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A simple connection between the motion in a constant magnetic field and the harmonic oscillator

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Abstract. We propose a very simple change of variables enlightening the connection between the motion in a constant magnetic field and the harmonic oscillator. The so-called kinematical and dynamical symmetry algebras are explicitly constructed for the magnetic context starting from the well known results on the harmonic oscillator.

1. Introduction

Since the contribution of Johnson and Lippmann [1], it is well known that the motion in a constant magnetic field can be studied by using some typical properties of the harmonic oscillator. Moreover when 2-dimensional space systems are considered, the corresponding nonrelativistic hamiltonians have also been recognized [2] as the generators of two nonconjugate 1-dimensional subalgebras of the inhomogeneous symplectic invariance algebra.

With the event of supersymmetric quantum mechanics [3], both of such physical and fundamental applications have recently been reconsidered [4, 5] with the purposes of determining their symmetries *and* supersymmetries. On the one hand, the harmonic oscillator is one of the simplest physical system admitting *all* the symmetries displayed in the so-called 'nonrelativistic conformal quantum mechanics' [6, 7] as shown by Niederer [8] in this journal. Such a system also admits a maximal set of supersymmetries in $N = 2$-supersymmetric quantum mechanics [5]. On the other hand, the motion in a constant magnetic field is another very fundamental type of an electromagnetic interaction when charged particles are concerned and its supersymmetric version has also been recently discussed [4].

In this note let us draw our attention on a simple connection between both of these applications with direct implications for the determination of their symmetries as well as their supersymmetries. Limiting here our discussion on

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symmetries we just want to give such a simple connection through a change of variables in the same way that Niederer [8] has related the harmonic oscillator to the free case [7] and has determined its so-called *m*aximal *k*inematical *i*nvariance (MKI) group SCHR(n) if the n-dimensional context is considered.

In Section 2, we give the announced change of variables, thus relating the problem of an interaction with a constant magnetic field with the one of the harmonic oscillator in two spatial dimensions. Both the classical and quantized points of view are considered. The kinematical [8] and dynamical [9] symmetries associated with these problems are respectively recovered in Sections 3 and 4 through our change of variables. In Section 5 some comments are presented in connection with the 3-dimensional case.

2. The change of variables and the one-to-one correspondence

2.a. *Classical point of view*

Let us here consider the 2-dimensional classical harmonic oscillator characterized by the equations

$$\ddot{x}_0 = -\omega^2 x_0, \qquad \ddot{y}_0 = -\omega^2 y_0, \tag{2.1}$$

where the subscript 0 will refer in the following to the harmonic *o*scillator context. The motion of an electron in a *constant* magnetic field is described by the equation

$$\ddot{\vec{x}}_M = e\dot{\vec{x}}_M \times \vec{B} \tag{2.2}$$

where $\vec{x}_M = (x_M, y_M, z_M)$. In the following, the subscript M will always refer to the *m*agnetic context. If the field $\vec{B}$ is chosen along the z-axis ($\vec{B} = (0, 0, B)$), we explicitly get

$$\ddot{x}_M = eB\dot{y}_M, \qquad \ddot{y}_M = -eB\dot{x}_M, \tag{2.3}$$

$$\ddot{z}_M = 0, \tag{2.4}$$

so that the particle moves freely along the z-axis as expected. We are then interested in the motion obtained from (2.3) in the plane (x_M, y_M) perpendicular to $\vec{B}$.

The change of variables $(t_0, x_0, y_0) \leftrightarrow (t_M, x_M, y_M)$ we are considering is the following one:

$$\begin{aligned} t_0 &= t_M, \\ x_0 &= \cos\omega t\, x_M - \sin\omega t\, y_M, \\ y_0 &= \sin\omega t\, x_M + \cos\omega t\, y_M, \end{aligned} \tag{2.5}$$

where we put $\omega = \frac{1}{2}eB$. Let us summarize such a transformation on the coordinates (x, y) on the form

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = R(\omega t, \vec{e}_3)\begin{pmatrix} x_M \\ y_M \end{pmatrix} \tag{2.6}$$

where R is a rotation around the z-axis with an angle ωt. It is easy to convince ourselves that such a change of variables leads to a 1–1 correspondence between the two contexts.

