\infty} \lambda_n\lambda_m|n-m|\right) + 0(\varepsilon^3).$$

If $\Sigma\lambda_n = 0$ the second term on the right-hand side is the leading one (it is positive by construction).

(4) If $f' \in \Lambda_\delta$ (e.g. if $\Sigma|n|^{1+\delta}|\lambda_n| < \infty$, $\delta > 0$) the number of bound states is *finite*. (Then $\hat{A}_{\mu_0}^{(2)}$ is compact and therefore has a finite number of eigenvalues bigger than 1.) We expect that the number of bound states is also finite if $\Sigma|n||\lambda_n| < \infty$ and that one can find an upper bound in terms of the $\lambda_n$ in analogy to the Schödinger case [8]. We won't do this here.

(5) Theorem 2.1 could be generalized to a large class of non nearest neighbor interactions.

## 3. Infinitely many bound states

Of course, as in the case of Schrödinger operators infinitely many bound states might well occur. As we saw in the last section the mere presence of infinitely many impurities alone is not yet sufficient for this to happen. However, if the $\lambda_n$ decay slowly enough one expects infinitely many bound states. We think that this point (and the related conjecture in [1]) is discussed exhaustively if we give an example which lies on the borderline with respect to the decay of the $\lambda_n$. Take

$$\lambda_n = \lambda_{-n} = \sigma\frac{1-(-1)^n e^{-\pi}}{1+n^2} \qquad n = 0, 1, \ldots, \sigma > 0 \tag{3.1}$$

and a nearest neighbor interaction is understood. Then (2.1) gives

$$f(p-p') = \frac{\sigma}{2} e^{-|p-p'|} \tag{3.2}$$

and this is nothing but the resolvent kernel (with the variables $p - p'$ taken mod $2\pi$ and $p, p'$ restricted to $BZ$) of

$$\sigma\left(-\frac{d^2}{dp^2} + 1\right)^{-1}. \tag{3.3}$$

So we have chosen the $\lambda_n$ in such a way that we get complete feed-back with Schrödinger operators.

We will show that the harmonic chain with masses given by (3.1) exhibits an infinite number of bound states as soon as $\sigma$ becomes sufficiently large. We list the steps through which the reader can get this result:

(1) Consider $\hat{A}_\mu$ on the invariant subspace $\mathscr{H}'$ of the even functions on $BZ$ (for the odd functions see Remark (1) below)

(2) After the sequence of unitary transformations

$$U_1 : \mathscr{H}' \to L^2[0, \pi]$$

$$(U_1 f)(p) = \sqrt{2} f(p), \quad p \in [0, \pi].$$

$$U_2 : L^2[0, \pi] \to L^2\left[0, \frac{\pi}{\delta}\right], \quad \delta^2 = \mu - \mu_0 > 0$$

$$(U_2 f)(p) = \sqrt{\delta} f(p\,\delta)$$

$$U_3 : L^2\left[0, \frac{\pi}{\delta}\right] \to L^2\left[0, \frac{\pi}{\delta}\right]$$

$$(U_3 f)(p) = f\left(\frac{\pi}{\delta} - p\right)$$

the operator $\hat{A}_\mu \restriction \mathscr{H}' = \hat{A}_{\mu_0 + \delta^2} \restriction \mathscr{H}'$ is converted into

$$\begin{aligned}(W\hat{A}_\mu W^{-1})(p, p') &= V^{1/2}(p) \frac{e^{-\delta|p-p'|}}{2\delta} V_\delta(p')^{1/2} + R_\delta(p, p') \\ &\equiv K_\delta(p, p') + R_\delta(p, p') \end{aligned} \tag{3.4}$$

where $W = U_3 U_2 U_1$
and

$$R_\delta(p, p') = \frac{1}{2\,\delta} V_\delta(p)^{1/2} e^{-\delta p} e^{-\delta p'} V_\delta(p')^{1/2} \tag{3.5}$$

and

$$V_\delta(p) \equiv \frac{\sigma J(1 + \cos p\delta)}{1 + \dfrac{J}{M} \dfrac{(1 - \cos p\delta)}{\delta^2}}, \quad p \in \left[0, \frac{\pi}{\delta}\right]. \tag{3.6}$$

The term $R_\delta(p, p')$ was picked up in the transformation $U_1$.

(3) *Set* $V_\delta(p) = 0$ for $p \notin [0, \pi/\delta]$ and interpret $V_\delta(p)$ as the potential in

$$H_\delta \equiv -\frac{d^2}{dp^2} - V_\delta(p) \tag{3.7}$$

as operator in $L^2(\mathbb{R})$. Note that for $p > 0$

$$V_\delta(p) \to V_0(p) = \frac{2J\sigma}{1 + \dfrac{J}{2M} p^2} \quad \text{as} \quad \delta \downarrow 0, \tag{3.8}$$

uniformly on compact sets and monotonically from below ($\partial V_\delta/\partial\delta \le 0$). When $p < 0$, $V_\delta(p) = V_0(p) \equiv 0$. This implies

(4) $H_\delta \to H_0$ as $\delta\downarrow 0$ in the strong resolvent sense. To see this note that $C_0^\infty(\mathbb{R})$ is a core of $H_0$ and we can apply Th. VIII.25 in [6]. Moreover, the negative eigenvalues of $H_\delta$ approach the eigenvalues of $H_0$ monotonically from above [10, p. 462].

