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A commutator approach to the limiting absorption principle

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Abstract. A limiting absorption principle for the free resolvent in non-relativistic quantum mechanics is derived, using commutator identities and the iteration of operator inequalities. The class of admissible perturbations is essentially the best possible, and asymptotically optimal bounds are also obtained for a class of related operators which localise to compact regions of position space.

1. Introduction

The so-called limiting absorption principle (L.A.P.) of mathematical physics has its origin [1], [2], in the possibility of selecting the physically relevant solution of a P.D.E. (Helmholtz equation, wave equation, etc.) through an appropriate limit of solutions at complex energy or wave number. In its most abstract and mathematical form (see for example [3], Chapter 14) one has to deal with the limit, as z approaches the real axis, of the resolvent $R(z)$ of some elliptic differential operator; although this limit does not usually exist in terms of operator norm convergence in the underlying Hilbert space, one can often regard $R(z)$ as a mapping from some suitable Banach space to its dual, in which case such a limit may be shown to exist under very general conditions. Alternatively, one can consider Hilbert space limits of $T_1R(z)T_2$ for some suitable choice of operators T_1, T_2.

In this paper, we present a new approach to the L.A.P. for the free Hamiltonian $H_0 = \mathbf{P}^2/2m$ in non-relativistic quantum mechanics, based on commutator methods and motivated by an important paper of Mourre [4]. In suitable units ($\hbar = 2m = 1$), $H_0 = -\Delta$ is the negative Laplacian operator, and we shall denote the resolvent in that case by $G_0(z) = (H_0 - z)^{-1}$. The question arises: for which class of multiplication operators $h(\mathbf{r})$ is it true that norm limits exist as $\varepsilon \to 0+$, $z = \lambda + i\varepsilon \to \lambda > 0$, for the operators

(A): $hG_0(\lambda + i\varepsilon)h$, and
(B): $h[G_0(\lambda + i\varepsilon) - G_0(\lambda - i\varepsilon)]h$.

Although there is an extensive literature on the L.A.P. as applied to the

Schrödinger operator (see for example the pioneering work of Agmon [5], [6]; for additional references see [4], [7]) there seems hitherto to be no self-contained "elementary" treatment of problems (A) and (B) which applies to the widest class of h-functions. Roughly, $h(\mathbf{r})$ has to decay at infinity more rapidly that $|\mathbf{r}|^{-1/2}$ (see for example [8]; other results are to be found in [7], [9]). Locally, $h(\mathbf{r})$ may have mild singularities; the two cases (A) and (B) are different in this respect.

The principal advantages of the approach presented here are as follows

(a) Bounds placed on the multiplicative functions $h(\mathbf{r})$ are optimal, in a sense to be described later.

(b) The method is operator theoretic, and therefore independent of a detailed knowledge of resolvent kernels, Fourier transforms, etc. The analysis is based on a single operator inequality (10). An analogous treatment of the L.A.P. in applications to scattering by singular, short-range and long-range potentials, will be published elsewhere. For other applications in this area see [10], Chapter 12.

For other developments of Mourre's work see [11], [12], [13], [14], [15].

(c) We shall derive (Lemma 2) estimates in operator norm, appropriate to cases (A) and (B) above, where $h(\mathbf{r})$ is multiplication by the characteristic function of a region of size R in position space. These estimates are optimal in the limit $R \to \infty$, and imply the L.A.P. of Hormander [3] for the resolvent as a mapping from one Banach space to another, (Hormander's analysis is, however, more general in that it applies to a wider class of elliptic operator).

Finally, let us cite a few of the many applications of the L.A.P. in quantum theory.

(i) With $h(\mathbf{r}) = |V(\mathbf{r})|^{1/2}$, a uniform norm estimate $\|hG_0(\lambda + i\varepsilon)h\| < 1$ implies the existence of wave and scattering operators for H_0, $H_0 + V$. See for example, [7] Chapter 13, where there are further references, as well as extensions and applications to many-body problems.

(ii) A uniform norm estimate in case (B) above implies that h is H_0-smooth, with important consequences for scattering and spectral theory. Wave operators exist whenever $V = h_1 h_2$, where h_1 is H_0-smooth and h_2 is H-smooth. For the theory of smooth perturbations, see [16], [17].

(iii) The norm limit in (B) above may be formally denoted by $2\pi i h\delta(H_0 - \lambda)h$. This limit is intimately related to the derivative, with respect to λ, of the spectral family of H_0. There is a corresponding expression with H instead of H_0; see for example [7], with applications to spectral analysis.

2. Principal results

Throughout this paper, H_0 will denote the unique self-adjoint extension, in the Hilbert space $L^2(\mathbb{R}^3)$, of the negative Laplacian operator $\hat{H}_0 = -\Delta$, defined with domain $D(\hat{H}_0) = C_0^\infty(\mathbb{R}^3)$. We can write $H_0 = \mathbf{P}^2$, where $P_j = -i(\partial/\partial x_j)$ $(j = 1, 2, 3)$ are the three components of the momentum operator. Each component of the momentum operator is self-adjoint in $L^2(\mathbb{R}^3)$, with $D(H_0^{1/2}) \subset D(P_j)$.

Elements f of $L^2(\mathbb{R}^3)$ will be represented in position space by wave functions $f(\mathbf{r})$, where $\mathbf{r}=(x_1, x_2, x_3)$, and the jth component X_j of the position operator is the operator of multiplication by x_j. In the momentum space representation of $L^2(\mathbb{R}^3)$, f is represented by $\hat{f}(\mathbf{k})$, the three-dimensional Fourier transform of f. In momentum space, P_j is implemented by multiplication by k_j, and H_0 corresponds to multiplication by $\mathbf{k}^2$. The position operator X_j in momentum space is given by $X_j = i(\partial/\partial k_j)$.

We shall use the same symbol h to denote both a positive real valued function $h(\mathbf{r})$ defined on $\mathbb{R}^3\backslash\{0\}$, and the operator of multiplication by $h(\mathbf{r})$, in position space. In particular, if $G_0(z)=(H_0-z)^{-1}$, $\operatorname{Im} z \neq 0$, is the resolvent operator for H_0, we shall be interested in the operator

$$M_0(z) = hG_0(z)h, \tag{1}$$

defined in $L^2(\mathbb{R}^3)$ by

$$(M_0(z)f)(\mathbf{r}) = h(\mathbf{r})(G_0(z)g)(\mathbf{r}),$$

where $g(\mathbf{r}) = h(\mathbf{r})f(\mathbf{r})$.

