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SCENARIOS FOR THE ONSET OF CHAOS

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I. INTRODUCTION

The aim of this paper is to give a simple and qualitative description of the transition to temporal chaos observed as well in experiments [1] as in dynamical systems [2,3]. Roughly speaking, we are going to present a classification of these routes according to the number of independent frequencies present just before the transition. Landau [4] proposed a scenario in which the destabilization of an infinite number of independent frequencies would lead to turbulence. Ruelle, Takens and Newhouse [5] showed that after the appearance of at most three independent frequencies in a given physical system, there exists arbitrary small deterministic perturbations which could give rise to chaotic behaviors, described by some stable solutions of ordinary differential equations : the so called "strange attractors". The picture which emerges from this result seems to be compatible with experimental results and numerical simulations [1]. This does not mean that the transition to chaos consists in a scenario where three frequencies necessarily appear. At the contrary, the transition has been many times observed after periodic and biperiodic states.

We are now going to describe the one and two-frequencies scenarios with some details. The three frequencies cases will be discussed in the conclusion.

II. SCENARIOS WITH ONE FREQUENCY : ABRUPT TRANSITION TO CHAOS, INTERMITTENT TRANSITION, THE CASCADES

Hopf bifurcation [6] is the natural mechanism which leads to the appearance of a frequency in a non-equilibrium system. The question we want to ask here is what happens when the resulting periodic state looses its stability. Floquet theory [6] states that a periodic motion is linearly stable if its Floquet multipliers have a modulus less than one. Let us consider an infinitesimally small perturbation $\vec{y}(t)$ of a periodic solution $\vec{\Phi}(t)$ of period T of an n^{th} order ordinary differential equation ($\vec{y} = y_1 \dots y_n$), $\vec{\Phi} = (\Phi_1 \dots \Phi_n)$. The eigenvalues of the nxn matrix M defined by

$$\vec{y}\,(t+T) = M\,\vec{y}(t)$$

are the Floquet multipliers associated with the solution $\vec{\Phi}(t)$. They give us the contracting and dilating factors in the eigendirections of M which multiply a small perturbation to $\vec{\Phi}$ after one period T. There are essentially three ways for a periodic state to loose its stability when only one external parameter is varied.

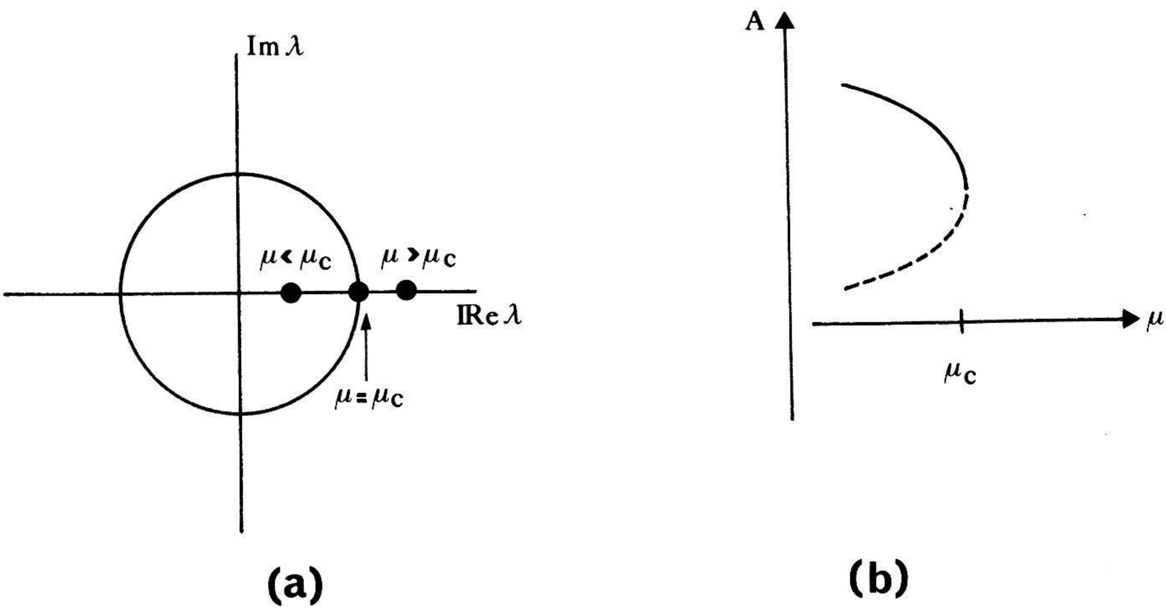

Fig. 1a represent the positions of the Floquet multiplier in the complex plane for different values of an external parameter

