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Autor: Antoine, J.-P. / Epifanio, G. / Trapani, C.

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Complete sets of unbounded observables

By J.-P. Antoine, Institut de Physique Théorique, Université Catholique de Louvain, B-1348–Louvain-la-Neuve (Belgium)
and G. Epifanio and C. Trapani, Istituto di Fisica dell' Università di Palermo, I-90123–Palermo (Italy)

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Abstract. The concept of complete set of commuting observables is formulated in algebraic terms, using the theory of V^*-algebras. These are a particular class of algebras of unbounded operators, and in many respects the analog of von Neumann algebras. We show that a complete parallelism exists between this approach and the familiar one, based on von Neumann algebras.

1. Introduction

Ever since the publication of Dirac's classical book [1], his elegant formulation of Quantum Mechanics has been almost universally adopted by physicists. In particular his concept of "complete set of commuting operators" (CSCO) has become standard. However Dirac's formulation is quite unsatisfactory from the mathematical point of view (and he was himself fully aware of this). He often translates automatically into an infinite dimensional Hilbert space propositions which are valid only in a finite dimensional one. In a nutshell, he simply ignores the difficulties created by unbounded operators and continuous spectra (for an elaboration of this point, see e.g. the careful analysis of Jauch [2]).

Of course the solution here is almost as old as the problem, since it goes back to the work of J. von Neumann. Observables are represented by self-adjoint operators in the Hilbert space of the system and, by the spectral theorem, each of these is fully equivalent to the set of its spectral projections. Thus any set $\mathfrak{O}$ of observables may be replaced by a von Neumann algebra of *bounded* operators, namely the smallest one containing all spectral projections of all elements of $\mathfrak{O}$, that is, the bicommutant $\mathfrak{O}''$. If $\mathfrak{O}$ consists of one operator only, or a family $\{A_1, A_2, \ldots\}$ of operators that commute with each other (in the strong sense, that is, all their spectral projections commute), then the corresponding von Neumann algebra $\mathfrak{A} = \mathfrak{O}''$ is *abelian*, i.e. $\mathfrak{A} \subseteq \mathfrak{A}'$. Then a CSCO may be defined as a family of commuting observables that generates a *maximal abelian* von Neumann algebra, $\mathfrak{A} = \mathfrak{A}'$. Using this concept, or the equivalent one of cyclic vector [3], the whole formalism may be developed, using the so-called spectral representation discussed in full detail by Jauch and Misra [4] (see also [20]).

Thus only bounded operators are considered in that approach, namely the spectral projections or, more generally, bounded functions of the observables.

However, there are very few instances where spectral projections can be obtained explicitly, as concrete operators, and anyway the observables themselves are usually much simpler. Think for instance of position or momentum operators! In fact there is more here than a matter of convenience, for quite often observables like those are imposed by the invariance properties of the system, through Noether's theorem (see for instance the discussion of Wightman [5]). Indeed, the invariance of a system under a Lie group G of symmetries is described, according to the Wigner–Bargmann theorems, by a continuous (projective) unitary representation of G in the Hilbert space $\mathscr{H}$ of the system. Then, by Stone's theorem and its generalizations, the (conserved) generators of G are represented in $\mathscr{H}$ by a Lie algebra of self-adjoint operators. These correspond to observables, and they cannot be all bounded, as shown by Doebner and Melsheimer [6]. Therefore it is useful to reformulate the spectral representation corresponding to a CSCO $\{A_1, A_2, \ldots, A_n\}$ directly in terms of these operators. This was done by Prugovečki [7]. He showed, in particular, that completeness of $\{A_1 \cdots A_n\}$ is equivalent to the existence of a cyclic vector belonging to the (dense) domain of all powers of $A_1, \ldots, A_n$, i.e. the largest domain invariant under $A_1, \ldots, A_n$.

Now a natural question arises. Is it possible to give an *algebraic* formulation of Prugovečki's results, thus generalizing to unbounded operators the elegant approach based on maximal abelian von Neumann algebras? The aim of the present paper is to give a positive answer to that question, using the notion of V^*-algebra developed by two of us in [8].

One feature that distinguishes unbounded operators from bounded ones is the importance of the domain of definition. Actually many mathematical difficulties disappear if one considers only a family of operators which have, together with their adjoints, a common, dense, invariant domain $\mathscr{D}$. Then they belong to an algebra, called $C_{\mathscr{D}}$ or $L^+(\mathscr{D})$, and, even if they are not norm-continuous, they are continuous for a suitable weak topology. In particular, a family of (strongly) commuting self-adjoint operators always has such a domain, namely the one considered by Prugovečki. This applies, in particular, to the elements of a CSCO for a given quantum mechanical system. Our discussion below is organized as follows. In Section 2 we study the abelian algebra of unbounded operators generated, on their common invariant domain, by a family of commuting self-adjoint operators. In Section 3 we give several characterizations for the completeness of such a family, including the existence of a cyclic vector. At this stage we have obtained the algebraic description of a CSCO we were looking for. Then in the last section we briefly comment on the relationship between this work and the so-called rigged Hilbert space approach to Quantum Mechanics, based on the concept of labeled observables. Finally in the Appendix we collect some technical facts about abelian algebras of unbounded operators.

