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Geometrical approach to the Aharonov–Bohm plus potential scattering

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Abstract. The ambiguity group of the canonical transformation which introduces the scattering variables ‘deflection angle’ and ‘time delay’ is not trivial. The restriction of the scattering operator to the invariant subspaces of the representations of this group leads exactly to the Aharonov–Bohm effect. Distinct values of the magnetic flux correspond to inequivalent representations.

1. Introduction

If you are interested in a non-linear canonical transformation $(q, p) \mapsto (\bar{q}, \bar{p})$ it will not be generally one-to-one. If this transformation has to be quantised, the operators associated to (\bar{q}, \bar{p}) will not have the same spectra than those associated to (q, p) . Therefore, it is impossible to find a quantum unitary operator which corresponds to the classical canonical transformation.

In certain cases however, a group may be associated to a non-one-to-one canonical transformation. This group, called ambiguity group, has been introduced by Moshinsky and Seligman [1, 2, 3]. Owing to its representations an unitary operator can be constructed which corresponds to the canonical transformation.

In a previous paper, the notion of ambiguity group was applied to the scattering theory and it was suggested there that this formalism could explain geometrically the Aharonov–Bohm effect [4]. This point of view is now presented here.

The Aharonov–Bohm effect consists in the scattering of charged particles arriving normally on a whisker of magnetic flux. Classically no effect is expected because the magnetic field vanishes outside the arbitrarily thin solenoid [5]. But not in quantum mechanics: some interferences occur between the parts of the incident plane wave which pass either at the right or at the left of the whisker. And these interferences lead to a non-zero cross section.

This paper describes the scattering of charged particles on a potential of cylindrical symmetry superposed to a whisker of magnetic flux. Two different approaches of the problem will be considered.

First an asymptotic expansion of the solution of the corresponding Schrödinger’s equation is calculated in order to find the phase shifts which are composed of two terms. One term represents exactly the Aharonov–Bohm phase

shift obtained by Henneberger [6], while the other one is equal to the phase shift due to the potential but evaluated for a non-integer value of the angular momentum. The fractional part of the angular momentum is proportional to the magnetic flux; so, by varying this flux, the phase shift becomes a function on the full real axis.

In the second approach, the Aharonov–Bohm fiber is removed, but at the same time a canonical transformation introduces some scattering variables (deflection angle, time delay). This transformation is not one-to-one and thus a non-trivial ambiguity group exists. To each irreducible representation of this group corresponds an invariant subspace. The restriction of the scattering operator to such a subspace coincides exactly with the result found in the first approach. Therefore, this restriction reintroduces naturally the Aharonov–Bohm effect, an effect which depends obviously on the chosen subspace. As pointed out by Roy et al. [7] and Ruijsenaars [8] it is important to remark that the operator $i\partial/\partial\theta$ ($\theta \in [0, 2\pi[$) is not essentially self-adjoint. In fact we shall see that the irreducible representations of the ambiguity group are related to the self-adjoint extensions of the operator $i\partial/\partial\theta$.

Finally, an example is presented in the last section. For a particular value of the magnetic flux and for the potential γ/q , the scattering amplitude can be calculated analytically. By changing γ we obtain an interesting comparison between the pure Aharonov–Bohm effect and the scattering on the potential γ/q alone.

Throughout the paper the following conventions will be made:

$$\sum_{l=-\infty}^{\infty} = \sum_l, \quad \int_{-\infty}^{\infty} d\lambda = \int d\lambda.$$

2. Conventional approach

Consider a solenoid along the z axis which is infinitely long, arbitrarily thin, impenetrable, containing a magnetic flux Φ_0 , and a potential V of cylindrical symmetry superposed to this whisker. We shall study the quantum scattering of charged particles arriving perpendicularly to the whisker. For such a system the Schrödinger's equation is

$$\frac{1}{2m} \left(\frac{\hbar}{i} \frac{\partial}{\partial \vec{q}} - \frac{e}{c} \vec{A} \right)^2 \psi(\vec{q}) + V\psi(\vec{q}) = E\psi(\vec{q}), \quad (2.1)$$

where \vec{A} describes the whisker:

$$\vec{A} = \frac{\Phi_0}{2\pi q^2} \begin{pmatrix} -q_y \\ q_x \end{pmatrix}, \quad \vec{q} \in \mathbb{R}^2. \quad (2.2)$$