2.b. *Quantized point of view*

In quantum mechanics, a 1–1 correspondence can be determined between the Schrödinger wave equation for the 2-dimensional harmonic oscillator:

$$i\,\partial_{t_0}\psi_0 = H_0\psi_0 \tag{2.7}$$

with

$$H_0 = \tfrac{1}{2}(\mathbf{p}_0^2 + \omega^2\mathbf{x}_0^2) \tag{2.8}$$

where we use the notation $\mathbf{a}$ for the bivector (a_x, a_y), and the Schrödinger equation describing the interaction with a constant magnetic field $\vec{B}$ along the z-axis:

$$i\,\partial_{t_M}\psi_M = H_M\psi_M \tag{2.9}$$

with

$$H_M = \tfrac{1}{2}(\mathbf{p}_M - e\mathbf{A}_M)^2 = \tfrac{1}{2}\mathbf{\Pi}_M^2. \tag{2.10}$$

Let us notice that we have taken as unity the mass m. If we choose the so-called gauge symmetric particular potential $\vec{A}^S = -\tfrac{1}{2}\vec{r}\times\vec{B}$ associated with the field $\vec{B}$, we have

$$A_x^S = -\tfrac{1}{2}By_M, \qquad A_y^S = \tfrac{1}{2}Bx_M \tag{2.11}$$

and we get the hamiltonian (2.10) on the form:

$$H_M = \tfrac{1}{2}\left\{\mathbf{p}_M^2 + \frac{e^2B^2}{4}\mathbf{x}_M^2 - eB(x_M p_{y_M} - y_M p_{x_M})\right\} \tag{2.12}$$

containing the terms associated with a 2-dimensional harmonic oscillator as well as a term associated with the orbital angular momentum L_M.

Now, using the change of variables (2.5) on the coordinates together with the corresponding transformations on the derivatives:

$$\begin{aligned} &\partial_{t_0} = \partial_{t_M} - \omega(x_M\,\partial_{y_M} - y_M\,\partial_{x_M}),\\ &\begin{pmatrix}\partial_{x_0}\\ \partial_{y_0}\end{pmatrix} = R(\omega t, \vec{e}_3)\begin{pmatrix}\partial_{x_M}\\ \partial_{y_M}\end{pmatrix}, \end{aligned} \tag{2.13}$$

the equations (2.7)–(2.8) *and* (2.9) with (2.12) clearly are in a 1–1 correspondence by taking

$$\psi_0(t_0, x_0, y_0) = \psi_M(t_M, x_M, y_M, \vec{A}^S) \tag{2.14}$$

where we notice that we have no non-trivial phase factor.

Let us point out that the correspondence is also realized for an arbitrary potential other than the gauge symmetric one. Indeed, choosing another potential $\vec{A}$ related to $\vec{A}^S$ by

$$\vec{A} = \vec{A}^S + \vec{\nabla}\lambda, \tag{2.15}$$

it is well-known (see [10] for example) that the equations containing respectively the potentials $\vec{A}$ and $\vec{A}^S$ are gauge invariant through the trivial (physically speaking) change in the wave functions

$$\psi_M(\vec{A}) = e^{ie\lambda}\psi_M(\vec{A}^S). \tag{2.16}$$

Then, finally, the correspondence with the wave function of the harmonic oscillator reads in general

$$\psi_0(t_0, x_0, y_0) = e^{-ie\lambda}\psi_M(t_M, x_M, y_M, \vec{A}). \tag{2.17}$$

3. Kinematical symmetries for the magnetic context

Let us first recall [8] that the MKI algebra of symmetries of the 2-dimensional harmonic oscillator is $ho(2)$ isomorphic to $schr(2)$, the Schrödinger algebra (or nonrelativistic conformal algebra) in two dimensions. It is generated by

$$\left.\begin{aligned} H_0 &= \tfrac{1}{2}(\mathbf{p}_0^2 + \omega^2\mathbf{x}_0^2),\\ C_0^1 &= \tfrac{1}{2}[\sin 2\omega t\,(\mathbf{p}_0^2 - \omega^2\mathbf{x}_0^2) - 2\omega\cos 2\omega t\,(\mathbf{x}_0\cdot\mathbf{p}_0 - i)],\\ C_0^2 &= \tfrac{1}{2}[\cos 2\omega t\,(\mathbf{p}_0^2 - \omega^2\mathbf{x}_0^2) + 2\omega\sin 2\omega t\,(\mathbf{x}_0\cdot\mathbf{p}_0 - i)],\end{aligned}\right\} \tag{3.1}$$

$$L_0 = x_0 p_{y_0} - y_0 p_{x_0} \equiv (\vec{x}_0 \times \vec{p}_0)_z, \tag{3.2}$$

$$\left.\begin{aligned} P_0^1 &= \cos\omega t\, p_{x_0} + \omega\sin\omega t\, x_0, & P_0^2 &= \cos\omega t\, p_{y_0} + \omega\sin\omega t\, y_0\\ K_0^1 &= \sin\omega t\, p_{x_0} - \omega\cos\omega t\, x_0, & K_0^2 &= \sin\omega t\, p_{y_0} - \omega\cos\omega t\, y_0\end{aligned}\right\} \tag{3.3}$$