For convenience we will recall first some basic definitions concerning algebras of unbounded operators and V^*-algebras, referring to [8–11] for more details. Let $\mathscr{D}$ be a prehilbert space, $\mathscr{H}$ its completion. We denote by $C(\mathscr{D}, \mathscr{H})$ the set of all closable operators A in $\mathscr{H}$ such that $\mathscr{D} \subseteq \mathscr{D}(A) \cap \mathscr{D}(A^*)$, where $\mathscr{D}(A)$ denotes the domain of A and A^* its adjoint. We call $C_{\mathscr{D}}$ $(\equiv L^+(\mathscr{D}))$ the *-algebra of all operators $A \in C(\mathscr{D}, \mathscr{H})$ such that both A and A^* map $\mathscr{D}$ into $\mathscr{D}$. Equivalently $C_{\mathscr{D}}$ consists of all operators from $\mathscr{D}$ into $\mathscr{D}$, continuous for the weak topology $\sigma(\mathscr{D}, \mathscr{D})$. The involution in $C_{\mathscr{D}}$ is defined by $A \mapsto A^+ \equiv A^* \restriction \mathscr{D}$. An *$Op^*$-algebra* $\mathfrak{A}$ on $\mathscr{D}$ is a *-subalgebra of $C_{\mathscr{D}}$, containing the identity. Its *bounded part* is $\mathfrak{A}_b \equiv \mathfrak{A} \cap \mathfrak{B}(\mathscr{H})$.

An Op*-algebra $\mathfrak{A}$ is said to be:

(i) *closed* if $\mathscr{D}=\tilde{\mathscr{D}}(\mathfrak{A})\equiv\bigcap_{A\in\mathfrak{A}}\mathscr{D}(\bar{A})$
(ii) *self-adjoint* if $\mathscr{D}=\tilde{\mathscr{D}}(\mathfrak{A})=\mathscr{D}^*(\mathfrak{A})\equiv\bigcap_{A\in\mathfrak{A}}\mathscr{D}(A^*)$
(iii) *standard* if each symmetric element $A=A^+$ of $\mathfrak{A}$ is essentially self-adjoint on $\mathscr{D}$, or equivalently if $\overline{A^+}=A^*$, $\forall A\in\mathfrak{A}$
(iv) *symmetric* if, for every $A\in\mathfrak{A}$, $(1+A^+A)^{-1}\in\mathfrak{A}_b$.

As in [8] we use the following notion of (weak, unbounded) commutant, for any *-invariant subset $\mathfrak{M}$ of $C(\mathscr{D},\mathscr{H})$:

$$\mathfrak{M}'_\sigma=\{X\in C(\mathscr{D},\mathscr{H})\mid(Xf,A^*g)=(Af,X^*g),\forall A\in\mathfrak{M},\forall f,g\in\mathscr{D}\}.$$

The commutants of higher order are defined as $\mathfrak{M}''_{\sigma\sigma}\equiv(\mathfrak{M}'_\sigma)'_\sigma$, etc. We will use also the following bounded commutants: $\mathfrak{M}'_w\equiv(\mathfrak{M}'_\sigma)_b=\mathfrak{M}'_\sigma\cap\mathfrak{B}(\mathscr{H})$ and $\mathfrak{M}'_s=\mathfrak{M}'_w\cap C_{\mathscr{D}}$. An Op*-algebra $\mathfrak{A}$ on $\mathscr{D}$ is said to be *regular* (resp. *completely regular*) if $\mathfrak{A}''_{\sigma\sigma}\subseteq C_{\mathscr{D}}$ (resp. $\mathfrak{A}''_{w\sigma}\subseteq C_{\mathscr{D}}$); in this case $\mathfrak{A}''_{\sigma\sigma}$ (resp. $\mathfrak{A}''_{w\sigma}$) is also an Op*-algebra on $\mathscr{D}$. We say that an Op*-algebra $\mathfrak{A}$ is a *V*-algebra* (resp. *SV*-algebra*) if $\mathfrak{A}=\mathfrak{A}''_{\sigma\sigma}$ (resp. $\mathfrak{A}=\mathfrak{A}''_{w\sigma}$).

2. V*-algebras generated by sets of commuting self-adjoint operators

In this section, we consider a set of self-adjoint operators $A_1\cdots A_n$, strongly commuting in the sense that all their spectral projections commute. Then, as shown below, these operators have a common dense invariant domain (already considered by Prugovečki [7]) and, on that domain, they generate an abelian, self-adjoint, standard Op*-algebra.

The first statement follows from Stone's theorem. Indeed under the assumptions made, $(t_1\cdots t_n)\mapsto\exp i(t_1A_1+\cdots+t_nA_n)$ is a strongly continuous unitary representation of $\mathbb{R}^n$ into $\mathscr{H}$ (see e.g. [12], Thm. VIII.12). For convenience we collect the relevant facts in the following proposition and give the easy proof explicitly.