In cylindrical coordinates $\vec{q} = (q, \theta)$, equation (2.1) reads

$$\left[\frac{\partial^2}{\partial q^2} + \frac{1}{q} \frac{\partial}{\partial q} + \frac{1}{q^2} \left(\frac{\partial}{\partial \theta} + i\alpha \right)^2 + \left(k^2 - \frac{2mV}{\hbar^2} \right) \right] \psi(q, \theta) = 0 \quad (2.3)$$

with $k^2 = 2mE/\hbar^2$ (scattering state) and $\alpha = -e\Phi_0/2\pi\hbar c$. For the sake of simplicity, α is assumed to satisfy the inequalities $0 \leq \alpha < 1$.

Next we expand the wave function ψ in partial waves by choosing the boundary conditions $\psi(\theta = 0) = \psi(\theta = 2\pi)$ which correspond to the conventional

extension of the operator $i\partial/\partial\theta$ used for example by Aharonov and Bohm [9]:

$$\psi = \sum_l e^{il\theta} R_{l+\alpha}(kq), \quad (2.4)$$

where $R_{l+\alpha}$ verifies

$$(kq)^2 R''_{l+\alpha} + kq R'_{l+\alpha} + \left[\left(1 - \frac{V}{E}\right) (kq)^2 - (l+\alpha)^2 \right] R_{l+\alpha} = 0. \quad (2.5)$$

If the potential V tends sufficiently rapidly to zero as $q \rightarrow \infty$, the asymptotic form of $R_{l+\alpha}$ is given by

$$R_{l+\alpha} \sim A_{l+\alpha} e^{i\delta_{|l+\alpha|}} [\cos \delta_{|l+\alpha|} J_{|l+\alpha|}(kq) - \sin \delta_{|l+\alpha|} Y_{|l+\alpha|}(kq)], \quad (2.6)$$

where we have used the conventions of Abramowitz and Stegun [10] for the Bessel functions J and Y . The phase shift $\delta_{|l+\alpha|}$ is a functional of the potential determined by the exact radial wave function with the boundary condition $R_{l+\alpha}(0) = 0$ (since the whisker is impenetrable). $A_{l+\alpha}$ is chosen in order to obtain the correct incident plane wave

$$\psi_i = e^{i(kq \cos \theta - \alpha \theta)} \quad (2.7)$$

which gives a constant current density in the positive x direction. The further calculations will show that we have to choose

$$A_{l+\alpha} = \exp \left[-i \left(|l+\alpha| \frac{\pi}{2} + (l+\alpha)\pi \right) \right]. \quad (2.8)$$

It is very useful to separate ψ in two parts, $\psi = \psi_1 + \psi_2$, where

$$\psi_1 = \sum_l e^{il\theta} A_{l+\alpha} J_{|l+\alpha|}(kq) \quad (2.9)$$

and

$$\psi_2 \sim \sum_l e^{il\theta} A_{l+\alpha} [(e^{i\delta_{|l+\alpha|}} \cos \delta_{|l+\alpha|} - 1) J_{|l+\alpha|}(kq) - e^{i\delta_{|l+\alpha|}} \sin \delta_{|l+\alpha|} Y_{|l+\alpha|}(kq)]. \quad (2.10)$$

ψ_1 is the exact Aharonov–Bohm wave function [9], while ψ_2 represents the contribution due to the potential, but modified by the whisker.