Fig. 1b represent the corresponding bifurcation diagrams . Crossing through +1 corresponds generically to the collision between two periodic orbits followed by their disappearance. The stable solutions will always be represented by full lines, the unstable ones by dashed lines.

1. A Floquet multipliers crosses the unit circle in the complex plane through + 1 (Fig. 1a).
2. A complex-conjugate couple of Floquet multipliers crosses the unit circle (Fig. 2a)
3. A Floquet multiplier crosses the unit circle through -1 (Fig. 3a).

In case 1, elementary bifurcation theory [6] tells us that the periodic orbit generically disappears.

Case 2 leads generally to the appearance of a quasiperiodic behavior with two independent frequencies.

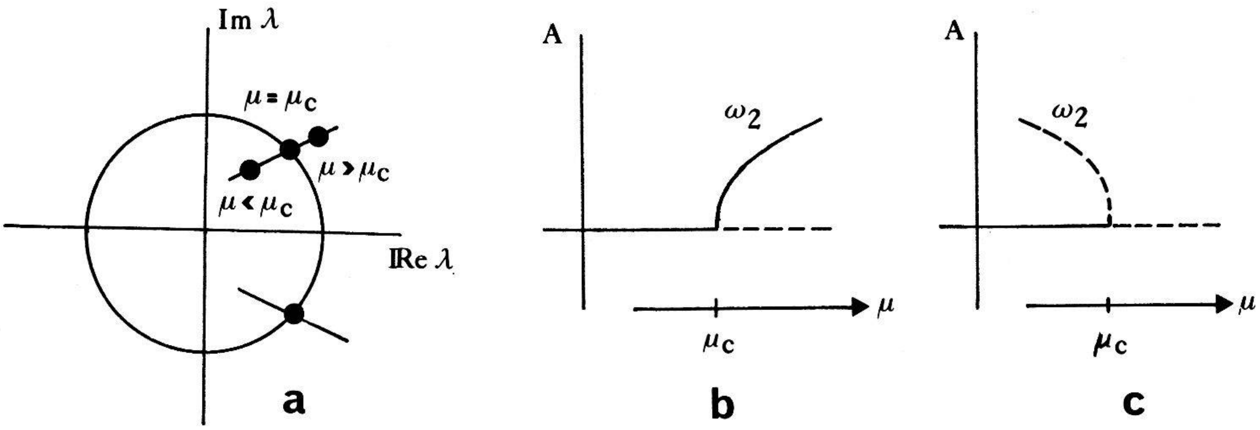

Fig. 2b corresponds to the supercritical situation. In this case, the nonlinear terms saturate the instability and give rise to a supercritical stable solution with two independent frequencies.

Fig. 2c corresponds to the subcritical case. Here, the nonlinear terms have opposite effects : they act to give a quasiperiodic solution before the bifurcation point, which is unstable.

Case 3 gives rise to a periodic solution with double period.