Proposition 2.1. *Let $A_1,A_2,\ldots,A_n$ be strongly commuting self-adjoint operators. Let $\mathscr{D}^\infty(A_1\cdots A_n)=\bigcap_{i=1}^n\mathscr{D}^\infty(A_i)$, where $\mathscr{D}^\infty(A_i)=\bigcap_{k\geq1}\mathscr{D}(A_i^k)$. Then we have:*

(i) *$\mathscr{D}^\infty(A_1\cdots A_n)$ contains a dense set of jointly analytic vectors for $A_1\cdots A_n$ (and thus it is dense in $\mathscr{H}$)*
(ii) *$A_i\mathscr{D}^\infty(A_1\cdots A_n)\subseteq\mathscr{D}^\infty(A_1\cdots A_n)$*
(iii) *Each A_i is essentially self-adjoint on $\mathscr{D}^\infty(A_1\cdots A_n)$*
(iv) *The operator $K=\sum_{i=1}^nA_i^2\restriction\mathscr{D}^\infty(A_1\cdots A_n)$ is also essentially self-adjoint and $\mathscr{D}^\infty(A_1\cdots A_n)=\mathscr{D}^\infty(\bar{K})$.*

Proof. (i) Let $\mathscr{D}^\omega$ be the span of all vectors $E_1(\Delta_1)\cdots E_n(\Delta_n)f$, where $f\in\mathscr{H}$, $E_i(\cdot)$ is the spectral measure of A_i, and $\Delta_1,\ldots,\Delta_n$ are bounded Borel subsets of $\mathbb{R}$. First we prove that $\mathscr{D}^\omega$ is dense, assuming for simplicity $n=2$. Were it not so, there would be a nonzero vector $g\in\mathscr{H}$ such that $(E_1(\Delta_1)E_2(\Delta_2)f,g)=0$ for all $f\in\mathscr{H}$ and all Δ_1,Δ_2. But this means that, for any Δ_1, $E_1(\Delta_1)g$ is orthogonal to all vectors $E_2(\Delta_2)f$, which are dense in $\mathscr{H}$ since A_2 is self-adjoint. Hence $E_1(\Delta_1)g=0$ for all Δ_1 and therefore $g=0$. Let us now prove that each vector in $\mathscr{D}^\omega$ is analytic for $A_1 \cdots A_n$. In fact

$$\sum_{k=0}^{\infty} \|A_i^k E_1(\Delta_1) \cdots E_n(\Delta_n) f\| \frac{t^k}{k!} = \sum_{k=0}^{\infty} \|A_i^k E_i(\Delta_i) \cdot E_1(\Delta_1) \cdots E_n(\Delta_n) f\| \frac{t^k}{k!} < \infty$$

Since $A_i^k E_i(\Delta_i)$ is bounded, all vectors in $\mathscr{H}$ are analytic for it. As a consequence $\mathscr{D}^\omega \subseteq \mathscr{D}^\infty(A_1 \cdots A_n)$, and the latter is dense.

(ii) We confine again ourselves to the case $n = 2$.

Because A_1 and A_2 commute there exists a self-adjoint (bounded) operator X such that $A_1 = F_1(X)$ and $A_2 = F_2(X)$. Let $\{E(\lambda)\}$ be the spectral family of X. We have:

$$\mathscr{D}^\infty(A_i) = \left\{ f \in \mathscr{H} \,\middle|\, \int_{-\infty}^{+\infty} F_i^{2k}(\lambda)\, d(E(\lambda)f, f) < \infty, \forall k \in \mathbb{N} \right\}.$$

Suppose $f \in \mathscr{D}^\infty(A_1) \cap \mathscr{D}^\infty(A_2)$, we will prove that $A_1 f \in \mathscr{D}^\infty(A_1) \cap \mathscr{D}^\infty(A_2)$. We have in fact:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} F_2^{2k}(\lambda)\, d(E(\lambda)A_1 f, A_1 f) \\
&= \int_{-\infty}^{+\infty} F_2^{2k}(\lambda)\, d(A_1^2 f, E(\lambda) f) \\
&= \int_{-\infty}^{+\infty} F_2^{2k}(\lambda)\, d\int_{-\infty}^{+\infty} F_1^2(\mu)\, d(E(\mu)f, E(\lambda)f) \\
&= \int_{-\infty}^{+\infty} F_2^{2k}(\lambda)\, d\int_{-\infty}^{+\infty} F_1^2(\mu)\chi_{(-\infty,\lambda]}(\mu)\, d(E(\mu)f, f) \\
&= \int_{-\infty}^{+\infty} F_2^{2k}(\lambda) F_1^2(\lambda)\, d(E(\lambda)f, f) \\
&\leqslant \frac{1}{2}\left\{ \int_{-\infty}^{+\infty} F_2^{4k}(\lambda)\, d(E(\lambda)f, f) + \int_{-\infty}^{+\infty} F_1^4(\lambda)\, d(E(\lambda)f, f) \right\} < \infty.
\end{aligned}$$

(In these calculations we take into account that $F_1^2(\lambda)$ is the Radon–Nikodym derivative of $(E(\lambda)A_1 f, A_1 f)$ with respect to $(E(\lambda)f, f)$.)

(iii) Follows from [12] §X.6, Coroll. 2

(iv) Let $K = \sum_{i=1}^n A_i^2 \upharpoonright \mathscr{D}^\infty(A_1 \cdots A_n)$. The operator K leaves $\mathscr{D}^\infty(A_1 \cdots A_n)$ invariant. Moreover each element of $\mathscr{D}^\omega$ is an analytic vector for K, because $K^k E_1(\Delta_1) \cdots E_n(\Delta_n)$ is a bounded operator $\forall k \in \mathbb{N}$. Hence K is essentially self-adjoint and $\bar{K} = \sum_{i=1}^n A_i^2$, as can be shown easily. Now since K leaves $\mathscr{D}^\infty(A_1 \cdots A_n)$ invariant, we get

$$\mathscr{D}^\infty(A_1 \cdots A_n) \subseteq \mathscr{D}^\infty(\bar{K}).$$

We will prove the converse inclusion. Assume $f \notin \mathscr{D}^\infty(A_1 \cdots A_n)$, then there exist $i, k \in \mathbb{N}$ such that $f \notin \mathscr{D}(A_i^k)$. Therefore $f \notin \mathscr{D}(A_i^{2k})$. This implies evidently that $f \notin \mathscr{D}(\bar{K}^k)$. Hence $f \notin \mathscr{D}^\infty(\bar{K})$. □