Now we have to find the asymptotic form of ψ_1 and ψ_2 in order to have

$$\psi \sim e^{i(kq \cos \theta - \alpha \theta)} + \frac{e^{ikq}}{\sqrt{q}} f(\alpha, \theta). \quad (2.11)$$

The asymptotic expansion of ψ_1 is well-known: for example Berry et al. [11] have proposed an elegant method to get it by using an integral representation for $J_{|l+\alpha|}$ in the complex plane. The result is

$$\psi_1 \sim e^{i(kq \cos \theta - \alpha \theta)} + \frac{e^{ikq}}{\sqrt{q}} \frac{1}{\sqrt{2\pi k}} \exp \left[-i \left(\frac{\pi}{4} + \alpha \pi \right) \right] e^{-i(\theta/2)} \frac{\sin(\alpha\pi)}{\sin \frac{\theta}{2}}. \quad (2.12)$$

A straightforward calculation leads to the result

$$\lim_{\varepsilon \rightarrow 0} \sum_l e^{il\theta} e^{-|l|\varepsilon} \left[\exp \left[-i \frac{l+\alpha}{|l+\alpha|} \alpha \pi \right] - 1 \right] = e^{-i(\theta/2)} \frac{\sin(\alpha\pi)}{\sin \frac{\theta}{2}} \quad (2.13)$$

and therefore

$$\psi_1 \sim e^{i(kq \cos \theta - \alpha \theta)} + \frac{e^{ikq}}{\sqrt{q}} \frac{1}{\sqrt{2\pi k}} \times \exp \left[-i \left(\frac{\pi}{4} + \alpha \pi \right) \right] \sum_l e^{il\theta} \left[\exp \left[-i \frac{l+\alpha}{|l+\alpha|} \alpha \pi \right] - 1 \right] \quad (2.14)$$

where the sum must be understood in the sense of equation (2.13). The linear combination of $J_{|l+\alpha|}$ and $Y_{|l+\alpha|}$ involved in equation (2.10) can be replaced by an Hankel function

$$\psi_2 \sim \frac{1}{2} \sum_l e^{il\theta} A_{l+\alpha} \left[e^{i2\delta_{|l+\alpha|}} - 1 \right] H_{|l+\alpha|}^{(1)}(kq) \quad (2.15)$$

and using the asymptotic behaviour of this Hankel function [10]

$$\psi_2 \sim \frac{e^{ikq}}{\sqrt{q}} \frac{1}{\sqrt{2\pi k}} \exp \left[-i \left(\frac{\pi}{4} + \alpha \pi \right) \right] \sum_l e^{il\theta} \exp \left[-i \frac{l+\alpha}{|l+\alpha|} \alpha \pi \right] \times [e^{i2\delta_{|l+\alpha|}} - 1]. \quad (2.16)$$

Finally, adding equation (2.14) and equation (2.16), and comparing with equation (2.11),

$$f(\alpha, \theta) = \frac{1}{\sqrt{2\pi k}} \exp \left[-i \left(\frac{\pi}{4} + \alpha \pi \right) \right] \times \sum_l e^{il\theta} \left[\exp \left[i \left(-\frac{l+\alpha}{|l+\alpha|} \alpha \pi + 2\delta_{|l+\alpha|} \right) \right] - 1 \right]. \quad (2.17)$$

This last expression shows that the phase shift due to the potential adds to the one produced by the whisker. Moreover, by varying from 0 to $-2\pi\hbar c/e$ the flux contained in the whisker, the phase shift δ becomes physically accessible for all real values of the angular momentum. We still remark that for θ different from zero, the equation (2.17) gives the relation

$$f(1-\alpha, 2\pi-\theta) = e^{i(\theta+2\alpha\pi)} f(\alpha, \theta). \quad (2.18)$$

Thus if we know the scattering amplitude for $\theta \in]0, \pi]$ and $\alpha \in [0, 1[$, we can deduce it for $\theta \in [\pi, 2\pi[$.

In the following section we shall be able to understand better the structure of the S -matrix. In particular we shall see how it depends on terms which represent the orbiting of the particle around the whisker.