It is well known [10] that any vector in $D(H_0^{1/2})$ belongs also to $D(1/|\mathbf{r}|)$; i.e. to the domain of the operator of multiplication by $1/|\mathbf{r}|$. Indeed, if we take account of the fact that the operator $H_0 - 1/4\,|\mathbf{r}|^2$ is positive we have, with $f=(H_0+1)^{1/2}u$,

$$\|f\|^2 - \left\|\frac{1}{2\,|\mathbf{r}|}(H_0+1)^{-1/2}f\right\|^2 = \langle (H_0+1)^{1/2}u,$$
$$\times (H_0+1)^{1/2}u\rangle - \left\langle \frac{1}{2\,|\mathbf{r}|}u, \frac{1}{2\,|\mathbf{r}|}u\right\rangle \geq 0$$

for any $u \in D(H_0^{1/2})$, from which it follows that

$$\left\|\frac{1}{|\mathbf{r}|}(H_0+1)^{-1/2}\right\| \leq 2.$$

(The norm is actually equal to 2, since $H_0 - c/|\mathbf{r}|^2$ is not even bounded below if $c > \frac{1}{4}$).

If, in (1), we make the assumption that $h(\mathbf{r})$ is bounded in any closed subset of $\mathbb{R}^3\backslash\{0\}$, and is bounded by const$/|\mathbf{r}|$ in the limit as $|\mathbf{r}| \to 0$, it follows that

$$\|M_0(z)\| = \|h(H_0+1)^{-1/2}\{(H_0+1)(H_0-z)^{-1}\}(H_0+1)^{-1/2}h\|$$
$$\leq \|h(H_0+1)^{-1/2}\|^2\,\|(H_0+1)(H_0-z)^{-1}\|,$$

so that $M_0(z)$ is bounded for fixed z such that $\operatorname{Im} z \neq 0$. On the other hand, if $h(\mathbf{r})$ is more singular than const$/|\mathbf{r}|$ at the origin (that is, if $\lim_{|\mathbf{r}|\to 0} |\mathbf{r}|\,h(\mathbf{r}) = \infty$), then $M_0(z)$ will be unbounded for all z. If $M_0(z)$ *is* bounded, we shall take $M_0(z)$ to mean the *closure* of the operator defined by the r.h.s. of (1). In that case, $M_0(z)$ is defined, by closure, as a bounded linear operator on the whole of $L^2(\mathbb{R}^3)$.

In order to consider the behaviour of $M_0(z)$ as z approaches the real axis from above or from below, we set $z=\lambda+i\varepsilon$ for some fixed $\lambda>0$, and estimate $M_0(\lambda+i\varepsilon)$, $M_0(\lambda-i\varepsilon)$ in the limits $\varepsilon\to 0+$. We shall do this by first of all obtaining a norm bound, as a function of ε, for $(d/d\varepsilon)M_0(\lambda+i\varepsilon)$ and $(d/d\varepsilon)M_0(\lambda-i\varepsilon)$, and then converting this, by integration with respect to ε, to corresponding bounds for M_0. Under the above assumption that h is bounded away from $\mathbf{r}=\mathbf{0}$, and that $h(\mathbf{r})=0(1/|\mathbf{r}|)$ as $|\mathbf{r}|\to 0$, these two derivatives exist as limits in operator norm, and we have

$$\begin{aligned}\frac{d}{d\varepsilon}M_0(\lambda+i\varepsilon)&=ih[G_0(\lambda+i\varepsilon)]^2h\\ \frac{d}{d\varepsilon}M_0(\lambda-i\varepsilon)&=-ih[G_0(\lambda-i\varepsilon)]^2h\end{aligned}\tag{3}$$

We shall also consider the asymptotic behaviour as $\varepsilon\to 0+$ of the operator

$$M_1(\lambda,\varepsilon)=h[G_0(\lambda+i\varepsilon)-G_0(\lambda-i\varepsilon)]h\tag{4}$$

Noting that

$$M_1(\lambda,\varepsilon)=2i\varepsilon h[(H_0-\lambda)^2+\varepsilon^2]^{-1}h=2i\varepsilon hG_0(\lambda+i\varepsilon)G_0(\lambda-i\varepsilon)h,\tag{5}$$

we may use equation (5) to define $M_1(\lambda,\varepsilon)$ as a bounded operator for a larger class of multiplicative operators h than that for which M_0 is defined. The r.h.s. of equation (5) may be written

$$2i\varepsilon hG_0(\lambda+i\varepsilon)\{hG_0(\lambda+i\varepsilon)\}^*,$$

and is bounded if and only if $hG_0(\lambda+i\varepsilon)$ is bounded. Suppose then, that h is bounded away from $\mathbf{r}=\mathbf{0}$, and that h satisfies in addition, for any finite R,

$$\int_{|\mathbf{r}|<R}|h(\mathbf{r})|^2\,d^3\mathbf{r}<\infty\tag{6}$$

Wave functions in the range of $G_0(\lambda+i\varepsilon)$ are locally bounded, and hence in the domain of h. Under the given hypothesis, $hG_0(\lambda+i\varepsilon)$ is defined as a linear operator on the entire Hilbert space, and hence bounded, by the Closed-Graph theorem. If h is bounded away from $\mathbf{r}=\mathbf{0}$, (6) is a necessary and sufficient condition for $M_1(\lambda,\varepsilon)$, defined by the r.h.s. of (5), to be bounded, and this is clearly weaker than the condition $h(\mathbf{r})\le \text{const}/|\mathbf{r}|$ for $M_0(\lambda+i\varepsilon)$ to be bounded.

The following Theorem summarises the main results of this paper, in the case of multiplicative functions $h(\mathbf{r})$ bounded by functions of $|\mathbf{r}|$.

Theorem 1. *Let $h(\mathbf{r})$ be a non-negative multiplicative function. Suppose that, for some $R>0$ (and hence for all R sufficiently large), $h(\mathbf{r})$ satisfies a bound of the form*

$$h(\mathbf{r})\le\frac{F(r)}{r^{1/2}},\quad |\mathbf{r}|\ge R,$$

where $F(r)$ is non-increasing for $r \geq R$ and

$$\int_R^\infty \frac{F^2(r)}{r}\,dr < \infty.$$

Then

(i) *if* $h(\mathbf{r}) \leq \frac{F(r)}{r^{1/2}}$ *for* $|\mathbf{r}| < R$, *where*

$$\int_0^R rF^2(r)\,dr < \infty,$$

then $\lim_{\varepsilon \to 0+} h[G_0(\lambda + i\varepsilon) - G_0(\lambda - i\varepsilon)]h$ *exists as a limit in operator norm*;

(ii) *if* $h(\mathbf{r}) \leq \frac{\text{const}}{r}$ *for* $|\mathbf{r}| \leq R$, *then* $\lim_{\varepsilon \to 0+} hG_0(\lambda + i\varepsilon)h$ *exists as a limit in operator norm. We shall defer the proof of Theorem* 1 *until Section* 5.