The operators (3.1), (3.2) and (3.3) respectively generate the algebras $so(2, 1)$, $so(2)$ and the so-called Heisenberg algebra $h(2)$.

Now, since the change of variables (2.5) is canonical, the MKI algebra of symmetries of the Schrödinger equations (2.9)–(2.10) is isomorphic to the one for the harmonic oscillator. Among the possible set of equations (2.8)–(2.10) with equivalent but different potentials, the one with the potential $\mathbf{A}^S \equiv (2.11)$ is privileged. Indeed the wave functions ψ_0 and ψ_M are related by (2.14) and then the generators are exactly obtained through the change of variables (2.5) only. We then get the 1–1 correspondences:

$$H_0 \leftrightarrow H_M + \omega L_M, \qquad C_0^1 \leftrightarrow C_M^1, \qquad C_0^2 \leftrightarrow C_M^2, \tag{3.4}$$

$$L_0 \leftrightarrow L_M, \tag{3.5}$$

$$\begin{aligned} P_0^1 &\leftrightarrow \tfrac{1}{2}(\pi_x + P_x)_M, & P_0^2 &\leftrightarrow \tfrac{1}{2}(\pi_y + P_y)_M,\\ K_0^1 &\leftrightarrow \tfrac{1}{2}(P_y - \pi_y)_M, & K_0^2 &\leftrightarrow \tfrac{1}{2}(\pi_x - P_x)_M,\end{aligned} \tag{3.6}$$

where π_x and π_y are the expected Johnson–Lippmann [1] constants of motion

$$\pi_x = p_{x_M} - \omega y_M, \qquad \pi_y = p_{y_M} + \omega x_M \tag{3.7}$$

and P_x, P_y are other constants of motion

$$\begin{aligned} P_x &= \cos 2\omega t(p_{x_M} + \omega y_M) - \sin 2\omega t(p_{y_M} - \omega x_M), \\ P_y &= \sin 2\omega t(p_{x_M} + \omega y_M) + \cos 2\omega t(p_{y_M} - \omega_{x_M}). \end{aligned} \tag{3.8}$$

With respect to the Johnson–Lippmann approach, this leads to additional constants of motion which are C_M^1, C_M^2, P_x and P_y. Such quantities have already been obtained by Durand [4] by studying directly the symmetries of the equation (2.9) using the Niederer method [7]. The interest of our presentation is that, through a simple change of variables, we get immediately the same results without tedious calculations.

Now, in order to obtain the symmetry generators for the equations (2.9)–(2.10) with an *arbitrary* potential we have to transform, beside the change of variables in the harmonic oscillator symmetry operators, these generators by the function $h = \exp(-ie\lambda)$ since the wave functions ψ_0 and $\psi_M(\vec{A})$ correspond to each other by (2.17). Then, if G_0 is a symmetry generator for the harmonic oscillator, we obtain the corresponding symmetry generator for the equation (2.9) with $\vec{A} = (2.15)$, on the form

$$G(\mathbf{x}_M, \mathbf{p}_M, \vec{A}) = h^{-1}G_0(\mathbf{x}_0(\mathbf{x}_M, \mathbf{p}_M), \mathbf{p}_0(\mathbf{x}_M, \mathbf{p}_M))h. \tag{3.9}$$

