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Thermodynamics of a relativistic Fermi gas in a strong magnetic field

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Abstract. The thermodynamical properties in the grand canonical ensemble for a relativistic spin $-\frac{1}{2}$ ideal gas including both the particles and anti-particles are investigated in the presence of strong constant homogeneous magnetic field. We calculate the magnetization and the static magnetic susceptibility for very strong fields at high temperatures, for which we present numerical results in the special case for equal numbers of particles and anti-particles.

Numerous aspects of relativistic statistical mechanics appear in many physical situations. One important example is a relativistic electron gas in a strong magnetic field, which has been studied by various authors [1, 2] in the context of astrophysical questions. A similar phenomenon is encountered in the case of the quark gas, that is supposed to be formed during heavy ion collisions at high energies [3], in which the existence of very strong magnetic fields is possible.

Another example of interest is the quark phase of the nucleons, which is supposed to be present at the core of neutron stars. In all such cases the theoretical understanding of the physics involved needs to be formulated on the basis of covariant formulation of the laws of thermodynamics. The existing works indeed satisfy this requirement. There is one aspect however, which although taken into account by many, is yet to be brought out explicitly in the calculations of various physical quantities – that is the contribution from the anti-particle states, which is a truly relativistic effect. In this work we shall describe this effect for the case of a relativistic electron gas in a very strong magnetic field.

A covariant formulation of the thermodynamical laws basing on Dirac's constraint concepts, [4], which builds the foundations of relativistic dynamics, has already been worked out by one of the authors previously [5]. We take this as the basis of our formulation here. Consider now a model where the electrons are in thermal equilibrium with a background blackbody radiation field and the temperature is in the relativistic domain. Such a situation is surely realized in the plasma state. The antiparticles, which are the positron states, originate due to the possibility of the background thermal photons materializing by pair production. An equilibrium will then be reached with certain distributions of electrons, positrons and photons. The thermodynamic parameters describing this state are

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then the temperature, the external magnetic field and the total charge of the system. A variation in temperature or the magnetic field will then imply a change in the electron and positron densities and also that of the photons, while the total charge of the system remains conserved. If the system is initially neutral with respect to a positive ionic charge background, then it remains so for any change of the physical parameters and hence the relativistic chemical potential is to be attributed to the lepton conservation law. With this view a study of the magnetic properties of the system is presented below.

The energy levels $E_{r,n}$ of the Dirac equation in the presence of an external magnetic field of strength B (parallel to z-direction) are given by [6]

$$E_{r,n} = [m^2c^4 + p_z^2c^2 + \epsilon^2(n+r-1)]^{1/2}$$

$$\epsilon^2 = 2\frac{B}{B_c}m^2c^4, \qquad B_c = \frac{m^2c^3}{|e|\,\hbar} \tag{1}$$

where $n = 0, 1, 2, \ldots$, and $r = 1$ or 2 according to the spin orientation up or down in the field. Here p_z is the z-component of the momentum. For quantizing in a box of column V one has $p_z = \hbar k_z$, $k_z = 2\pi V^{-1/3}n_z$ with $n_z = 0, \pm 1, \pm 2, \ldots$. For $r = 1$ and 2 separately $E_{1,n} = (m^2c^4 + p_z^2c^2 + \epsilon^2 n)^{1/2}$ and $E_{2,n} = (m^2c^4 + p_z^2c^2 + \epsilon^2(n+1))^{1/2}$ so that $E_{1,n+1} = E_{2,n}$. The statistical weight $g_{r,n}$ of each energy eigenstate $E_{r,n}$ is independent of r and n and has the value $g_{r,n} = \dfrac{V^{2/3}|e|\,B}{2\pi\hbar c}$.

In the covariant formulation of quantum statistics the grand canonical partition function takes the expression:

$$Z_g = \mathrm{Tr}\, e^{-\beta(H-\mu Q)}$$

where μ is the chemical potential considered as a Lagrangian multiplier corresponding to the globally conserved charge Q, which is the time component of a conserved four vector, and H is the Hamiltonian of the system with $\beta = (kT)^{-1}$. In terms of the number operators of the particles N_+ and anti-particles N_- one can write $Q = N_+ - N_-$. This implies that the chemical potentials μ_+ and μ_- for the particles and anti-particle states are related by $\mu_+ = -\mu_- = \mu$. Thus the grand canonical partition function for our system can be written as

$$\ln Z_g = \sum_{n,r,p_z} g_{r,n}\{\ln(1+e^{-\beta(E_{r,n}-\mu)}) + \ln(1+e^{-\beta(E_{r,n}+\mu)})\} - \frac{16\pi^3}{h^3}\sum_{p}\ln(1-e^{-\beta\omega_p}). \tag{3}$$

where the last term denotes the contribution due to the background photons. For a large volume V one can write equation (3) as

$$\ln Z_g = \frac{V|e|\,B}{(2\pi)^2\hbar e}\sum_{r=1}^{2}\sum_{n=0}^{\infty}\int_{-\infty}^{+\infty} dk\{\ln(1+e^{-\beta(E_{r,n}(k)-\mu)}) + \ln(1+e^{-\beta(E_{r,n}+\mu)})\} - \frac{8\pi V}{h^3c^3}\int_0^{\infty}\varepsilon^2\, d\varepsilon\, \ln(1-e^{-\beta\varepsilon}) \tag{4}$$

where the first term in the bracket is the contribution from the electrons and the second term from the positrons respectively. Writing $E_n(k, B) = E_{1,n}(k, B) = (m^2c^4 + c^2\hbar^2k^2 + n\epsilon^2)^{1/2}$ one has $\frac{\partial E_n}{\partial B} = \frac{n}{E_n}\frac{m^2c^4}{|e|\,\hbar}$.