3. Geometrical aspect

We first briefly recall the results obtained by us in a previous paper [4], but by introducing them in a slightly different manner. Consider the one-to-one canonical transformation

$$f: \mathbb{R}^2 - \{\vec{0}\} \times \mathbb{R}^2 - \{\vec{0}\} \rightarrow \mathbb{R} \times \mathbb{R} - \{0\} \times [0, 2\pi[\times \mathbb{R}^+ \quad (3.1)$$

$$(\vec{q}, \vec{p}) \mapsto (\tau, \lambda; \kappa, \varepsilon)$$

defined by

$$\begin{aligned} \tau &= m \frac{\vec{q} \cdot \vec{p}}{p^2} & \lambda &= q_x p_y - q_y p_x \\ \kappa &= \operatorname{tg}^{-1} \frac{p_y}{p_x} & \varepsilon &= \frac{p^2}{2m}. \end{aligned} \quad (3.2)$$

In order to be able to take $(\tau, \lambda; \kappa, \varepsilon) \times \mathbb{R} \times \mathbb{R} - \{0\} \times \mathbb{R} \times \mathbb{R} - \{0\}$ we have to consider the ambiguity group of f [1, 2, 3]. This group is given by the direct product of \mathbb{Z} with the inversion group $\{\pm 1\}$:

$$[n, j](\tau, \lambda; \kappa, \varepsilon) = (\tau, \lambda; \kappa + 2\pi n, j\varepsilon), \quad (n, j) \in \mathbb{Z} \times \{\pm 1\}. \quad (3.3)$$

Thus, it is equivalent to know $(\vec{q}, \vec{p}; n, j)$ or $(\tau, \lambda; \kappa, \varepsilon) \in \mathbb{R} \times \mathbb{R} - \{0\} \times \mathbb{R} \times \mathbb{R} - \{0\}$. Now we would like to quantise the transformation f . To this end we define two orthonormal basis of $L^2(\mathbb{R}^2)$ (in Dirac's sense), $|\tau, \lambda\rangle$ and $|\kappa, \varepsilon\rangle$, related by

$$\langle \kappa, \varepsilon | \tau, \lambda \rangle = \frac{1}{2\pi\hbar} e^{i/h(\kappa\lambda - \varepsilon\tau)}. \quad (3.4)$$

The operators $(T, L; K, H)$ having respectively $|\tau, \lambda\rangle$ and $|\kappa, \varepsilon\rangle$ as eigenfunctions will be associated to the classical variables $(\tau, \lambda; \kappa, \varepsilon)$, which are also the eigenvalues of these operators. We shall write the momentum operator \vec{P} in cylindrical coordinates by means of K and H :

$$P = \sqrt{2m |H|}, \quad \Phi = K - 2\pi \left[\frac{K}{2\pi} \right] \quad (3.5)$$

where $[x]$ denotes the integer part of x . P and Φ do not represent a complete set of commuting observables in $L^2(\mathbb{R}^2)$. This fact is related with the ambiguity group introduced previously: indeed the unitary operators \mathcal{A} and \mathcal{P} ,

$$\begin{aligned} \langle \kappa, \varepsilon | \mathcal{A} | \psi \rangle &= \langle \kappa + 2\pi, \varepsilon | \psi \rangle \\ \langle \kappa, \varepsilon | \mathcal{P} | \psi \rangle &= \langle \kappa, -\varepsilon | \psi \rangle \end{aligned} \quad (3.6)$$

commute both with P and Φ . Thus, we can construct a new ket $|p, \varphi; \nu, j\rangle$ ($\nu \in [0, 1[, j \in \{\pm 1\}$)

$$\begin{aligned} \langle \kappa, \varepsilon | p, \varphi; \nu, j \rangle &= \frac{1}{\sqrt{2m}} \left[\delta\left(\varepsilon - j \frac{p^2}{2m}\right) \right. \\ &\quad \left. + j \delta\left(\varepsilon + j \frac{p^2}{2m}\right) \delta\left(\kappa - 2\pi \left[\frac{\kappa}{2\pi} \right] - \varphi\right) e^{i2\pi\nu \left[\frac{\kappa}{2\pi} \right]} \right] \quad (3.7) \end{aligned}$$