Remarks. 1. The square integrability condition governing the large-r behaviour of $F(r)$ is optimal, in the sense that if $h(\mathbf{r}) = F(r)/r^{1/2}$, where

$$\int_R^\infty \frac{F^2(r)}{r}\,dr = \infty,$$

then both $M_0(\lambda, \varepsilon)$ and $M_1(\lambda, \varepsilon)$ are unbounded in the limit as $\varepsilon \to 0+$. The condition allows a decay at infinity like $(\log r)^{-\beta}$ provided $\beta > \frac{1}{2}$.

The requirement that $F(r)$ decrease for $r > R$ may be relaxed somewhat, but the Theorem no longer holds if this requirement is dropped altogether.

2. The distinction between cases (i) and (ii) of the Theorem is a real one; under that hypotheses (i), $hG_0(\lambda + i\varepsilon)h$ need not be bounded for $\varepsilon > 0$.

3. The Theorem may be extended to cover limits such as $\lim_{\varepsilon \to 0+} h_1 G_0(\lambda + i\varepsilon)h_2$, involving a pair of multiplicative functions h_1, h_2; see also Lemma 4 below.

4. The methods developed in this paper may also be applied to estimate norms where the limit $\varepsilon \to 0+$ does not exist. As an example of such a result, we have

$$\left\| \frac{1}{r^{1/2}} G_0(\lambda + i\varepsilon) \frac{1}{r^{1/2}} \right\| = O(-\log \varepsilon), \quad \text{as} \quad \varepsilon \to 0+.$$

3. Operator bounds and the L.A.P.

The following Lemma shows how to estimate other norms, in terms of a norm bound for the operator $hG_0(\lambda + i\varepsilon)h$.

Lemma 1. *Let h be a real non-negative multiplication operator such that $D(H_0^{1/2}) \subseteq D(h)$ and define the function $m(\varepsilon)$ for $\varepsilon > 0$ by*

$$\|hG_0(\lambda + i\varepsilon)h\| = m(\varepsilon) \tag{7}$$

Then

$$\|G_0(\lambda+i\varepsilon)h\| \le (m(\varepsilon)/\varepsilon)^{1/2}, \tag{8}$$

$$\|H_0^{1/2}G_0(\lambda+i\varepsilon)h\| \le \left(\frac{|\lambda+i\varepsilon|\, m(\varepsilon)}{\varepsilon}\right)^{1/2} \tag{8'}$$

Moreover, if T_1, T_2 are any two bounded operators satisfying

$$\left.\begin{array}{l}\text{range } (T_1^*) \subseteq D(rh)\\ \text{range } (T_2) \subseteq D(rh)\end{array}\right\}. \tag{9}$$

where $D(T)$ denotes the domain of a linear operator T, we have the estimate, valid for $\operatorname{Im} z \ne 0$,

$$\begin{aligned}&\|T_1\{h[G_0(z)]^2h\}T_2 - (2z)^{-1}T_1\{hG_0(z)h\}T_2\|\\ &\quad \le (2|z|)^{-1}\{\|(rh)T_1^*\|\ \|\{H_0^{1/2}(H_0-z)^{-1}h\}T_2\|\\ &\qquad + \|(rh)T_2\|\ \|\{H_0^{1/2}(H_0-\bar z)^{-1}h\}T_1^*\|\}\end{aligned} \tag{10}$$

Proof. On taking adjoints, note that

$$\|hG_0(\lambda-i\varepsilon)h\| = m(\varepsilon).$$

From the resolvent identity

$$G_0(\lambda-i\varepsilon)G_0(\lambda+i\varepsilon) = \frac{1}{2i\varepsilon}\{G_0(\lambda+i\varepsilon) - G_0(\lambda-i\varepsilon)\},$$

we have

$$\begin{aligned}\|G_0(\lambda+i\varepsilon)h\|^2 &= \|\{G_0(\lambda+i\varepsilon)h\}^*G_0(\lambda+i\varepsilon)h\|\\ &= \|hG_0(\lambda-i\varepsilon)G_0(\lambda+i\varepsilon)h\|\\ &\le \frac{1}{2\varepsilon}\{\|hG_0(\lambda+i\varepsilon)h\| + \|hG_0(\lambda-i\varepsilon)h\|\} = \frac{m(\varepsilon)}{\varepsilon}.\end{aligned}$$

Hence (8) follows immediately. We obtain the same estimate for $\|G_0(\lambda-i\varepsilon)h\|$.

To prove (8′), note that

$$\|H_0^{1/2}G_0(\lambda+i\varepsilon)h\|^2 = \|hH_0G_0(\lambda-i\varepsilon)G_0(\lambda+i\varepsilon)h\|,$$

and use the resolvent identity

$$H_0G_0(\lambda-i\varepsilon)G_0(\lambda+i\varepsilon) = \frac{1}{2i\varepsilon}\{(\lambda+i\varepsilon)G_0(\lambda+i\varepsilon) - (\lambda-i\varepsilon)G_0(\lambda-i\varepsilon)\}$$

Now let T_1, T_2 be two bounded linear operators satisfying (9). Consider first the case $h=1$.

We rely on the identity, valid for any f, $g \in D(r)$ (domain of operator of multiplication by $|\mathbf{r}|$), $2iz\langle f, (H_0-z)^{-2}g\rangle$

$$= \sum_j \{\langle P_j(H_0-\bar z)^{-1}f, X_jg\rangle - \langle X_jf, P_j(H_0-z)^{-1}g\rangle\} + i\langle f, (H_0-z)^{-1}g\rangle \tag{11}$$

Equation (11) follows from evaluation of the commutator $[X_j, P_j(H_0 - z)^{-1}]$. This commutator can most readily be evaluated in the momentum space representation, with $X_j = i(\partial/\partial k_j)$, by considering the derivative with respect to k_j of the function $k_j(k^2 - z)^{-1}$.

With $h = 1$, the hypotheses (9) imply $R(T_1^*) \subseteq D(r)$, $R(T_2) \subseteq D(r)$, so that (11) holds with f, g replaced by $T_1^* f$, $T_2 g$ respectively. Dividing throughout by $2iz$, we then have

$$|\langle f, T_1[G_0(z)]^2 T_2 g\rangle - (2z)^{-1}\langle f, T_1 G_0(z) T_2 g\rangle|$$
$$\leq \frac{1}{2|z|} \sum_j \{|\langle P_j(H_0 - \bar{z})^{-1} T_1^* f, X_j T_2 g\rangle| + |\langle X_j T_1^* f, P_j(H_0 - z)^{-1} T_2 g\rangle|\} \quad (12)$$

By Schwarz's inequality,

$$\left\{\sum_j |\langle P_j(H_0 - \bar{z})^{-1} T_1^* f, X_j T_2 g\rangle|\right\}^2$$
$$\leq \left(\sum_j \|P_j(H_0 - \bar{z})^{-1} T_1^* f\|^2\right)\left(\sum_j \|X_j T_2 g\|^2\right)$$
$$= \|H_0^{1/2}(H_0 - \bar{z})^{-1} T_1^* f\|^2 \, \||\mathbf{r}|\, T_2 g\|^2.$$

Thus the first contribution to the r.h.s of (12) is bounded by $\frac{1}{2|z|}\|H_0^{1/2}(H_0 - \bar{z})^{-1} T_1^*\| \, \||\mathbf{r}|\, T_2\| \, \|f\| \cdot \|g\|$. The second contribution to the r.h.s. may similarly be estimated. Since each bound is proportional to $\|f\| \cdot \|g\|$, we can convert the inequality (12) into a bound for the operator norm in equation (10), which is now verified in the case $h = 1$.