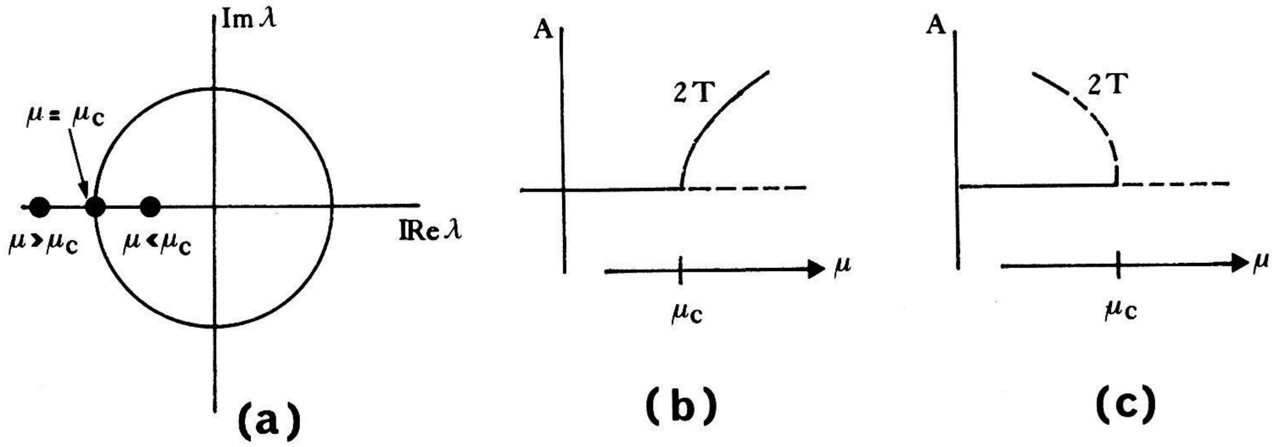

Fig. 3 (b) and (c) correspond respectively to the supercritical and subcritical cases. A represents the amplitude of the subharmonic with half frequency.

Let us consider now each case in the context of the transition to chaos.

1- A Floquet multiplier crosses the unit circle through +1

In the generic case the disappearence of the periodic orbit can lead to any kind of behavior. If it is chaotic we expect an abrupt transition to chaos with hysteresis effects. Actually although this transition exists, another scenario has also been observed : the intermittent route to chaos.[7] This scenario can be described as follows. Although the periodic orbit disappears, just after the crossing through +1 of its Floquet multiplier, we still observe on the signal some periodic part separated by chaotic bursts. This phenomenon is reminiscent of the intermittency effects observed in fully devellopped turbulence.[8] As we increase a control parameter μ , the duration of the periodic parts of the signal decreases. The transition is a smooth transition described by the inverse of the mean time τ of the periodic part of the signal

$$\tau^{-1} \sim (\mu - \mu_c)^{1/2} \qquad (\tau^{-1}=0 \text{ for } \mu<\mu_c)$$

The explanation of this scenario has been given by Manneville and Pomeau [9] . The basic idea is that a conflict occurs between a chaotic attractor and the periodic orbit when this orbit still exists and is stable. This competition is won by the periodic orbit, and the chaotic set looses its stability until the disappearence of the periodic state

With an additional control parameter we can go continuously from the abrupt transition to the intermittent one [10]

In the case of a symmetry (in contrast to the generic case) for the mode of destabilization through +1, the bifurcation leads to two periodic orbits with the same frequency which exchange under the symmetry. If the bifurcation is subcritical, we have again either an abrupt transition with hysteresis or an intermittent one. In the supercritical case, we can ask the question of a next crossing through +1 for the newly-born orbits. Since they are no more symmetric we come back to the generic case previously discussed. A possible scenario appears if in-between the symmetry has been restored by a different mechanism.

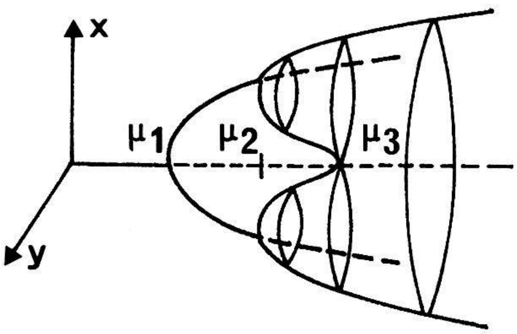

Fig. 4 Illustration of the spontaneously broken, via bifurcation, and restored, via homoclinic bifurcation, mechanism. The reflection is here $x \to -x$. For $\mu > \mu_1$ a pair of non-symmetric stable equilibrium solutions appear which undergo a Hopf bifurcation for $\mu = \mu_2$. For $\mu > \mu_3$ we observe a homoclinic bifurcation which gives rise to a unique symmetric periodic solution.