If M be the magnetization (or the magnetic moment) of the total system, one has by the consideration of the quantum statistics

$$M = \frac{1}{Z_g}\mathrm{Tr}\left\{e^{-\beta(\hat{H}(B)-\mu Q)}\left(-\frac{\partial \hat{H}}{\partial B}\right)\right\} = \frac{1}{\beta}\frac{\partial}{\partial B}\ln Z_g \tag{5}$$

where $\hat{H} = \hat{H}(B)$ is the Hamiltonian. Thus one obtains from (3) the magnetization per unit volume as

$$\begin{aligned}\frac{M}{V} = \frac{2\,|e|\,c\hbar}{(2\pi)^2}\Bigg\{&\int_0^\infty dk\,\frac{k^2}{E_n}\left(\frac{1}{e^{\beta(E_0-\mu)}+1}+\frac{1}{e^{\beta(E_0+\mu)}+1}\right)\\ &+2\sum_{n=1}^{\infty}\int_0^\infty dk\,\frac{k^2}{E_n}\left(\frac{1}{e^{\beta(E_0-\mu)}+1}+\frac{1}{e^{\beta(E_0+\mu)}+1}\right)\\ &-\frac{4\,|e|\,Bm^2c^4}{(2\pi)^2\hbar cB_c}\sum_{n=1}^{\infty} n\int_0^\infty \frac{dk}{E_n}\left(\frac{1}{e^{\beta(E_n-\mu)}+1}+\frac{1}{e^{\beta(E_n+\mu)}+1}\right).\end{aligned} \tag{6}$$

The magnetic susceptibility χ is related by $\chi = \frac{\partial}{\partial B}\left(\frac{M}{V}\right)$. One has then explicitly

$$\begin{aligned}\chi = {}&\frac{1}{\pi^2}\frac{m^2c^4}{B_c}\frac{|e|}{\hbar c}\sum_{n=1}^{\infty} n\int_0^\infty \frac{dk}{E_n}\left(\frac{1}{e^{\beta(E_n-\mu)}+1}+\frac{1}{e^{\beta(E_n+\mu)}+1}\right)\\ &\times\left(-\frac{c^2\hbar^2k^2}{E_n^2}-1+B\frac{n}{E_n^2}\frac{m^2c^4}{B_c}\right)\\ &+\frac{\beta}{\pi^2}\frac{m^2c^4}{B_c}\frac{|e|}{\hbar c}\sum_{n=1}^{\infty} n\int_0^\infty dk\left(\frac{e^{\beta(E_n-\mu)}}{(e^{\beta(E_n-\mu)}+1)^2}+\frac{e^{\beta(E_n+\mu)}}{(e^{\beta(E_n+\mu)}+1)^2}\right)\\ &\times\left(-\frac{c^2\hbar^2k^2}{E_n^2}+\frac{nB}{E_n^2}\frac{m^2c^4}{B_c}\right).\end{aligned} \tag{7}$$

Here the free parameters of the system are $\beta = 1/kT$, the external field B and the total charge Q or the charge density Q/V. One must choose μ so that one can match a given value for Q. One has explicitly

$$\begin{aligned}\frac{Q}{V} = \frac{2\,|e|^2\,B}{(2\pi)^2\hbar c}\Bigg[&\int_0^\infty dk\left(\frac{1}{e^{\beta(E_0(k)-\mu)}+1}-\frac{1}{e^{\beta(E_0(k)+\mu)}+1}\right)\\ &+2\int_0^\infty dk\sum_{n=1}^{\infty}\left(\frac{1}{e^{\beta(E_n(k)-\mu)}+1}-\frac{1}{e^{\beta(E_n(k)+\mu)}+1}\right)\Bigg].\end{aligned} \tag{8}$$

The number density $\rho = N/V$ (i.e. particles and anti-particles per unit volume) i similarly given by

$$\rho=\frac{2|e|^2 B}{(2\pi)^2\hbar c}\left[\int_0^\infty dk\left(\frac{1}{e^{\beta(E_0(k)-\mu)}+1}+\frac{1}{e^{\beta(E_0(k)+\mu)}+1}\right)\right.$$

$$\left.+2\int_0^\infty dk\sum_{n=1}^\infty\left(\frac{1}{e^{\beta(E_n(k)-\mu)}+1}+\frac{1}{e^{\beta(E_n(k)+\mu)}+1}\right)\right] \tag{9}$$