The proof of (10) in the general case for h is not quite so clear cut, because the stated hypotheses do not imply range $(T_1^*) \subseteq D(h)$, range $(T_2) \subseteq D(h)$. However, equation (9) does imply range $(E_{|\mathbf{r}|>R} T_1^*) \subseteq D(h)$ and range $(E_{|\mathbf{r}|>R} T_2) \subseteq D(h)$, where $E_{|\mathbf{r}|>R}$ is the projection operator associated with the region $|\mathbf{r}| > R$. (I.e. $E_{|\mathbf{r}|>R}$ is the operator of multiplication by the characteristic function of the region $|\mathbf{r}| > R$; note that $\|hE_{|\mathbf{r}|>R} T_1^*\| = \||\mathbf{r}|^{-1} E_{|\mathbf{r}|>R} rhT_1^*\| \leq (1/R)\|rhT_1^*\|$, and similarly for T_2). We can therefore apply equation (11), with f, g replaced by $hE_{|\mathbf{r}|>R} T_1^* f$, $hE_{|\mathbf{r}|>R} T_2 g$, respectively.

Proceeding as before, we arrive at the inequality

$$|\langle f, T_1 E_{|\mathbf{r}|>R} h[G_0(z)]^2 h E_{|\mathbf{r}|>R} T_2 g\rangle - (2z)^{-1}\langle f, T_1 E_{|\mathbf{r}|>R} h G_0(z) h E_{|\mathbf{r}|>R} T_2 g\rangle|$$
$$\leq \frac{1}{2|z|}\{\|H_0^{1/2}(H_0 - \bar{z})^{-1} h E_{|\mathbf{r}|>R} T_1^* f\| \, \|E_{|\mathbf{r}|>R}(rh) T_2 g\|$$
$$+ \|E_{|\mathbf{r}|>R}(rh) T_1^* f\| \, \|H_0^{1/2}(H_0 - z)^{-1} h E_{|\mathbf{r}|>R} T_2 g\| \quad (13)$$

Under the stated hypotheses, $(H_0^{1/2}(H_0 - \bar{z})^{-1} h)^* = h(H_0 + 1)^{-1/2}[H_0^{1/2}(H_0 + 1)^{1/2}(H_0 - z)^{-1}]$ is defined on the entire Hilbert space, and hence is bounded, by the Closed-Graph theorem. It follows that $H_0^{1/2}(H_0 - \bar{z})^{-1} h$ is bounded, and similarly with z instead of $\bar{z}$. The operators $hG_0(z)h$, $h[G_0(z)]^2 h$, are also bounded. In the limit as $R \to 0$, $E_{|\mathbf{r}|>R}$ converges strongly to the identity operator. Taking the limit $R \to 0$, we can remove the projection operator from both sides of the inequality (13). Proceeding as before, (10) now follows in the general case.

The use of (10) in asymptotic analysis of the resolvent operator near the real axis lies in the fact that estimates for $[G_0(z)]^2$ may be reduced to estimates for $G_0(z)$. The following Lemma gives two applications of this idea.

Lemma 2. *Let $E_{|\mathbf{r}|<R}$ denote the projection operator, in position space, associated with the region $|\mathbf{r}|<R$; thus $E_{|\mathbf{r}|<R}$ is the operator of multiplication by the characteristic function of a ball of radius R, centered at the origin.*

Then the limits

$$\lim_{\varepsilon\to 0+} \frac{1}{r} G_0(\lambda+i\varepsilon)\frac{1}{r} \quad \text{and} \quad \lim_{\varepsilon\to 0+} E_{|\mathbf{r}|<R} G_0(\lambda+i\varepsilon) E_{|\mathbf{r}|<R}$$

exist as limits in operator norm, for every $\lambda>0$. We have, moreover, the following uniformly valid estimates, for λ, ε, R, R' all positive,

$$\text{(i)} \quad \|G_0(\lambda+i\varepsilon)E_{|\mathbf{r}|<R}\| \le C_1(\lambda)(R/\varepsilon)^{1/2}, \tag{14}$$

$$\text{(ii)} \quad \|E_{|\mathbf{r}|<R}G_0(\lambda+i\varepsilon)E_{|\mathbf{r}|<R'}\| \le C_2(\lambda)(RR')^{1/2} \tag{14'}$$

where the constants $C_1(\lambda)$, $C_2(\lambda)$ depend on λ but are independent of ε, R, R'.

Proof. The existence of the second norm limit in the statement of the Lemma follows from that of the first norm limit on noting that the operator $rE_{|\mathbf{r}|<R}$ and its adjoint are bounded. To prove existence of the first limit, start from (10) with $h=1/r$ and $T_1=T_2=I=$ identity operator in $L^2(\mathbb{R}^3)$. Since $rh=I$ in this case, (9) is trivially satisfied. We then have, from (10),

$$\left\|\frac{1}{r}[G_0(\lambda+i\varepsilon)]^2\frac{1}{r}\right\| \le \frac{1}{2|\lambda+i\varepsilon|}\left\{\left\|\frac{1}{r}G_0(\lambda+i\varepsilon)\frac{1}{r}\right\| + 2\left\|H_0^{1/2}G_0(\lambda+i\varepsilon)\frac{1}{r}\right\|\right\}$$

Using (8)' with $m(\varepsilon)=\|(1/r)G_0(\lambda+i\varepsilon)1/r\|$, this gives

$$\left\|\frac{1}{r}[G_0(\lambda+i\varepsilon)]^2\frac{1}{r}\right\| \le \frac{1}{2|\lambda|}\left(m(\varepsilon)+2\left(\frac{|\lambda|\, m(\varepsilon)}{\varepsilon}\right)^{1/2}\right) \tag{15}$$

From equation (3), starting from the bound $m(\varepsilon)\le \text{const}/\varepsilon$ as $\varepsilon\to 0$, we can integrate the inequality (15) with respect to ε to obtain the improved estimate

$$m(\varepsilon) \le \text{const} + \int_{\varepsilon}^{1} \frac{1}{2|\lambda|}\left(m(\varepsilon')+2\left(\frac{|\lambda|\, m(\varepsilon')}{\varepsilon'}\right)^{1/2}\right) d\varepsilon' \tag{16}$$

showing that $m(\varepsilon)$ is at worst logarithmically divergent in this limit. Repeating this argument with $m(\varepsilon')\le \text{const}\,|\log \varepsilon'|$ on the r.h.s. of (16), we find that in fact $m(\varepsilon)$ is bounded.