(5)
$$\begin{aligned}\dim(\operatorname{Ran} P_{(-\infty,0)}(H_0)) &= \lim_{\delta\downarrow 0}\dim(\operatorname{Ran} P_{(-\infty,-\delta^2)}(H_0))\\ &= \lim_{\delta\downarrow 0}\dim(\operatorname{Ran} P_{(-\infty,-\delta^2)}(H_\delta)) \quad \text{(by step 4)}\\ &= \lim_{\delta\downarrow 0}\dim(\operatorname{Ran} P_{(1,\infty)}(K_\delta)) \qquad (3.9)\\ &= \infty \quad \text{if } \sigma > \sigma_0 = 1/16M \quad \text{(by [9, 12])} \qquad (3.10)\\ &= 1 \quad \text{if } \sigma \le \sigma_0 \qquad (3.11)\end{aligned}$$

Moreover, by (3.4)

$$\lim_{\delta\downarrow 0}\dim(\operatorname{Ran} P_{(1,\infty)}(\hat{A}_{\mu_0+\delta^2})) = \infty \quad \text{if } \sigma > \sigma_0, \tag{3.12}$$

whereas

$$\lim_{\delta\downarrow 0}\dim(\operatorname{Ran} P_{(1,\infty)}(\hat{A}_{\mu_0} + \delta^2)) = 1 \quad \text{if } \sigma \le \sigma_0. \tag{3.13}$$

The reader should keep in mind that one gets (3.9)–(3.11) by writing

$$K_\delta = R_\delta + (K_\delta - R_\delta) \tag{3.14}$$

and noticing that $R_\delta$ is a rank one projection, $R_\delta \ge 0$ and $\|R_\delta\| \to \infty$ as $\delta\downarrow 0$. The bracket term in (3.14) stays, however, bounded as $\delta\downarrow 0$. One also gets

$$W\hat{A}_{\mu_0+\delta^2}W^{-1} = 2R_\delta + (K_\delta - R_\delta). \tag{3.15}$$

This differs from (3.14) only by a factor 2 in front of $R_\delta$ and this does not affect the analysis. In essence $R_\delta$ is responsible for (3.11) and (3.14) whereas the behavior of $K_\delta - R_\delta$ as $\delta\downarrow 0$ is responsible for (3.10) and (3.12). One has

$$\lim_{\delta\downarrow 0}\dim(\operatorname{Ran} P_{(1,\infty)}(K_\delta - R_\delta)) = \infty \quad \text{if } \sigma > \sigma_0.$$

For more details of the method we refer to [9, 12].

(6) *Conclusion.* (3.12) and (3.13) show that the harmonic chain with masses given by (3.1) has infinitely many bound states if and only if $\sigma > \sigma_0 = 1/16M$ and a unique (simple) bound state if $\sigma \le \sigma_0$, $\sigma > 0$.

*Remarks:* (1) If we had started with odd functions we had obtained $-R_\delta$ in (3.4). This entails that the single bound state for $\sigma \le \sigma_0$ is absent but for $\sigma > \sigma_0$ we again get infinitely many bound states.

(2) The variational principle and (3.1) imply that, if $\lambda_n \ge a/n^2$ for $|n| \ge n_0 > 0$ and $a > \sigma_0$, the number of bound states is infinite.

## 4. A remark about random masses

One can ask whether some of our results hold with *probability 1* for a harmonic chain whose masses depend on a stochastic parameter. We only consider briefly Theorem 2.1. It turns out that we can weaken the assumptions of the decay of the $\lambda_n$ if the $\lambda_n$ (not the real masses) are symmetrically distributed around 0. We have opportunity to apply the theory of random series of functions [11]. Let us consider the following model

$$\tilde{\lambda}_n(w) = \varepsilon_n(w)\lambda_n \quad n = 0, \pm 1, \dots \tag{4.1}$$

where $w = (\dots, w_{-1}, w_0, w_1, \dots)$, $w_i \in \{0, 1\}$, denotes a point in the probability space $\Omega = X_{n=-\infty}^{\infty} \{0, 1\}$ and $\varepsilon_n(w)$ is a random variable obeying

$$\varepsilon_n(w) = \begin{cases} 1 & w_n = 0 \\ -1 & w_n = 1 \end{cases}. \tag{4.2}$$

That means we change the sign of the $\lambda_n$ at random. Suppose (for brevity) that

$$\lambda_n = \lambda_{-n}. \tag{4.4}$$

Define

$$s_j = \left( \sum_{2^j \le n < 2^{j+1}} n^2 \lambda_n^2 \right)^{1/2} \tag{4.5}$$

and suppose that

$$s_j = 0(2^{-\beta j}), \quad \beta > 0. \tag{4.6}$$

Then Theorem 3 in [11, p. 68] implies that with (2.1) and (2.6)

$$f' \in \Lambda_\delta \quad \text{with probability 1.} \tag{4.7}$$

But if $f' \in \Lambda_\delta$ the proof of Theorem 2.1 works and its implications hold with probability 1, in particular, the number of bound states is *finite* with prob. 1 (Remark 4 to Theorem 2.1). Notice that the $\lambda_n$ of (3.1) obey (4.6) despite the fact that this chain can have infinitely many bound states. (4.6) is, for example, implied by

$$|\lambda_n| \le \frac{c}{n^{3/2+\delta}}, \quad \delta > 0.$$

*Note*. After this work was finished, we learned that J. E. Avron in [13] discussed applications of Schrödinger techniques to other threshold problems. Our decomposition (2.4) (2.5) also appears in that paper. I am indebted to B. Simon for pointing out this reference to me.

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