If such mechanism occurs, we can imagine a cascade of spontaneously broken symmetry through a bifurcation followed by a resymmetrization, via a homoclinic bifurcation, which could lead to chaos. It turns out that this scenario, proposed on the basis of simple dynamical systems [11] has been observed in ordinary differential systems [11] and in partial-derivative equations [12] as well, and is likely to occur in real systems when the needed symmetry is present. This scenario is reminiscent of the period-doubling one. Renormalization-group arguments can be used to predict and compute numerically various exponents describing the transition. The analysis follows closely those of the cascade of period-doubling discussed below.

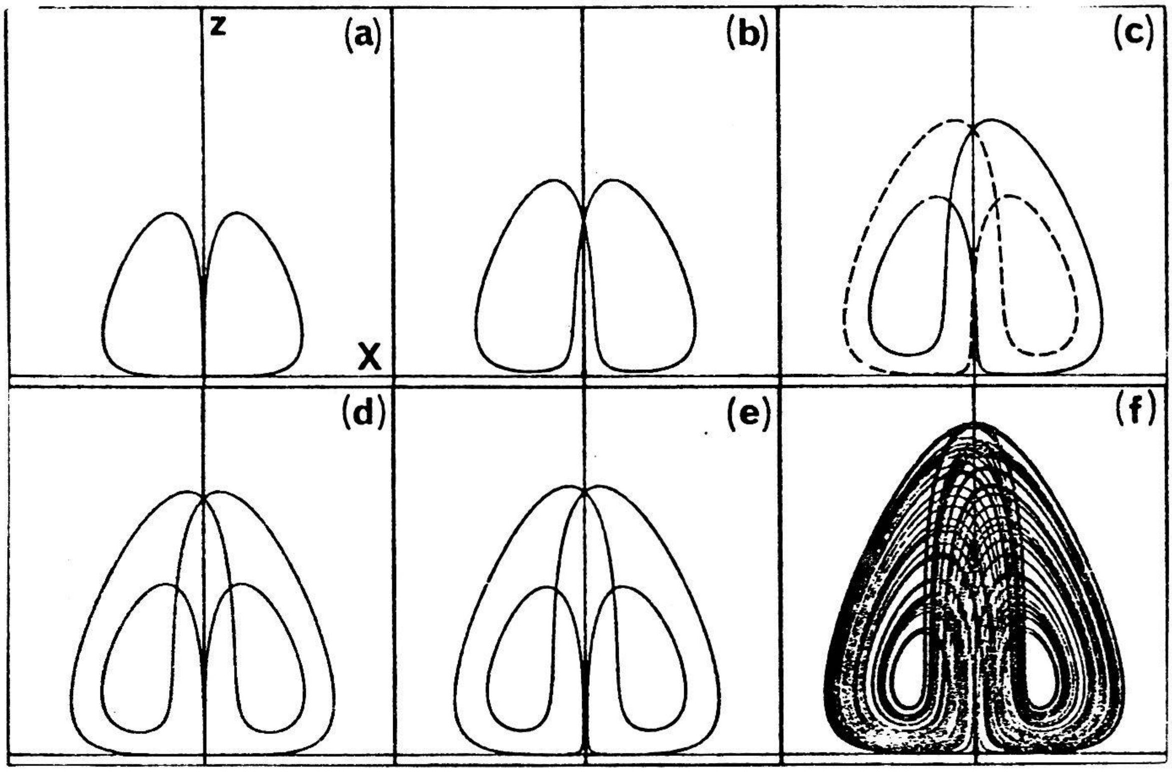

Fig. 5

A numerical investigation of the system of differential equations

$$\dot{x} = \alpha(x - y)$$
$$\dot{y} = -4\alpha y + xz + \mu x^3$$
$$\dot{z} = -\delta\alpha z + xy + \beta z^2$$

with $\alpha = 1.8$, $\beta = -0.07$, $\delta = 1.5$ yields

(a) a pair of stable homoclinic orbits ($\mu = .0760710$)