Hence

$$\left\|\frac{1}{r}[G_0(\lambda+i\varepsilon)]^2\frac{1}{r}\right\| \le \frac{\text{const}}{\varepsilon^{1/2}},$$

and this norm is integrable with respect to ε down to $\varepsilon = 0$. So $(1/r)G_0(\lambda + i\varepsilon)(1/r)$, and hence also $E_{|\mathbf{r}|<R}G_0(\lambda + i\varepsilon)E_{|\mathbf{r}|<R}$, converges to a limit in operator norm.

To derive (14), we take a linear combination of equation (11) with $z = \lambda + i\varepsilon$, $\lambda - i\varepsilon$ respectively to obtain

$$\frac{d}{d\varepsilon}\langle E_{|\mathbf{r}|<R}f, [G_0(\lambda + i\varepsilon) - G_0(\lambda - i\varepsilon)]E_{|\mathbf{r}|<R}g\rangle$$
$$= \langle E_{|\mathbf{r}|<R}f, i\{[G_0(\lambda + i\varepsilon)]^2 + [G_0(\lambda - i\varepsilon)]^2\}E_{|\mathbf{r}|<R}g\rangle$$
$$= \sum_j \{\langle E_{|\mathbf{r}|<R}f, P_jW(\lambda, \varepsilon)X_jE_{|\mathbf{r}|<R}g\rangle - \langle X_jE_{|\mathbf{r}|<R}f, P_jW(\lambda, \varepsilon)E_{|\mathbf{r}|<R}g\rangle\}$$
$$+ i\langle E_{|\mathbf{r}|<R}f, W(\lambda, \varepsilon)F_{|\mathbf{r}|<R}g\rangle, \tag{17}$$

where the operator $W(\lambda, \varepsilon)$ is given by

$$W(\lambda, \varepsilon) = \frac{1}{2(\lambda + i\varepsilon)}(H_0 - \lambda - i\varepsilon)^{-1} + \frac{1}{2(\lambda - i\varepsilon)}(H_0 - \lambda + i\varepsilon)^{-1}$$
$$= \frac{1}{(\lambda^2 + \varepsilon^2)}\{\lambda(H_0 - \lambda)[(H_0 - \lambda)^2 + \varepsilon^2]^{-1} + \varepsilon^2[(H_0 - \lambda)^2 + \varepsilon^2]^{-1}\}.$$

For fixed $\lambda > 0$, we can write

$$W(\lambda, \varepsilon) = G(H_0, \varepsilon) = \int_0^\infty G(x, \varepsilon)\, dE_x, \quad \text{where } \{E_x\}$$

is the spectral family of the operator H_0, and

$$G(x, \varepsilon) = \frac{1}{(\lambda^2 + \varepsilon^2)}\{\lambda(x - \lambda) + \varepsilon^2\}[(x - \lambda)^2 + \varepsilon^2]^{-1}.$$

Since

$$|G(x, \varepsilon)| \le \text{const} + \frac{\text{const}\,|x - \lambda|}{(x - \lambda)^2 + \varepsilon^2},$$

it follows that $\int_0^1 |G(x, \varepsilon)|\, d\varepsilon$ is bounded uniformly in x. By considering values of x satisfying respectively the inequalities $|x - \lambda| \le 1$ and $|x - \lambda| > 1$, we find also that $\int_0^1 x^{1/2}\,|G(x, \varepsilon)|\, d\varepsilon$ is bounded uniformly in x. The integrals $\int_1^\infty |G(x, \varepsilon)|\, d\varepsilon$ and $\int_1^\infty x^{1/2}\,|G(x, \varepsilon)|\, d\varepsilon$ are also bounded uniformly in x.

Thus

$$\left\|\int_\varepsilon^\infty d\varepsilon' W(\lambda, \varepsilon')\right\| = \left\|\int_0^\infty \left\{\int_\varepsilon^\infty d\varepsilon' G(x, \varepsilon')\right\} dE_x\right\| \le \sup_x \int_0^\infty |G(x, \varepsilon')|\, d\varepsilon',$$

so that this norm, and similarly the norm of $\int_\varepsilon^\infty d\varepsilon' H_0^{1/2}W(\lambda, \varepsilon')$, is bounded uniformly in ε. Since $P_jW(\lambda, \varepsilon) = (P_jH_0^{-1/2})H_0^{1/2}W(\lambda, \varepsilon)$, we have a similar bound for $\int_\varepsilon^\infty d\varepsilon' P_jW(\lambda, \varepsilon')$.

Using the fact that $\|X_jE_{|\mathbf{r}|<R}\| = R$, we can integrate equation (17) with respect to ε from ε to ∞, to obtain

$$\|E_{|\mathbf{r}|<R}[G_0(\lambda + i\varepsilon) - G_0(\lambda - i\varepsilon)]E_{|\mathbf{r}|<R}\| \le \text{const}\, R \tag{18}$$

The estimate (14) now follows on using the resolvent identity, as in the proof of (8). Actually, (18) follows from (14)′, and the corresponding result with $\lambda - i\varepsilon$, on setting $R = R'$. We shall argue in the reverse order, that is derive (14)′ from (14).

Suppose first that we have $\varepsilon \geq 1/R$. Then $\|E_{|\mathbf{r}|<R} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'}\| \leq \|G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'}\| \leq \text{const}\,(R'/\varepsilon)^{1/2} \leq \text{const}\,(RR')^{1/2}$, where we have used (14) with R' for R. Hence (14)′ holds provided $\varepsilon \geq 1/R$, and similarly provided $\varepsilon \geq 1/R'$. We need therefore only to consider the case in which both $\varepsilon < 1/R$ and $\varepsilon < 1/R'$. We shall suppose $\varepsilon < 1/R \leq 1/R'$; a similar proof holds with $\varepsilon < 1/R' \leq 1/R$.