which is eigenfunction of $(P, \Phi; \mathcal{A}, \mathcal{P})$ with eigenvalues $(p, \varphi; e^{i2\pi\nu}, j)$. This ket constitutes a generalisation of the momentum representation and, using the Poisson summation formula, we easily verify that it forms a new orthonormal basis of $L^2(\mathbb{R}^2)$. We remark also that all the irreducible representations of the (abelian) ambiguity group $\mathbb{Z} \times \{\pm 1\}$ are exactly characterised by $\nu \in [0, 1[$ and $j \in \{\pm 1\}$. Moreover, ν has a simple physical signification: by calculating the scalar product $\langle \tau, \lambda | p, \varphi; \nu, j \rangle$ we see that it is just equal to the fractional part of the

angular momentum in \hbar units:

$$\langle \tau, \lambda \mid p, \varphi; \nu, j \rangle = \frac{1}{2\pi\hbar} \frac{1}{\sqrt{2m}} \exp \left[-\frac{i}{\hbar} \lambda \varphi \right] \times \left[\exp \left[\frac{i}{\hbar} \tau j \frac{p^2}{2m} \right] + j \exp \left[-\frac{i}{\hbar} \tau j \frac{p^2}{2m} \right] \right] \delta \left(\frac{\lambda}{\hbar} - \left[\frac{\lambda}{\hbar} \right] - \nu \right). \quad (3.8)$$

In addition from equation (3.7) we remark:

$$\langle \kappa + 2\pi, \varepsilon \mid p, \varphi; \nu, j \rangle = e^{i2\pi\nu} \langle \kappa, \varepsilon \mid p, \varphi; \nu, j \rangle. \quad (3.9)$$

This relation is identical to the formula (14) of [7]; hence ν fixes at the same time the self-adjoint extensions of the operator $i\partial/\partial\theta$.

With the variables $(\tau, \lambda; \kappa, \varepsilon)$ we would like to study the scattering of particles by a potential of cylindrical symmetry.

Classically the S-matrix applies canonically the straight line $(\tau, \lambda; \kappa, \varepsilon)$ on the straight line $(\tau', \lambda; \kappa', \varepsilon)$ ($(\lambda, \kappa, \varepsilon)$ and $(\lambda, \kappa', \varepsilon)$ determine the asymptotes of the motion while τ and τ' fix a point on them). It is described by the generating function

$$G(\kappa', \varepsilon'; \tau, \lambda) = \kappa' \lambda - \varepsilon' \tau - 2\pi\lambda\theta(\lambda) + 2\hbar\delta_{|\lambda|}(|\varepsilon'|) \quad (3.10)$$

where $\hbar\delta_{|\lambda|}(|\varepsilon'|)$ is the classical phase shift [12]. If we differentiate G , the time delay and the deflection angle are obtained:

$$\tau' - \tau = -2\hbar \frac{\partial}{\partial \varepsilon'} \delta_{|\lambda|}(|\varepsilon'|), \quad \kappa' - \kappa = 2\pi\theta(\lambda) - 2\hbar \frac{\partial}{\partial \lambda} \delta_{|\lambda|}(|\varepsilon'|). \quad (3.11)$$

To understand the term $2\pi\theta(\lambda)$ in equation (3.11) we recall that the direction of the incident particle is determined by \vec{p} , while $\vec{p}' = p(\vec{q}'/q')$ ($q' \rightarrow \infty$) gives the outgoing asymptote. Now it is sufficient to consider free particles: figure 1 shows that the deflection angle is either 0 or 2π and thus it depends on the sign of the angular momentum, which agrees with equation (3.11). In quantum mechanics we propose the following expression for the S-matrix:

$$\langle \kappa', \varepsilon' \mid S \mid \tau, \lambda \rangle = \frac{1}{2\pi\hbar} \exp \left[\frac{i}{\hbar} (\kappa' \lambda - \varepsilon' \tau - 2\pi\lambda\theta(\lambda) + 2\hbar\delta_{|\lambda|}(|\varepsilon'|)) \right] \quad (3.12)$$