(b) a stable symmetric orbit ($\mu = .05$)

(c) a pair of stable orbits ($\mu = 0.034$)

(d) a pair of stable homoclinic orbits ($\mu =.0321825$)

(e) a stable symmetric periodic orbit ($\mu =.0321$)

(f) chaotic attractor ($\mu = .02$)

2- A complex pair of Floquet multipliers crosses the unit circle

In the supercritical situation we observe generally the appearance of a quasiperiodic behavior with two independent frequencies which will be discussed in the next section. In the subcritical case, we can have either an abrupt or intermittent transition to chaos.

3. A Floquet multiplier crosses the unit circle through -1

The subcritical case corresponds again to the possibility of an abrupt or intermittent transition [13]. In the supercritical case, we can imagine a scenario of repeated period-doubling bifurcations. This scenario exists and is probably one of the most famous routes to weak turbulence [14,15]. The cascade of period-doubling bifurcation or the parametric cascade arises in simple one-dimensional non-invertible maps, in ordinary differential equations, partial-derivative equations, and finally in real systems. This cascade has been successfully described by renormalization-group methods [17,18,19,20,21].

The main consequence of this analysis is the existence of universal exponents describing for example the "speed of bifurcations" or the low-frequency behavior of the Fourier spectrum at the onset of chaotic behavior .

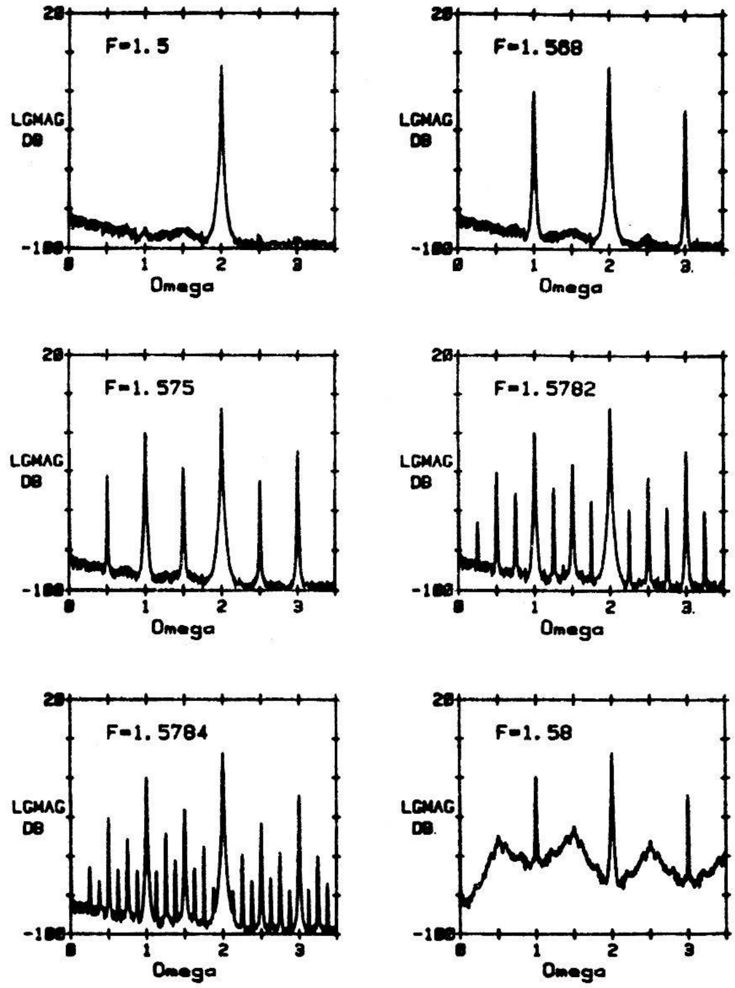

Illustration of the period-doubling cascade on the parametrically pumped pendulum [22,23]
The equations of the system are :

$$\ddot{\theta} + \nu\dot{\theta} + \omega_0^2(1+F\cos\Omega t)\sin\theta = 0$$

We have choosen for the numerical simulation

$\Omega = \omega_0 = 2$, $\nu = .18\ s^{-1}$

and showed on Fig.6(a,b,c,d,e,f) the power spectrum for respectively the frequencies 2, 1, 1/2, 1/4, 1/8 and a chaotic state.