Proposition 2.2. *Let $\mathfrak{A}$ be the abelian *-algebra generated by the restrictions of* *the A_i's to $\mathscr{D}^\infty(A_1 \cdots A_n)$. We have:*

(i) *$\mathfrak{A}$ is a self-adjoint Op*-algebra on $\mathscr{D}^\infty(A_1 \cdots A_n)$*
(ii) *$\mathfrak{A}$ is standard i.e. each symmetric element of $\mathfrak{A}$ is essentially self-adjoint.*

Proof. Both statements follow from $\mathscr{D}^\infty(A_1 \cdots A_n) = \mathscr{D}^\infty(\bar{K})$ and Proposition A.1 in the Appendix. □

Corollary 2.3. *$\mathfrak{A}$ is a completely regular Op*-algebra on $\mathscr{D}^\infty(A_i \cdots A_n)$ and $\mathfrak{B} \equiv \mathfrak{A}''_{\sigma\sigma}$ is a SV*-algebra.*

The corollary follows from Proposition A.2. In the sequel we will call $\mathfrak{B} \equiv \mathfrak{A}''_{\sigma\sigma}$ the *canonical SV*-algebra generated by* $A_1 \cdots A_n$. This algebra is a very natural object, for it contains all "reasonable" functions of $A_1 \cdots A_n$, namely all "reasonable" operators $u(A_1 \cdots A_n)$ which have $\mathscr{D}^\infty(A_1 \cdots A_n)$ in their domain (and automatically leave it invariant). The precise mathematical statement is given in the Appendix and also in Proposition 5.2 and 6.2 of [8]. We shall come back to this point in Section 3 below.

Proposition 2.4. *Let $\{A_1 \cdots A_n\}$ be a set of commuting self-adjoint operators. The von Neumann algebra associated to it coincides with the bounded part of the V*-algebra $\mathfrak{B}$ generated by $A_1 \cdots A_n$ on $\mathscr{D}^\infty(A_1 \cdots A_n)$ and it is dense in $\mathfrak{B}$ with respect to the $\mathscr{D}$-strong topology of $C(\mathscr{D}, \mathscr{H})$, defined by the set of seminorms $X \mapsto \|Xf\|$, $f \in \mathscr{D}$.*

Proof. First we have $\mathfrak{A}'_w = \mathfrak{A}'_s = \mathfrak{A}'$, where $\mathfrak{A}'$ denotes the usual (bounded). commutant. The first equality results from the self-adjointness of $\mathfrak{A}$, whereas the second one follows directly from the definition, in the case of a closed Op*-algebra.

We have to prove that $(\mathfrak{A}''_{\sigma\sigma})_b = \mathfrak{A}''_{\sigma w} = \mathfrak{A}''$. Since $\mathfrak{A}$ is standard, both $\mathfrak{A}'_\sigma$ and $\mathfrak{A}''_{\sigma\sigma}$ are symmetric Op*-algebras (see Proposition A.1). Then $\mathfrak{A}''_{\sigma\sigma} = \mathfrak{A}''_{w\sigma}$ and therefore $\mathfrak{A}''_{\sigma w} = \mathfrak{A}''_{ww} \equiv \mathfrak{A}''$. The density of the von Neumann algebra follows from Proposition A.4. □

3. Complete sets of commuting observables

So far we know that any set $(A_1 \cdots A_n)$ of commuting self-adjoint operators generates on its invariant domain $\mathscr{D}^\infty(A_1 \cdots A_n)$ a canonical V*-algebra $\mathfrak{B}$. As stated in the introduction we want to characterize the completeness of $(A_i \cdots A_n)$ directly in terms of $\mathfrak{B}$. We simply follow the familiar pattern [3][4].

Definition 3.1. Let $\mathfrak{W}$ be an abelian V*-algebra on $\mathscr{D}$. We say that $\mathfrak{W}$ is a *maximal abelian* if $\mathfrak{W} = \mathfrak{W}'_\sigma$.

Proposition 3.2. *Let $\mathfrak{W}$ be a closed, standard abelian V*-algebra. $\mathfrak{W}$ is a maximal abelian V*-algebra if, and only if, its bounded part is a maximal abelian von Neumann algebra.*

Proof. If $\mathfrak{W} = \mathfrak{W}'_\sigma$, then $\mathfrak{W}_b = \mathfrak{W}'_w$. Since $\mathfrak{W}$ is a closed standard abelian V*-algebra, it is symmetric and the same holds true for $\mathfrak{W}'_\sigma$. Thus $\mathfrak{W}'_w = (\mathfrak{W}_b)'_w$.