We know that $\|(1/r) G_0(\lambda + i\varepsilon) 1/r\|$ is bounded in the limit as $\varepsilon \to 0+$, and it is also straightforward to verify that this norm is a bounded function of ε for $\varepsilon \geq \text{const} > 0$. Hence $\|(1/r) G_0(\lambda + i\varepsilon) 1/r\| \leq \text{const}$ for *all* $\varepsilon > 0$. Using the results $\|rE_{|\mathbf{r}|<R}\| = R$, $\|rE_{|\mathbf{r}|<R'}\| = R'$. we may deduce the estimate

$$\|E_{|\mathbf{r}|<R} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'}\| \leq \text{const}\,(RR').$$

This is a *better* estimate than (14)′ whenever $RR' < 1$. We shall therefore consider the case $RR' \geq 1$. We shall prove that

$$\int_0^{1/R} d\varepsilon \left\| \frac{d}{d\varepsilon} E_{|\mathbf{r}|<R} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'} \right\| \leq \text{const}\,(RR')^{1/2}. \tag{19}$$

Assuming (19), we have

$$\left\| E_{|\mathbf{r}|<R} \left[G_0\left(\lambda + \frac{i}{R}\right) - G_0(\lambda + i\varepsilon) \right] E_{|\mathbf{r}|<R'} \right\|$$
$$= \left\| \int_\varepsilon^{1/R} d\varepsilon \frac{d}{d\varepsilon} E_{|\mathbf{r}|<R} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'} \right\| \leq \text{const}\,(RR')^{1/2}.$$

We may then deduce (14)′ from the fact that this inequality has been shown to hold for $\varepsilon = 1/R$. It remains, then, only to prove (19). To do so, use (10) with $h = I$, $T_1 = E_{|\mathbf{r}|<R}$, $T_2 = E_{|\mathbf{r}|<R'}$, to obtain

$$\left\| \frac{d}{d\varepsilon} E_{|\mathbf{r}|<R} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'} \right\| \leq \text{const}\, \|E_{|\mathbf{r}|<R} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'}\|$$
$$+ \text{const}\, R\, \|H_0^{1/2} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'}\| + \text{const}\, R'\, \|H_0^{1/2} G_0(\lambda - i\varepsilon) E_{|\mathbf{r}|<R}\| \tag{20}$$

Using (14), the first term on the r.h.s. is bounded by $\text{const}\,(R/\varepsilon)^{1/2}$; for this term we have $\int_0^{1/R} d\varepsilon\, \|\cdot\| \leq \text{const} \leq \text{const}\,(RR')^{1/2}$ since we have $RR' \geq 1$.

For the second term on the r.h.s. of (20), use (14) with R' for R, together with (18), and proceed as in the proof of (8′) to obtain

$$\|H_0^{1/2} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R'}\|^2 \leq \text{const} \left(\left(\frac{R'}{\varepsilon}\right) + \left(\frac{R'}{\varepsilon}\right)^{1/2} \right).$$

This gives the bound $\text{const}\,\{R(R'/\varepsilon)^{1/2} + R(R'/\varepsilon)^{1/4}\}$ for the second term, and again we find

$$\int_0^{1/R} d\varepsilon \, \|\cdot\| \le \text{const}\,(RR')^{1/2}.$$

The same estimate applies on integrating the third term on the r.h.s. of (20); here we have R and R' exchanged, and we can write

$$\int_0^{1/R} d\varepsilon \, \|\cdot\| \le \int_0^{1/R'} d\varepsilon \, \|\cdot\| \le \text{const}\,(RR')^{1/2}.$$

Thus (19) is verified, and with it we have completed the proof of the Lemma.

Both of the bounds (14), (14)′ are optimal in the limits R, $R' \to \infty$, in the sense that the power $R^{1/2}$ cannot be replaced by any other function of R which is $o(R^{1/2})$ as $R \to \infty$. In the limits R, R', $\to 0$, both estimates can be improved. We have already seen that that r.h.s. of (14)′ can then be replaced by (RR'). On the other hand, for $h \in L^2(\mathbb{R}^3)$, $G_0(\lambda + i\varepsilon)h$ is an integral operator having kernel $(2\pi)^{-3/2}K(\mathbf{r} - \mathbf{r}')h(\mathbf{r}')$, where $K(\cdot)$ is the inverse Fourier transform of the function $(\mathbf{k}^2 - \lambda - i\varepsilon)^{-1}$. Since

$$\int \frac{d^3\mathbf{k}}{|\mathbf{k}^2 - \lambda - i\varepsilon|^2} \le \frac{\text{const}}{\varepsilon} \text{ in the limit } \varepsilon \to 0,$$

the kernel is Hilbert Schmidt and we have

$$\|G_0(\lambda + i\varepsilon)h\| \le \frac{\text{const}}{\varepsilon^{1/2}} \left(\int |h(\mathbf{r})|^2 \, d^3\mathbf{r} \right)^{1/2}. \tag{21}$$

This corresponds to the improved bound $C(\lambda)(R^3/\varepsilon)^{1/2}$ on the r.h.s. of (14), in the limit as $R \to 0$, and for $0 < \varepsilon < 1$.

Finally, in this section, let us note the estimate

$$\left\| \frac{1}{r} G_0(\lambda + i\varepsilon) E_{|\mathbf{r}|<R} \right\| \le C(\lambda) R^{1/2}. \tag{22}$$

The inequality (22) may be proved from (14) in the same way as for (14)′; here we need also the bound

$$\left\| \frac{1}{r} G_0(\lambda + i\varepsilon) \right\| \le \frac{\text{const}}{\varepsilon^{1/2}},$$

which follows from (7) and (8) and our bound for $\|(1/r)G_0(\lambda + i\varepsilon)(1/r)\|$.

4. An L.A.P. for short-range perturbations

As an application of Lemma 1, we shall prove the L.A.P. for a class of multiplicative functions h decaying more rapidly that $|\mathbf{r}|^{-1/2}$ at infinity. This corresponds, for example, to the existence of wave operators for scattering by potentials having a power decay better than $|\mathbf{r}|^{-1}$. In Lemma 4, we shall obtain a general criterion which includes these short-range perturbations as a special case. What is needed is a precise norm estimate for the square of the resolvent.

The technique used will be the iteration of operator inequalities.

Lemma 3. *Suppose that the multiplicative function h satisfies the conditions* $D(H_0^{1/2}) \subseteq D(h)$ *and* $D(H_0) \subseteq D(rh^2)$. *Then*

$$\|h[G_0(\lambda+i\varepsilon)]^2h\| \leq \frac{1}{2|\lambda+i\varepsilon|}\|hG_0(\lambda+i\varepsilon)h\| + \frac{\varepsilon^{-1/2}}{|\lambda+i\varepsilon|^{1/4}}\|hG_0(\lambda+i\varepsilon)h\|^{1/2}\|rh^2G_0(\lambda+i\varepsilon)\|^{1/2} \tag{23}$$

Proof. Let $T = h[G_0(\lambda+i\varepsilon)]^2h$. Then

$$T^*TT^* = h[G_0(\lambda-i\varepsilon)]^2h \cdot h[G_0(\lambda+i\varepsilon)]^2h \cdot h[G_0(\lambda-i\varepsilon)]^2h.$$

Applying (10), with $T_1 = T_2 = T^*$, we have

$$\|T^*TT^*\| \leq \frac{1}{2|\lambda+i\varepsilon|}\{\|T\|^2\|hG_0(\lambda+i\varepsilon)h\| + \|rh^2[G_0(\lambda+i\varepsilon)]^2h\|\,\|H_0^{1/2}(H_0-\lambda-i\varepsilon)^{-1}h\|\,\|T\| + \|rh^2[G_0(\lambda-i\varepsilon)]^2h\|\,\|H_0^{1/2}(H_0-\lambda+i\varepsilon)^{-1}h\|\,\|T\|\}.$$