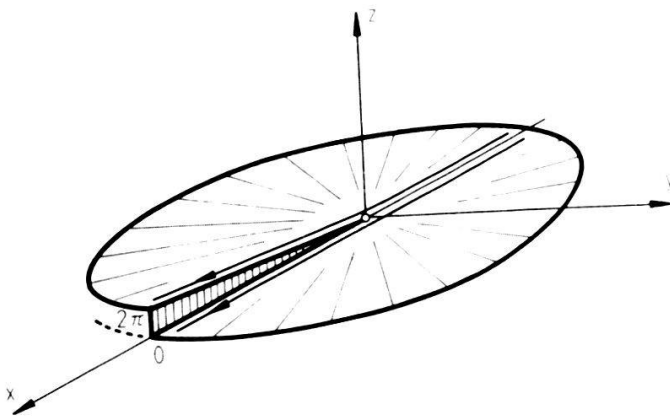


Figure 1

Classical trajectories for free particles showing that the deflection angle depends on the sign of the angular momentum: $\kappa' - \kappa = 2\pi\theta(\lambda)$

where δ has to be calculated now from the Schrödinger's equation. This form is chosen in order to correspond to an unitary operator and to reduce to the generating function (3.10) in the semi-classical approximation. In a real scattering experiment an incident plane wave (given by \vec{p}) is sent on the target and a scattered wave is detected in the direction fixed by \vec{p}' . Therefore, the interesting S -matrix elements are $\langle p', \varphi'; \nu', j' | S | p, \varphi; \nu, j \rangle$. From equation (3.12) a short calculation shows that S commutes with the ambiguity operators \mathcal{A} and \mathcal{P} . So we have

$$\langle p', \varphi'; \nu', j' | S | p, \varphi; \nu, j \rangle = \frac{1}{p} \delta(p' - p) \delta(\nu' - \nu) \delta_{j'j} S\nu(\varphi', \varphi). \quad (3.13)$$

From a mathematical point of view the S -matrix has been defined on the sheeted phase space $(\kappa, \varepsilon; \tau, \lambda)$ by equation (3.12). Thus, we can consider equation (3.13) as the restrictions of the scattering operator to the subspaces labelled by the indices (ν, j) which distinguish the irreducible representations of the ambiguity group.

Making use of the completeness relations

$$\iint d\kappa d\varepsilon |\kappa, \varepsilon\rangle\langle\kappa, \varepsilon| = \iint d\tau d\lambda |\tau, \lambda\rangle\langle\tau, \lambda| = \mathbb{1} \quad (3.14)$$

$S\nu(\varphi', \varphi)$ reads:

$$S\nu(\varphi', \varphi) = \frac{1}{2\pi\hbar} \sum_n \int d\lambda \exp \left[i2\pi n \left(\nu - \frac{\lambda}{\hbar} \right) \right] \times \exp \left[\frac{i}{\hbar} \left[(\varphi' - \varphi)\lambda - 2\pi\lambda\theta(\lambda) + 2\hbar\delta_{|\lambda|} \left(\frac{p^2}{2m} \right) \right] \right]. \quad (3.15)$$

And finally, again with the Poisson summation formula, we obtain

$$S\nu(\varphi', \varphi) = \frac{1}{2\pi} e^{i\nu(\varphi' - \varphi - \pi)} \sum_l e^{il(\varphi' - \varphi)} \times \exp \left[i \left(-\frac{l + \nu}{|l + \nu|} \nu\pi + 2\delta_{|l + \nu|} \left(\frac{p^2}{2m} \right) \right) \right]. \quad (3.16)$$

Now comes the most important step of this paper: the comparison of the two expressions (2.17) and (3.16). For θ different from zero S_ν is proportional to f if the following choices are made: $\nu \equiv \alpha$ and $\theta \equiv \varphi' - \varphi$. From these identifications we see that the S -matrix (3.12) describes exactly the scattering by a whisker of magnetic flux and a potential, although the whisker was not introduced explicitly in equation (3.12)! The effect due to the solenoid depends on the subspace (ν, j) which is chosen to project the S -matrix. More precisely, in absence of the potential (i.e. when all the phase shifts are zero) the S -matrix (3.12) do not reduce to the identity operator. In this case only its restriction to the subspace $\nu = 0$ equals the identity operator, while its other restrictions give a pure Aharonov-Bohm scattering.