Then $\mathfrak{W}_b = (\mathfrak{W}_b)'_w$. Conversely let $\mathfrak{W}_b = (\mathfrak{W}_b)'_w$. Because $\mathfrak{W}$ is symmetric, we have $(\mathfrak{W}_b)'_\sigma = \mathfrak{W}'_\sigma$ and $(\mathfrak{W}_b)'_w = \mathfrak{W}'_w$. Since $\mathfrak{W}'_\sigma$ is also symmetric, $\mathfrak{W}''_{w\sigma} = \mathfrak{W}''_{\sigma\sigma} = \mathfrak{W}$. Therefore $\mathfrak{W} = \mathfrak{W}'_\sigma$. □

Comparing now Propositions 2.4 and 3.2, we see that the two approaches, using either bounded or unbounded operators, are fully equivalent. Indeed:

Corollary 3.3. *Let $\{A_1 \cdots A_n\}$ be a set of commuting self-adjoint operators, $\mathfrak{A}$ the Op*-algebra generated by $A_1 \cdots A_n$ on $\mathcal{D}^\infty(A_1 \cdots A_n)$ and $\mathfrak{B}$ the V*-algebra generated by them on $\mathcal{D}^\infty(A_1 \cdots A_n)$, i.e. $\mathfrak{B} = \mathfrak{A}''_{\sigma\sigma}$ on $\mathcal{D}^\infty(A_1 \cdots A_n)$. Then the following conditions are equivalent:*

(i) $\mathfrak{B} = \mathfrak{B}'_\sigma$
(ii) $\mathfrak{A}' = \mathfrak{A}''$ *(where $\mathfrak{A}'$ and $\mathfrak{A}''$ are the commutants in the sense of von Neumann algebras.* □

This result now yields the natural definition of CSCO implicit in Dirac's words: A set of compatible observables is said to be *complete* if either of the conditions of Corollary 3.3 is satisfied. Another characterization yet, in terms of a cyclic vector, will be given in Proposition 3.6 below.

The notion of completeness has an intuitive meaning of maximality. If $\{A_1 \cdots A_n\}$ is a CSCO, it contains all possible informations on the system. Hence "a linear operator commuting with each observable of a complete system of commuting observables is a function of them" (Dirac [1]) i.e. it is affiliated with the maximal abelian von Neumann algebra generated by $\{A_1 \cdots A_n\}$.

If the CSCO consists of a single operator A with simple spectrum, Dirac's sentence has a direct algebraic meaning in terms of V*-algebras, for the bicommutant consists, in this case, exactly of the functions of A whose domain contains $\mathcal{D}^\infty(A)$ (Prop. A.1). But in the case of more than one operator, exactly as in the bounded case, one cannot say that each element of the (weak, unbounded) bicommutant is a function of the given operators.

If we add to a CSCO $\{A_1 \cdots A_n\}$ a further operator, also in the case where the CSCO consists of only one operator with simple spectrum, a relationship between the additional operator and the V*-algebra generated by the CSCO can be found, provided one takes the domain into account. Indeed, the additional operator need not, a priori, be defined on $\mathcal{D}^\infty(A_1 \cdots A_n)$. Thus we get:

Proposition 3.4. *Let $\{A_1 \cdots A_n\}$ be a complete set of commuting self-adjoint operators. If A_{n+1} is another self-adjoint operator commuting with them, such that*

$$\mathcal{D}^\infty(A_1 \cdots A_n) \subseteq \mathcal{D}(A_{n+1}),$$

then

$$\mathcal{D}^\infty(A_1 \cdots A_n, A_{n+1}) = \mathcal{D}^\infty(A_1 \cdots A_n)$$

and the V-algebras generated respectively by $\{A_1 \cdots A_n\}$ and by $\{A_1 \cdots A_{n+1}\}$ are the same.*

Proof. Since $\{A_1 \cdots A_n\}$ is complete, the usual commutants verify the relation $\mathfrak{A}' = \mathfrak{A}''$, where as before $\mathfrak{A}$ is the polynomial algebra generated by $A_1 \cdots A_n$ on $\mathcal{D}^\infty(A_1 \cdots A_n)$. Therefore the additional operator A_{n+1} is affiliated with $\mathfrak{A}'' = (\mathfrak{A}'_w)'$ and, since $\mathcal{D}(A_{n+1}) \supseteq \mathcal{D}^\infty(A_1 \cdots A_n)$, $A_{n+1} \in C(\mathcal{D}^\infty(A_1 \cdots A_n), \mathcal{H})$. This implies that $A_{n+1} \in \mathfrak{A}''_{w\sigma} = \mathfrak{A}''_{\sigma\sigma}$. Thus A_{n+1} leaves $\mathcal{D}^\infty(A_1 \cdots A_n)$ invariant. Hence $\mathcal{D}^\infty(A_1 \cdots A_n) \subseteq \mathcal{D}^\infty(A_{n+1})$, which proves the assertion about the domains. The equality of the V*-algebras generated is now straightforward. □

3.5. *Examples.* At this stage it is useful to give a few examples. For simplicity we restrict ourselves to the case of one non-relativistic particle, but generalizations are straightforward. Thus we take $\mathcal{H} = L^2(\mathbb{R}^3, d\mathbf{x})$ throughout, in the position representation.

(*i*) *Momentum operators*: The three components $\{p_1, p_2, p_3\}$ of the momentum $\mathbf{p} = -i\nabla$ constitute a CSCO. The domain $\mathcal{D}^\infty(p_1, p_2, p_3)$ consists of all C^∞ functions which are, together with their partial derivatives of all orders, square integrable over $\mathbb{R}^3$, i.e. it is the Sobolev space of infinite order $H^2_\infty(\mathbb{R}^3)$. The algebra $\mathfrak{A}$ generated by p_1, p_2, p_3 consists of all polynomials in p_j, i.e. all partial differential operators with constant coefficients. The V*-algebra $\mathfrak{B} = \mathfrak{A}''_{\sigma\sigma}$ contains a large class of non-polynomial functions $u(p_1, p_2, p_3)$, which corresponds to pseudo-differential operators. Typical are e.g. arbitrary powers Δ^α $(\alpha > 0)$ of the Laplacian $\Delta = -\mathbf{p}^2$.