Writing

$$\|rh^2[G_0(\lambda+i\varepsilon)]^2h\| \leq \|rh^2G_0(\lambda+i\varepsilon)\|\,\|G_0(\lambda+i\varepsilon)h\|,$$

with the corresponding result for $-\varepsilon$ instead of ε, and using (8), (8)′ to estimate $\|G_0(\lambda \pm i\varepsilon)h\|$ and $\|H_0^{1/2}G_0(\lambda \pm i\varepsilon)h\|$, we obtain

$$\|T^*TT^*\| \leq \frac{1}{2|\lambda+i\varepsilon|}\{\|T\|^2\|hG_0(\lambda+i\varepsilon)h\| + 2|\lambda+i\varepsilon|^{1/2}\|T\|\,\|hG_0(\lambda+i\varepsilon)h\|\,\|rh^2G_0(\lambda+i\varepsilon)\|/\varepsilon\} \tag{24}$$

Since T^*T is a positive operator, we have

$$\|(T^*T)^2\| = \|T^*T\|^2 = \|T\|^4$$

Also $\|(T^*T)^2\| = \|T^*TT^*T\| \leq \|T^*TT^*\|\,\|T\|$. Combining these two results gives

$$\|T\|^3 \leq \|T^*TT^*\|$$

From the explicit bound (24), this leads to an inequality of the form

$$\|T\|^2 \leq A\|T\| + B,$$

with prescribed positive values for A and B. Such an inequality implies

$$\|T\| \leq \frac{A+(A^2+4B)^{1/2}}{2} \leq A + B^{1/2},$$

which gives (23) on substituting for A and B. The inequality (23) may be "iterated" to give a precise estimate for the norm $m(\varepsilon) = \|hG_0(\lambda + i\varepsilon)h\|$. Suppose, then, that we can obtain a bound

$$m(\varepsilon) \leq m_n(\varepsilon).$$

On integration of (23) with respect to ε we then have $m(\varepsilon) \leq m_{n+1}(\varepsilon)$, where

$$m_{n+1}(\varepsilon) = a + b \int_\varepsilon^1 d\varepsilon' \{m_n(\varepsilon') + ((\varepsilon')^{-1} m_n(\varepsilon') \, \|rh^2 G_0(\lambda + i\varepsilon')\|)^{1/2}\} \tag{25}$$

We can thus generate a sequence of upper bounds to $m(\varepsilon)$. If, for n sufficiently large, the integral on the r.h.s. of (25) converges even in the limit $\varepsilon \to 0$, it follows that $\|h[G_0(\lambda + i\varepsilon)]^2 h\|$ is integrable with respect to ε from 0 to 1. In that case we may deduce the existence of the norm limit of $hG_0(\lambda + i\varepsilon)h$ as $\varepsilon \to 0$. As an illustration of these ideas, suppose for example that

$$h(\mathbf{r}) = |\mathbf{r}|^{-(1/2+\delta)}$$

for some δ with $0 < \delta < \frac{1}{2}$.

On the r.h.s. of (25), we have then,

$$\begin{aligned} \|rh^2 G_0(\lambda + i\varepsilon)\| &= \|E_{|\mathbf{r}| \leq \varepsilon^{-1/2}} r^{1-2\delta}(r^{-1} G_0(\lambda + i\varepsilon)) + E_{|\mathbf{r}| > \varepsilon^{-1/2}} r^{-2\delta} G_0(\lambda + i\varepsilon)\| \\ &\leq \varepsilon^{\delta - 1/2} \, \|r^{-1} G_0(\lambda + i\varepsilon)\| + \varepsilon^{\delta} \, \|G_0(\lambda + i\varepsilon)\| \\ &\leq \text{const } \varepsilon^{\delta - 1}, \text{ in view of (8) with } h = \frac{1}{r}. \end{aligned}$$

Assume inductively that $m(\varepsilon) \leq m_n(\varepsilon) = \text{const}/\varepsilon^{\gamma_n}$ for some $\gamma_n > 0$; h is less singular that $1/|\mathbf{r}|$ at the origin, so that it is straightforward to establish this inequality with $\gamma_1 = 1$.

On the r.h.s. of (25) the integrand is of order $(\varepsilon')^{-(1+\gamma_n/2-\delta/2)}$. On integration, we obtain

$$m(\varepsilon) \leq m_{n+1}(\varepsilon) = \text{const}/\varepsilon^{\gamma_{n+1}}, \quad \text{where} \quad \gamma_{n+1} = (\gamma_n)/2 - \delta/2 \tag{26}$$

if $\gamma_n > \delta$. If $\gamma_n < \delta$, then $\|h[G_0(\lambda + i\varepsilon)]^2 h\|$ is integrable; if $\gamma_n = \delta$ we obtain a logarithmic bound for m_{n+1} and again $\|h[G_0(\lambda + i\varepsilon)]^2 h\|$ is integrable on estimating the r.h.s. of (25) with $n + 1$ for n.

In any case, (26) shows that γ_n decreases with n until ultimately we arrive at an estimate of the form $m(\varepsilon) \leq \text{const}/\varepsilon^{\gamma}$ for some $\gamma < \delta$. It then follows that $hG_0(\lambda + i\varepsilon)h$ has a norm limit as $\varepsilon \to 0+$, and similarly as $\varepsilon \to 0-$.

This result is a special case of Theorem 1 in which h has a power decay at infinity. The following Lemma gives a criterion for norm convergence which sets this result in a more general context.

Lemma 4. *Suppose h satisfies the domain conditions*

$$D(H_0^{1/2}) \subseteq D(h) \quad \textit{and} \quad D(H_0) \subseteq D(rh^2),$$

and that

$$\int_0^1 d\varepsilon(\varepsilon)^{-1/2}\,\|rh^2G_0(\lambda+i\varepsilon)\|^{1/2}<\infty.$$

Then $\lim_{\varepsilon\to 0+} hG_0(\lambda+i\varepsilon)h$ *exists as a limit in operator norm. If* h_1, h_2 *are two multiplicative functions, each satisfying the above domain conditions, and such that*

$$\int_0^1 d\varepsilon(\varepsilon)^{-1/2}\,\|rh_i^2G_0(\lambda+i\varepsilon)\|^{1/2}<\infty \quad (i=1,2),$$

then $\lim_{\varepsilon\to 0+} h_1G_0(\lambda+i\varepsilon)h_2$ *exists in operator norm.*

Proof. Consider first the case of a single multiplicative function h, satisfying the conditions of the Lemma. Then $h(H_0+1)^{-1/2}$ is bounded so that

$$\begin{aligned}\|hG_0(\lambda+i\varepsilon)h\| &\le \|h(H_0+1)^{-1/2}\|\;\|(H_0+1)^{1/2}G_0(\lambda+i\varepsilon)h\|\\ &\le \text{const}/\varepsilon \quad \text{as} \quad \varepsilon\to 0+.\end{aligned}$$

Iterate the inequality (25) with $m_1(\varepsilon)=\text{const}/\varepsilon$.