Fig. 6

This closes our section on the scenarios to chaos after one frequency. Other one-frequency scenarios could emerge from numerical simulation or experiments but our feeling is that the ones described above are, with an exception for the spontaneously broken-and-restored-symmetry cascade, the most probable.

III. TWO FREQUENCY SCENARIOS

When a periodic orbit looses its stability with a couple of complex Floquet multipliers crossing the unit circle in the supercritical case, a quasiperiodic behavior with two independent frequencies generally appears [6].

In the phase space a quasiperiodic behavior is represented as an endless motion on a torus. This invariant surface is materialized by the winding ergodic trajectory. The existence of quasiperiodic motion is a difficult mathematical problem. In a more physical language we are considering the problem of the interaction between two oscillators. The natural tendency of such an interaction is towards the mode-locking : i.e. one oscillator is always trying to slave or lock-in the other. When they succeed to do so, the trajectory is no more winding an infinite number of times around the torus. Indeed, we get a periodic solution. Varying one external parameter one can succeed to unlock the two oscillators until we get a new locked state. The picture which emerges from this is complicated. We expect locked states with quasiperiodic states in between, the locking occuring everytime the ratio of the two frequencies becomes rational.

This apparently does not leave a lot of chance to observe quasiperiodic states. Indeed, such a chance exists and relies on a deep mathematical result [25]. Nevertheless, the probability of finding two independent frequencies decreases when we increase the coupling between the oscillators. Let us first discuss the case where the transition to chaos occurs during a locked state. The problem is then reduced to that of the transition after one frequency previously observed [24]. In particular, cascades of period-doubling have been frequently observed in this case. A non-trivial transition occurs if we are able to keep fixed the ratio of the two frequencies to an irrational value. We then get a smooth transition to chaos again characterized by critical indices, where the invariant torus looses its regularity and becomes a "fractal". The signature of this irregularity can be seen in the signal as an irregularity of the phase. The self-similarity property of the solution at the transition, involves universal numbers depending on the irrational ratio of the frequencies. A renormalization group again describes this transition [26,27].

Up to now this transition has never been experimentally observed. This is essentially due to the difficulty to keep fixed the ratio ω_1/ω_2. The only reasonable way to see the prediction of this theory would consist to consider a self-oscillatory system on which a frequency would be externally imposed and varied continuously as we vary some other control parameter in order to keep the ratio ω_1/Ω fixed. To close this section of the transition to chaos after two frequencies, let us mention the possibility of bifurcation for a quasiperiodic solution [28]. More precisely, when the frequencies are sufficiently independent we can consider them as uncoupled, and then imagine for such a quasiperiodic solution all the bifurcations of periodic solutions. In particular intermittent transition after the disappearance of the invariant torus has been experimentally observed [14]. Cascade of periodic-doubling for quasiperiodic solutions can also be imagined. Numerical simulations on systems [29] displaying such a cascade suggests that such a scenario is rather unstable against the tendency to lock-in and undergoes a transition to chaos after a locked state. Our feeling is that the transition after a quasiperiodic behavior is very rich and also, except when it reduces to the one-frequency scenario via locked modes, probably difficult to observe quantitatively in real experiments.

IV. CONCLUSION

We could be tempted to discuss now the three-frequency scenarios. We can imagine successive lockings in two-frequency regimes and a transition after such states. We probably can also describe a transition where the three-dimensional torus looses progressively its regularity with critical indices. Bifurcation could lead to intermittencies, first-order transitions and even period-doubling scenarios. In some of the cases alluded to, the three-dimensional torus is not destroyed, for chaotic behavior needs only a three-dimensional space to set in. This is one of the ingredients of the Ruelle-Takens-Newhouse result on the existence of strange attractors near (in parameter space) a three-frequency behavior.

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