The symmetry generators for the Schrödinger equation in the magnetic case are then obtained in a form independent on an explicit choice of the potential:

$$\left.\begin{aligned} H_M &= \tfrac{1}{2}(\mathbf{p}_M - e\mathbf{A}_M)^2 = \tfrac{1}{2}\mathbf{\Pi}_M^2, \\ C_M^1 &= \tfrac{1}{2}[\sin 2\omega t\{\Pi_{x_M}\pi_x + \Pi_{y_M}\pi_y\} - \cos 2\omega t\{\Pi_{x_M}\pi_y - \Pi_{y_M}\pi_x\}], \\ C_M^2 &= \tfrac{1}{2}[\cos 2\omega t\{\Pi_{x_M}\pi_x + \Pi_{y_M}\pi_y\} + \sin 2\omega t\{\Pi_{x_M}\pi_y - \Pi_{y_M}\pi_x\}], \\ L_M &= x_M\Pi_{y_M} - y_M\Pi_{x_M} + \omega(x_M^2 + y_M^2), \\ \pi_x &= \Pi_{x_M} - 2\omega y_M, \qquad \pi_y = \Pi_{y_M} + 2\omega x_M, \\ P_x &= \cos 2\omega t\,\Pi_{x_M} - \sin 2\omega t\,\Pi_{y_M}, \qquad P_y = \sin 2\omega t\,\Pi_{x_M} + \cos 2\omega t\,\Pi_{y_M}. \end{aligned}\right\} \tag{3.10}$$

These are the generators of the *kinematical* symmetries. Their commutation relations are evidently obtained from the ones of the algebra $schr(2) \equiv [so(2, 1) \oplus so(2)] \,\square\, h(2)$ where $so(2, 1) \equiv \{H_M, C_M^1, C_M^2\}$, $so(2) = \{L_M\}$ and $h(2) = \{\pi_x, \pi_y, P_x, P_y\}$.

The contents of the algebra $h(2)$ has been examined [1], [4], [11] in connection with creation $(a_{x_0}^\dagger, a_{y_0}^\dagger)$ and annihilation (a_{x_0}, a_{y_0}) operators of the harmonic oscillator in order to study the correspondence between the two problems.

Concerning again the kinematical symmetries (3.10), let us finally notice that they correspond to coordinates transformations which leave the equations of motion invariant. We get for the interaction with the magnetic field the transformations $t' = t + \delta t$, $x'_M = x_M + \delta x_M$, $y'_M = y_M + \delta y_M$ where

$$\left.\begin{aligned}
\delta t &= c_1 \sin 2\omega t + c_2 \cos 2\omega t + b. \\
\delta_{x_M} &= -\omega(c_1 \cos 2\omega t - c_2 \sin 2\omega t)x_M - \omega(c_1 \sin 2\omega t + c_2 \cos 2\omega t)y_M \\
&\quad - \theta y_M + a_x + b_x \cos 2\omega t + b_y \sin 2\omega t, \\
\delta y_M &= \omega(c_1 \sin 2\omega t + c_2 \cos 2\omega t)x_M - \omega(c_1 \cos 2\omega t - c_2 \sin 2\omega t)y_M \\
&\quad + \theta_{x_M} + a_y - b_x \sin 2\omega t + b_y \cos 2\omega t,
\end{aligned}\right\} \tag{3.11}$$

b, c_1, c_2, θ, a_x, a_y, b_x and b_y being respectively associated with the generators H_M, C^1_M, C^2_M, L_M, π_x, π_y, P_x and P_y.

4. Dynamical symmetries for the magnetic context

The largest *dynamical* algebra [9] of symmetries of the 2-dimensional harmonic oscillator is the fifteen dimensional algebra $sp(4, \mathbb{R}) \square h(2)$. It corresponds to the well-known degeneracy group of the harmonic oscillator. In terms of the creation and annihilation operators given explicitly by

$$a_i^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_{0i} - ip_{0i}), \qquad a_i = \frac{1}{\sqrt{2\omega}}(\omega x_{0i} + ip_{0i}) \quad (i, j = 1, 2) \tag{4.1}$$

the generators of $sp(4, \mathbb{R}) \square h(2)$ are obtained on the form

$$T_{ij} = \frac{\omega}{2}\{a_i, a_j^\dagger\}, \qquad C_{+ij} = \frac{i\omega}{2}\{a_i^\dagger, a_j^\dagger\}, \qquad C_{-ij} = -\frac{i\omega}{2}\{a_i, a_j\} \tag{4.2}$$

and

$$P^i_+ = i\sqrt{2\omega}\, a_i^\dagger, \qquad P^i_- = -i\sqrt{2\omega}\, a_i, \tag{4.3}$$

the coefficients being chosen according to [5]. Let us notice that the four T_{ij}'s generate $u(2)$ and together with the six $C_{\pm ij}$'s they form $sp(4, \mathbb{R})$ while the four $P_{\pm i}$'s plus the identity generator evidently generate $h(2)$.