In order to understand better the final formula (3.16) we shall still note that the expression (3.15) is particularly convenient to make a semi-classical approximation. If we apply the stationary phase method to each term of equation (3.15)

we have to find some λ which verify

$$\varphi' - \varphi = 2\pi n + 2\pi\theta(\lambda) - 2\hbar \frac{\partial}{\partial \lambda} \delta_{|\lambda|} \left(\frac{p^2}{2m} \right). \tag{3.17}$$

So the n th term represents the probability amplitude to detect the scattered particule after an orbiting of n turns around the whisker. And because the particle entwines n times the flux of the solenoid the total phase is modified by the value of $2\pi n\nu$ (see equation (3.15)).

4. An example

It is well known that an analytic expression for the scattering amplitude can be found for the potential γ/q . In this section we shall show that the same calculations are always possible if a whisker containing a flux $-\pi\hbar c/e$ is added to the potential. To this flux corresponds a flux parameter α equal to one half, which leads to a maximum Aharonov–Bohm cross-section (see equation (2.12)).

For θ different from zero, the equation (2.17) gives

$$f_{(\frac{1}{2}, \theta)} = -\frac{1}{\sqrt{2\pi k}} e^{i(\pi/4)} \sum_l e^{il\theta} \exp \left[i \left(-\frac{l+\frac{1}{2}}{|l+\frac{1}{2}|} \frac{\pi}{2} + 2\delta_{|l+\frac{1}{2}|} \right) \right] \tag{4.1}$$

$$f_{(\frac{1}{2}, \theta)} = -\frac{2}{\sqrt{2\pi k}} e^{i(\pi/4)} e^{-i(\theta/2)} \sum_0^\infty \exp [i2\delta_{|l+\frac{1}{2}|}] \sin (l+\frac{1}{2})\theta. \tag{4.2}$$

In our example, the phase shifts $\delta_{|l|}$ have to be determined from the Schrödinger's equation

$$\Delta\psi + \left(k^2 - \frac{2m\gamma}{\hbar^2} \frac{1}{q} \right) \psi = 0. \tag{4.3}$$

We verify easily that the regular solution at the origin reads

$$\psi = \sum_l C_l e^{il\theta} e^{ikq} q^{|l|} M \left(|l| + \frac{1}{2} + i \frac{m\gamma}{\hbar^2 k}, 2|l| + 1; -2ikq \right) \tag{4.4}$$

where M denotes the confluent hypergeometric function. Using the asymptotic expansion for M [10], we obtain the phase shifts:

$$e^{i2\delta_{|l|}} = \frac{\Gamma \left(|l| + \frac{1}{2} + i \frac{m\gamma}{\hbar^2 k} \right)}{\Gamma \left(|l| + \frac{1}{2} - i \frac{m\gamma}{\hbar^2 k} \right)}. \tag{4.5}$$

Evaluating these phase shifts at $|l+\frac{1}{2}|$ rather than at $|l|$, we find an explicit formula for the scattering amplitude (4.2):

$$f_{(\frac{1}{2}, \theta)} = -\frac{2}{\sqrt{2\pi k}} e^{i(\pi/4)} e^{-i(\theta/2)} \sum_0^\infty \frac{\Gamma \left(l+1 + i \frac{m\gamma}{\hbar^2 k} \right)}{\Gamma \left(l+1 - i \frac{m\gamma}{\hbar^2 k} \right)} \sin (l+\frac{1}{2})\theta. \tag{4.6}$$

But this last sum is proportional to the trigonometric expansion of a Legendre function [10], so

$$f\left(\frac{1}{2}, \theta\right) = -\frac{1}{\sqrt{2k}} e^{i(\pi/4)} 2^{-\mu} \Gamma\left(\mu + \frac{1}{2}\right) e^{-i(\theta/2)} \left(\sin \frac{\theta}{2}\right)^{-\mu} P_{-\mu}^{\mu}\left(\cos \frac{\theta}{2}\right), \quad \mu = \frac{1}{2} + i \frac{m\gamma}{\hbar^2 k}. \quad (4.7)$$