(ii) *Position operators*: The situation is identical to (i), with the Fourier transform $p_j \to x_j$ providing a unitary equivalence between the two. The domain $\mathcal{D}^\infty(x_1, x_2, x_3)$ consists of all square integrable functions of fast decrease, $\mathfrak{A}$ is the polynomial algebra and $\mathfrak{A}''_{\sigma\sigma}$ contains a large class of functions $u(x_1, x_2, x_3)$.

(iii) *Hamiltonian and angular momentum*: Let $H = \mathbf{p}^2 + V(|\mathbf{x}|)$ be the Hamiltonian for a particle in a central potential V. Then $\{H, L^2, L_z\}$ is a CSCO, where L^2 is the squared angular momentum and L_z its third component (see any textbook on Quantum Mechanics and also Fredricks [13] for a detailed analysis). If V is a C^∞ function of slow increase (in particular $V \equiv 0$), $\mathcal{D}^\infty(H, L^2, L_z)$ is simply Schwartz's space $\mathcal{S}(\mathbb{R}^3)$. Otherwise the domain is difficult to describe explicitly, except when H has a purely discrete spectrum $\{E_n, n = 0, 1, 2 \cdots\}$ such as in the case of the harmonic oscillator or the bound hydrogen atom. Then the common eigenfunctions $\psi_{nlm}(\mathbf{x})$ are indexed by a countable set of indices, which implies that $\mathcal{H}$ is isomorphic to a space of square integrable sequences:

$$\psi = \sum_{n,l,m} a_{nlm}\psi_{nlm} \in \mathcal{H} \Leftrightarrow \sum_{n,l,m} |a_{nlm}|^2 < \infty.$$

As a consequence the domain $\mathcal{D}^\infty(H, L^2, L_Z)$ consists in that case of all vectors ψ with coefficients a_{nlm} fast decreasing in all three indices n, l, m:

$$\psi = \sum_{n,l,m} a_{nlm}\psi_{nlm} \in \mathcal{D}^\infty(H, L^2, L_z)$$

$$\Leftrightarrow \sum_{n,l,m} |a_{nlm}|^2 [E_n]^r [l(l+1)]^s m^t < \infty \quad \text{for all} \quad r, s, t = 0, 1, 2 \cdots$$

(if H is actually bounded, as for the negative energy hydrogen atom, the condition with $r = 0$ suffices).

Of course the same situation occurs for any CSCO consisting of operators with purely discrete spectra, such as the three components of momentum of a particle confined in a box, as one finds for instance in solid state physics.

We conclude now this section with the concept of cyclic vector. The definition is, of course, identical to the usual one [3], [7]:

Definition 3.6. Let $\mathfrak{V}$ be a V*-algebra on $\mathscr{D}$. We say that a vector f is *cyclic* for $\mathfrak{V}$ if $\mathfrak{V}f$ is dense in $\mathscr{H}$ with respect to the norm topology.

With this definition, we get another characterization of a CSCO. This result has been obtained previously by Prugovečki [7], but the proof given below is totally different, and, in our opinion, much simpler.

Proposition 3.7. *Let $\{A_1 \cdots A_n\}$ be a set of commuting self-adjoint operators in the separable Hilbert space $\mathscr{H}$ and $\mathfrak{V}$ the V*-algebra generated by them on $\mathscr{D}^\infty(A_1 \cdots A_n)$. The set $\{A_1 \cdots A_n\}$ is complete if, and only if, there exists a vector f in $\mathscr{D}^\infty(A_1 \cdots A_n)$ cyclic for $\mathfrak{V}$.*

Proof. Suppose that $f \in \mathscr{D}^\infty(A_1 \cdots A_n)$ is cyclic for $\mathfrak{V}$; because $\mathfrak{V}$ is a standard abelian V*-algebra it is symmetric, then $(\mathfrak{V}_b)'_\sigma = \mathfrak{V}'_\sigma$ and, by Proposition 2.4, $\mathfrak{V}_b$ is strongly dense in $\mathfrak{V}$. As a consequence we get

$$\overline{\mathfrak{V}_b f} = \overline{\mathfrak{V} f} = \mathscr{H}.$$

Therefore f is a cyclic vector for the abelian von Neumann algebra $\mathfrak{V}_b$. Consequently $\mathfrak{V}_b$ is maximal abelian and the set $\{A_1 \cdots A_n\}$ is complete, by Proposition 3.2.

Conversely, suppose that $\{A_1 \cdots A_n\}$ is complete. Then $\mathfrak{V}_b = \{A_1 \cdots A_n\}''$ admits a cyclic vector $f \in \mathscr{H}$. We will prove that starting from f, it is possible to find a vector $g \in \mathscr{D}^\infty(A_1 \cdots A_n)$ cyclic for $\mathfrak{V}$.

By Proposition 2.1, $\mathscr{D}^\infty(A_1 \cdots A_n) = \mathscr{D}^\infty(\bar{K})$ where

$$K = \textstyle\sum_{i=1}^n A_i^2 \restriction \mathscr{D}^\infty(A_1 \cdots A_n).$$

Let $\{E(\Delta)\}$ be the spectral measure associated to $\bar{K}$ and put $E(n) = E([n, n+1))$. It is obvious that $E(n)E(m) = 0$ for $n \neq m$ and, since $\bar{K}$ is positive, we get $\sum_{n=0}^\infty E(n) = 1$ (in strong sense).