The generators (4.2) and (4.3) correspond to constants of motion for the harmonic oscillator hamiltonian but taken when the time t equals zero. In fact since the commutation relations between the hamiltonian $H_0 = (2.8)$ and the generators (4.2) and (4.3) are

$$\begin{gathered}
[H_0, T_{ij}] = 0, \qquad [H_0, C_{\pm ij}] = \pm 2\omega C_{\pm ij}, \\
[H_0, P^i_\pm] = \pm\omega P^i_\pm,
\end{gathered} \tag{4.4}$$

we obtain the time dependent constants of motion on the form

$$\begin{gathered}
T_{ij}(t) = T_{ij}(0), \qquad C_{\pm ij}(t) = e^{\mp 2i\omega t} C_{\pm ij}(0), \\
P^i_\pm(t) = e^{\mp i\omega t} P^i_\pm(0).
\end{gathered} \tag{4.5}$$

Among the operators (4.2)–(4.3) some are directly related to those of the MKI algebra. In fact, we have

$$\left.\begin{aligned} &H_0 = T_{11} + T_{22}, \\ &C_{0\pm}(t) \equiv C_0^1 \pm iC_0^2 = C_{\pm 11}(t) + C_{\pm 22}(t), \\ &L_0 = \frac{i}{\omega}(T_{12} - T_{21}), \\ &P_{0\pm}^1 \equiv K_0^1 \pm iP_0^1 = P_\pm^1(t), \qquad P_{0\pm}^2 \equiv K_0^2 \pm iP_0^2 = P_\pm^2(t). \end{aligned}\right\} \tag{4.6}$$

We then conclude that the kinematical algebra is completely contained in the dynamical one and that six generators belonging to $sp(4, \mathbb{R})$ cannot be associated with any coordinate transformations.

Through the change of variables (2.5), we immediately see that in the magnetic context, we get once again the dynamical algebra $sp(4, \mathbb{R}) \square h_2$ containing the kinematical one generated by (3.10) and six additional generators obtained through (3.9) from the generators $T_{11} - T_{22}$, $T_{12} + T_{21}$, $C_{+11} - C_{+22}$, $C_{-11} - C_{-22}$, C_{+12} and C_{+21}. Let us notice that these last symmetries cannot evidently be determined by the Niederer method since they do not correspond to coordinate transformations. In fact, in the work of Durand [4], they are obtained through the supersymmetric context *after* the knowledge of the supersymmetry generators in order to close the superalgebra. Here such operators are completely justified without supersymmetric considerations but only by using completely the 1–1 correspondence.

5. Comments

Here we have considered the 2-dimensional context since along the z-axis the particle moves freely. Nevertheless, Johnson and Lippmann [1] have considered the 3-dimensional context and have used the symmetries associated with L_M, π_x, π_y and p_z in order to study the energy spectrum of the hamiltonian

$$H = \tfrac{1}{2}(\mathbf{\Pi}_M^2 + p_z^2). \tag{5.1}$$

These symmetries have recently [12] been completed by using the symmetry properties [13] of the constant magnetic field. Let us only recall that the corresponding symmetry algebra is

$$\mathbb{G} = \{H, L_M, \pi_x, \pi_y, p_z, K_z\} \tag{5.2}$$

when the usual galilean coordinate transformations are considered, K_z corresponding to pure galilean transformations along the z-axis. It is easy to show that we can now add the generators P_x, P_y given in (3.10): they also correspond to coordinate transformations generalizing the usual galilean ones. We cannot add the generators C_M^1 and C_M^2 since, while they are constants of motion for the hamiltonian $H = (5.1)$, they do not correspond to kinematical symmetries and do not close under commutation. All these results were expected when analysed in connection with the work of Boyer [14]. Indeed, the extension of the change of variables (2.5) to three dimensions leads to an *anisotropic* harmonic oscillator whose hamiltonian reads:

$$H_0 = \tfrac{1}{2}(\mathbf{p}_0^2 + \omega^2\mathbf{x}_0^2 + p_{0z}^2). \tag{5.3}$$

From Boyer's considerations [14], we then get an algebra isomorphic to $[so(2) \oplus t_1] \square h(3)$, $so(2)$ being generated by $L_0 \equiv (3.2)$, t_1 by $H_0 \equiv (5.3)$ and $h(3)$ by P_0^1, P_0^2, K_0^1, K_0^2, p_z and K_z where C_0^1 and C_0^2 do not appear.

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