Next, we remark that the function $P_{-\mu}^{\mu}$ can be expressed by means of a more elementary function:

$$f\left(\frac{1}{2}, \theta\right) = -\frac{1}{\sqrt{2k}} e^{i(\pi/4)} \frac{\Gamma\left(1 + i \frac{m\gamma}{\hbar^2 k}\right) e^{-i(\theta/2)}}{\Gamma\left(\frac{1}{2} - i \frac{m\gamma}{\hbar^2 k}\right) \sin \frac{\theta}{2}} \left(\sin \frac{\theta}{2}\right)^{-i(2m\gamma/\hbar^2 k)} \quad (4.8)$$

And finally, separating modulus and phase

$$|f\left(\frac{1}{2}, \theta\right)|^2 = \frac{\gamma}{4E} \frac{1}{\sin^2 \frac{\theta}{2}} \operatorname{cth} \frac{\pi m\gamma}{\hbar\sqrt{2mE}} \quad (4.9)$$

$$\arg f\left(\frac{1}{2}, \theta\right) = \frac{5\pi}{4} - \frac{\theta}{2} - \frac{2m\gamma}{\hbar\sqrt{2mE}} \ln\left(\sin \frac{\theta}{2}\right) + \arg \Gamma\left(1 + i \frac{m\gamma}{\hbar\sqrt{2mE}}\right) - \arg \Gamma\left(\frac{1}{2} - i \frac{m\gamma}{\hbar\sqrt{2mE}}\right). \quad (4.10)$$

This scattering amplitude possesses the following properties:

- i) The change of sign of γ affects only the phase (4.10).
- ii) If both \hbar and Φ_0 tend to zero in such a way that the flux parameter α remains equal to one half, then $|f|^2$ approaches the classical cross section due to the potential γ/q :

$$|f\left(\frac{1}{2}, \theta\right)|^2 = \frac{|\gamma|}{4E} \frac{1}{\sin^2 \frac{\theta}{2}}. \quad (4.11)$$

- iii) The limit $\gamma = 0$ leads naturally to a pure Aharonov–Bohm scattering:

$$|f\left(\frac{1}{2}, \theta\right)|^2 = \frac{\hbar}{2\pi\sqrt{2mE}} \frac{1}{\sin^2 \frac{\theta}{2}}. \quad (4.12)$$

- iv) Inversely, for $|\gamma|$ which goes to infinity, only γ/q contributes to $|f|^2$; in this case $|f|^2$ equals again the classical cross section (4.11).

Now we understand why this example can be solved completely in an analytic way: the scattering on the potential γ/q and the Aharonov–Bohm scattering give some proportional cross sections. And if both interactions are present, the factor

$$\operatorname{cth} \frac{\pi m\gamma}{\hbar\sqrt{2mE}} \quad (4.13)$$

in equation (4.9) leads to the correct cross section.

We have also to remark that in the limits ii) and iv) the Aharonov–Bohm effect subsists in the phase (4.10):

$$\arg f(\frac{1}{2}, 2\pi - \theta) = \arg f(\frac{1}{2}, \theta) + \theta - \pi. \quad (4.14)$$

But unfortunately this phase remains inaccessible to the experience!

5. Conclusion

In this paper we have given a geometrical (group-theoretical) interpretation of the Aharonov–Bohm effect.

The crucial point of the above considerations consists in the introduction of the ambiguity group to which are associated the variables ‘deflection angle’ and ‘time delay’; in fact this group reflects the topological structure of the phase space. In this frame it should be certainly interesting to analyse the situation where the configuration space is bored by more than one solenoid.

As pointed out recently by Amiet and Huguenin [13, 14], we remark that the concept of generating functions clarifies certain analogies between the classical and quantum mechanics. We shall note particularly the similitude between the formulas (3.10) and (3.12) and the possibility to make a semi-classical approximation with equation (3.15).

In a next stage we think to replace the cylindrical potential by a spherical potential to treat the three-dimensional scattering. In order to use the formalism developed in this paper, the spherical harmonics have to be generalised for non-integer indices.

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