We define now

$$g = \sum_{n=0}^\infty \frac{1}{n!} E(n)f$$

Since $E(n)f$ is an analytic vector for $\bar{K}$, $E(n)f \in \mathscr{D}^\infty(\bar{K})$. We will prove that $g \in \mathscr{D}^\infty(\bar{K})$. It suffices to show that, for all $r \in \mathbb{N}$, the sequence $h_N = \sum_{n=0}^N 1/n!\, K^r E(n)f$ is convergent in $\mathscr{H}$ for $N \to \infty$.

By the functional calculus, one gets

$$\|K^r E(n)f\| < (n+1)^r \, \|f\|$$

and therefore:

$$\left\| \sum_{n=N'}^{N} \frac{1}{n!} K^r E(n) f \right\| \leq \sum_{n \in N'}^{N} \frac{1}{n!} \|K^r E(n) f\|$$
$$\leq \|f\| \sum_{n=N'}^{N} \frac{(n+1)^r}{n!} \to 0$$

for $N' \to \infty$.

It remains only to prove that g is cyclic for $\mathfrak{B}_b$, or equivalently (because of the maximality of $\mathfrak{B}_b$) that g is separating for $\mathfrak{B}_b$. Let $X \in \mathfrak{B}_b$ and assume $Xg = 0$. It is easy to check that X commutes with each $E(n)$. Thus:

$$Xg = \sum_{n=0}^{\infty} \frac{1}{n!} XE(n)f = \sum_{n=0}^{\infty} \frac{1}{n!} E(n)Xf = 0$$

Since the terms of the series are orthogonal, we get $XE(n)f = E(n)Xf = 0$, and therefore $Xf = \sum_{n=0}^{\infty} E(n)Xf = 0$. By assumption, f is separating for $\mathfrak{B}_b$, so that $X = 0$. We conclude that g is cyclic for $\mathfrak{B}_b$. □

4. Connection with the rigged Hilbert space formulation

One of the highlights of Dirac's approach is the so-called bra and ket formalism. But the latter, taken literally, cannot be reconciled with the Hilbert space language, as emphasized for instance by Jauch [2]. Thus it is not surprising that many authors have endeavoured to build up a mathematically precise version of Dirac's formalism. The best known of these is probably the rigged Hilbert space (RHS) formulation [14]–[16] and it has much in common with the present work, as we shall see.

The starting point is to realize that most physical systems are characterized by a limited number of observables,[1]) called *labeled observables* by Roberts [14] and *fundamental observables* by Prugovečki [7], which have both a mathematical definition (self-adjoint operator) and a physical one (essentially in terms of measurement). Such are, for instance, position, momentum, energy, total angular momentum, and so on. In fact this list shows that most labeled observables derive from the invariance properties of the system, as we have seen in Sec. 1 [16]. At this stage a crucial assumption is made: the family $\mathfrak{O}_I$ of labeled observables must possess a common dense invariant domain, and usually one considers the largest possible one, namely $\mathscr{D} = \bigcap_{A \in \mathfrak{O}_I} \mathscr{D}^\infty(A)$. From there on the procedure is well-known. One chooses on $\mathscr{D}$ a suitable locally convex topology; calling Φ the domain $\mathscr{D}$ with that topology and $\Phi^\times$ the space of continuous antilinear functionals on Φ, one gets the familiar triplet of spaces $\Phi \subset \mathscr{H} \subset \Phi^\times$, with all maps $A : \Phi \to \Phi (A \in \mathfrak{O}_I)$ continuous. Variants of this formulation exist, such as the *tight rigging* version of Babbitt [18] and Fredricks [13], based on five spaces: $\Phi \subset \mathscr{H}_+ \subset \mathscr{H} \subset \mathscr{H}_- \subset \Phi^\times$, where $\mathscr{H}_\pm$ are Hilbert spaces, dual of each other, and the labeled observables are only required to map Φ continuously into $\mathscr{H}_+$.

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[1]) A notable exception to this class of physical systems is a quantum field theory (uncountably many field observables!); however it does fit in the RHS framework, but in a different fashion [17].

Coming back to the present work, it is clear that all elements of any CSCO $\{A_1 \cdots A_n\}$ must belong to the labeled observables of the system. It follows that the corresponding domain $\mathscr{D}^\infty(A_1 \cdots A_n)$ must contain $\mathscr{D}$, and both are invariant under $A_1 \cdots A_n$. But whereas such an invariant domain $\mathscr{D}^\infty(A_1 \cdots A_n)$ always exists, the existence of a suitable $\mathscr{D}$ is an *assumption*.

On the other hand a given system may have many different CSCO's (see Examples 3.5), which play the rôle of coordinate systems ("representations" in Dirac's terminology). Each of the corresponding domains $\mathscr{D}^\infty(\cdots)$ contains $\mathscr{D}$, which may be or not be the intersection of them all. For instance, if V is a C^∞ potential of slow increase, Schwartz's space $\mathscr{S}$ is the intersection of the natural domains of all three CSCO's $\{p_1, p_2, p_3\}$, $\{x_1, x_2, x_3\}$ and $\{H, L^2, L_z\}$ discussed in 3.5. But if V fails to be C^∞ on some set S of measure zero, then one may take e.g. the space $C_0^\infty(\mathbb{R}^3 \setminus S)$ of C^∞ functions with compact support contained in $\mathbb{R}^3 \setminus S$, or some other domain of the same type [14][16].