Finally, for the sake of ease in the explicit computations now we consider the very special case of equally distributed electrons and positrons in equilibrium [7]. Although such an assumption is seldom warranted for any realizable physical system, it nevertheless leads to a vanishing total charge Q in (8) and its corresponding chemical potential μ. In this case the magnetization per unit volume M/V in (6) together with its corresponding susceptibility χ in (7) may be computed numerically for very strong fields at high temperatures. We have displayed the quantities in a plot as a function of the magnetic field B in Fig. 1. On the left-hand axis we can see the plot of the falling magnetization with increasing fields B in the range $0.01B_c \leq B \leq 100B_c$, where B_c is $B_c = 4.414\times 10^{13}$ Gauss as given directly from the constants in (1). We have chosen the temperature to be equal to the electron rest mass energy, which gives $k_B T$ as 511 KeV. Under the same conditions on the right-hand side axis we have drawn the magnetic susceptibility as a function of the magnetic field. It should be further remarked that the value of M/V at a field equal to $100B_c$ is very near to its saturation magnetization for the free electron–positron gas at this temperature,

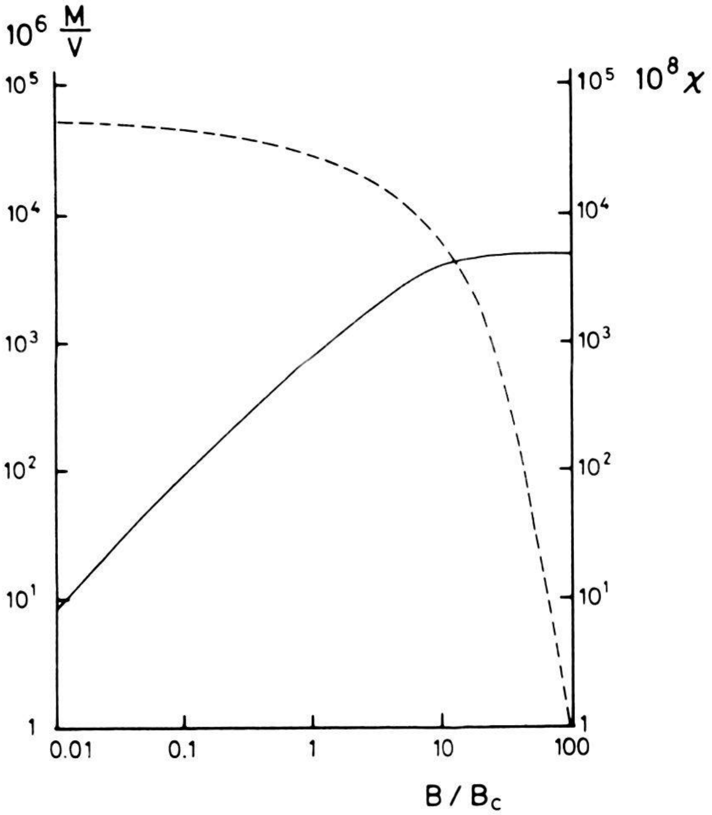

Figure 1
The magnetization per volume ($M/V\times 10^6$ on the left axis) and the magnetic susceptibility ($\chi\times 10^8$ on the right axis) are plotted as a function of the magnetic field B in the high field range between $0.01B_c$ and $100B_c$. M/V is plotted with the solid curve and χ is drawn with broken lines.

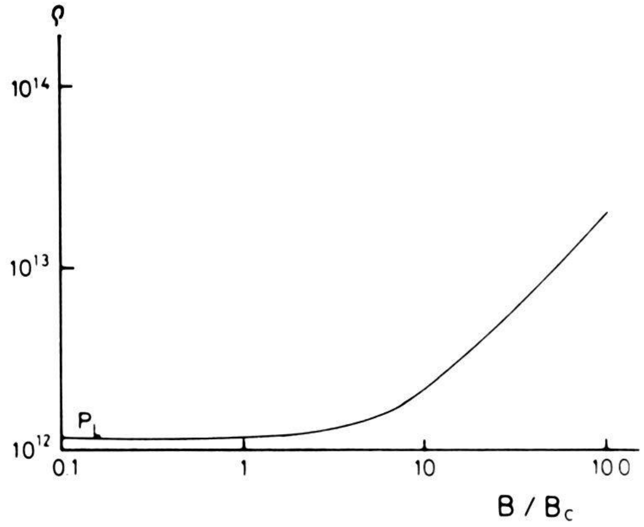

Figure 2
Variation of particle and anti-particle number density as a function of external magnetic field B for the fixed temperature of 511 KeV.

which is equal to 4.69875×10^{-3} magnetons per cm^3. This effect is also seen from the very rapid fall in the susceptibility at high fields. Furthermore, the results presented here are clearly very different from the rapid variation of the magnetization at fields around B_c, which were previously calculated for the relativistic electron gas alone.[1])

Finally we have also shown in the accompanying figure 2 the variation in the number density ρ as a function of the external field B. With rising values of B one finds the effect of pair production as p also increases.

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[1]) See Fig. 2 in Ref. 1, p. 1234.