The conclusion is that a given quantum mechanical system has, in the RHS approach, a specific domain $\mathscr{D}$, the existence of which must be *postulated*. On the other hand, each CSCO $\{A_1 \cdots A_n\}$ for that system has its own canonical domain $\mathscr{D}^\infty(A_1 \cdots A_n)$ and the corresponding abelian SV*-algebra. However two such algebras need not be unitarily equivalent, nor even isomorphic. This raises the (difficult) mathematical problem of the classification of abelian SV*-algebras and homomorphisms between them. Here again the standard theory of von Neumann algebras will probably be the guide to follow.

Appendix

For convenience of the reader, we collect here, without proofs, some results on algebras of unbounded operators, mostly taken from [8].

Proposition A.1. *Let T be a self-adjoint operator in $\mathscr{H}$ and $\mathfrak{T}$ the Op*-algebra generated by its restriction to $\mathscr{D}^\infty(T) = \bigcap_{n>0} \mathscr{D}(T^n)$. We have:*

(i) *$\mathfrak{T}$ is a closed and standard (therefore self-adjoint) Op*-algebra.*
(ii) *Both $\mathfrak{T}'_\sigma$ and $\mathfrak{T}''_{\sigma\sigma}$ are symmetric and therefore standard.*
(iii) *Both $\mathfrak{T}'_\sigma$ and $\mathfrak{T}''_{\sigma\sigma}$ are closed, $\mathscr{D}$-strongly closed SV*-algebras.*
(iv) *If $u(T)$ is a function of T, defined in the usual way by the functional calculus, and $\mathscr{D}(u(T)) \supseteq \mathscr{D}^\infty(T)$ then $u(T) \restriction \mathscr{D}^\infty(T) \in \mathfrak{T}''_\sigma$ and thus it leaves $\mathscr{D}^\infty(T)$ invariant.*
(v) *If $\mathscr{H}$ is separable, $\mathfrak{T}''_{\sigma\sigma}$ consists only of functions of T.*

(see [8] Prop. 6.1 and 6.2).

Proposition A.2. *Let $\mathfrak{A}$ be a closed abelian standard (and therefore self-adjoint) Op*-algebra on $\mathscr{D}$. Then:*

(i) *Both $\mathfrak{A}'_\sigma$ and $\mathfrak{A}''_{\sigma\sigma}$ are symmetric and therefore standard.*
(ii) $\mathfrak{A}''_{w\sigma} = \mathfrak{A}''_{\sigma\sigma}$.
(iii) *Both $\mathfrak{A}'_\sigma$ and $\mathfrak{A}''_{\sigma\sigma}$ are $\mathscr{D}$-strongly closed SV*-algebras*

(see [8], Prop. 6.3.)

The above proposition applies in particular to the Op*-algebra generated by a set $A_1 \cdots A_n$ of commuting self-adjoint operators on $\mathscr{D}^\infty(A_1 \cdots A_n)$, because it is standard. This fact follows by the following proposition proved by Inoue and Takesue [19].

Proposition A.3. *Let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$. Let A and B be hermitian elements of $C_{\mathcal{D}}$ satisfying $AB = BA$, and $\mathfrak{P}(A, B)$ be the commutative Op*-algebra on $\mathcal{D}$ generated by A and B. Assume that $\mathfrak{P}(A, B)$ is closed on $\mathcal{D}$. Then the following statements are equivalent:*

(i) *$\mathfrak{P}(A, B)$ is standard*
(ii) *$\mathfrak{P}(A, B)$ is self-adjoint and there exists a normal operator C which is an extension of $A + iB$*
(iii) *A and B are essentially self-adjoint on $\mathcal{D}$, $\mathcal{D} = \mathcal{D}^\infty(\bar{A}) \cap \mathcal{D}^\infty(\bar{B})$ and there is a normal operator C which is an extension of $A + iB$.*
(iv) *$\bar{A}$ and $\bar{B}$ are self-adjoint with mutually commuting spectral projections and $\mathfrak{P}(A, B)'_w \mathcal{D} \subseteq \mathcal{D}$.*

As remarked in [19], this proposition extends easily to the Op*-algebra generated by n commuting hermitian elements $A_1 \cdots A_n$.

In the applications of Proposition A.2, we have made use of the following result, remembering that $\mathfrak{A}$ symmetric implies $(\mathfrak{A}_b)'_\sigma = \mathfrak{A}'_\sigma$.

Proposition A.4. *Let $\mathfrak{A}$ be an Op*-algebra such that $\mathfrak{A}''_{\sigma\sigma}$ is $\mathcal{D}$-strongly closed. If $(\mathfrak{A}_b)'_\sigma = \mathfrak{A}'_\sigma$, then $\mathfrak{A}_b$ is $\mathcal{D}$-strongly dense in $\mathfrak{A}$.*
(Slight modification of [8], Prop. 3.10).

Remark A.5. If $\mathfrak{A}$ is the Op*-algebra generated by a set $\{A_1 \cdots A_n\}$ of commuting self-adjoint operators on $\mathcal{D}^\infty(A_1 \cdots A_n)$, $\mathfrak{A}''_{\sigma\sigma}$ is always $\mathcal{D}$-strongly closed, since $\mathfrak{A}$ is self-adjoint.

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