Since $m_n(\varepsilon)\ge\text{const}>0$ for all n, we have $|m_n^{1/2}-m_{n-1}^{1/2}|\le\text{const}\,|m_n-m_{n-1}|$. By Schwarz's inequality we find, for constants b, c,

$$\begin{aligned}|m_{n+1}(\varepsilon)-m_n(\varepsilon)| &\le b\int_\varepsilon^1 d\varepsilon'\,|m_n(\varepsilon')-m_{n-1}(\varepsilon')|\\ &\quad + c\left(\int_\varepsilon^1 d\varepsilon'\,|m_n(\varepsilon')-m_{n-1}(\varepsilon')|^2\right)^{1/2},\end{aligned}$$

uniformly for ε in any fixed interval $[\alpha, 1]$ with $\alpha>0$. This leads by induction to an estimate of the form

$$|m_{n+1}(\varepsilon)-m_n(\varepsilon)|\le\frac{c^n(1-\varepsilon)^{(n-1)/2}}{\sqrt{(n-1)!}},$$

and standard iterative techniques for Volterra integral equations allow us to deduce the uniform convergence, as $n\to\infty$, of $m_n(\varepsilon)$ to a limiting function $\bar{m}(\varepsilon)$ in $[\alpha, 1]$ for any $\alpha>0$.

Thus $m(\varepsilon)\le\lim_{n\to\infty} m_n(\varepsilon)$ where, on taking the limit $n\to\infty$ in (25), $\bar{m}=\lim_{n\to\infty} m_n$ satisfies the integral equation

$$\bar{m}(\varepsilon)=a+b\int_\varepsilon^1 d\varepsilon'\{\bar{m}(\varepsilon')+((\varepsilon')^{-1}\bar{m}(\varepsilon')\,\|rh^2G_0(\lambda+i\varepsilon')\|)^{1/2}\}$$

The function $\bar{m}$ is therefore the solution of the differential equation

$$\frac{d\bar{m}(\varepsilon)}{d\varepsilon}=-b\bar{m}(\varepsilon)-b\left(\frac{\bar{m}(\varepsilon)}{\varepsilon}\right)^{1/2}\|rh^2G_0(\lambda+i\varepsilon)\|^{1/2},$$

subject to the initial condition $\bar{m}(1)=a$. This is actually a *linear* differential equation for $\bar{m}^{1/2}$, of which the solution is

$$\bar{m}^{1/2}(\varepsilon) = a^{1/2} e^{(1-\varepsilon)b/2} + b \int_\varepsilon^1 \frac{\exp(b(\varepsilon' - \varepsilon)/2) \, \|rh^2 G_0(\lambda + i\varepsilon')\|^{1/2}}{2(\varepsilon')^{1/2}}.$$

Provided $\int_0^1 d\varepsilon (\varepsilon)^{-1/2} \|rh^2 G_0(\lambda + i\varepsilon)\|^{1/2} < \infty$, $\bar{m}(\varepsilon)$ is bounded in the limit $\varepsilon \to 0$. It follows in that case that $\|h[G_0(\lambda + i\varepsilon)]^2 h\|$ is integrable from 0 to 1, so that $hG_0(\lambda + i\varepsilon)h$ has a norm limit as $\varepsilon \to 0+$. In the case of *two* multiplicative functions h_1, h_2, proceed as in the proof of Lemma 3, with, now

$$T = h_1[G_0(\lambda + i\varepsilon)]^2 h_2.$$

Instead of (24), we now have

$$\begin{aligned} \|T^*TT^*\| \le \frac{1}{2\,|\lambda + i\varepsilon|} \{ &\|T\|^2 \, \|h_1 G_0(\lambda + i\varepsilon) h_2 \\ &+ |\lambda + i\varepsilon|^{1/2} \, \|T\| \, \|h_1 G_0(\lambda + i\varepsilon) h_1\| \, \|rh_2^2 G_0(\lambda + i\varepsilon)\| / \varepsilon \\ &+ |\lambda + i\varepsilon|^{1/2} \, \|T\| \, \|h_2 G_0(\lambda + i\varepsilon) h_2\| \, \|rh_1^2 G_0(\lambda + i\varepsilon)\| / \varepsilon \}. \end{aligned}$$

Moreover, since we can take $h = h_i$ $(i = 1, 2)$ in the first part of the Corollary, we know that $\|h_i G_0(\lambda + i\varepsilon) h_i\|$ is bounded in the limit $\varepsilon \to 0+$. Proceeding again as in the proof of Lemma 3, (23) becomes

$$\begin{aligned} \|h_1[G_0(\lambda + i\varepsilon)]^2 h_2\| &\le \text{const} \, \|h_1 G_0(\lambda + i\varepsilon) h_2\| \\ &+ \text{const} \, \varepsilon^{-1/2} \{ \|rh_1^2 G_0(\lambda + i\varepsilon)\|^{1/2} + \|rh_2^2 G_0(\lambda + i\varepsilon)\|^{1/2} \}. \end{aligned}$$

This leads to a linear differential equation for $\bar{m}^{1/2}(\varepsilon)$, and the proof of the second part of the Lemma follows as before. The following Corollary shows that the conditions of Lemma 4 are satisfied by a large class of functions $h(\mathbf{r})$.

Corollary. *For some $R > 0$, let $h(\mathbf{r})$ be a non-negative multiplicative function such that*

$$h(\mathbf{r}) \le \frac{\text{const}}{|\mathbf{r}|} \quad \textit{for} \quad |\mathbf{r}| < R, \quad \textit{and}$$

$$h(\mathbf{r}) \le \frac{F(r)}{r^{1/2}} \quad \textit{for} \quad |\mathbf{r}| \ge R,$$

where $F(r)$ is non increasing and

$$\int_R^\infty \frac{F(r)}{r} \, dr < \infty.$$

Then h satisfies the conditions of Lemma 4; in particular, $\lim_{\varepsilon \to 0+} hG_0(\lambda + i\varepsilon)h$ exists as a limit in operator norm.

Proof. The domain conditions on h are easily verified, and it remains only to verify the integrability, near $\varepsilon = 0$, of $\varepsilon^{-1/2} \|rh^2 G_0(\lambda + i\varepsilon)\|^{1/2}$, under the stated conditions. To do so, we divide position space into three disjoint regions. We take ε sufficiently small that $\varepsilon \leq 1/R^2$.

(a) Region $|\mathbf{r}| < R$. In this region, $h(\mathbf{r}) \leq \text{const}/|\mathbf{r}|$, so that $\|E_{|\mathbf{r}|<R} rh^2 G_0(\lambda + i\varepsilon)\| \leq \text{const}\, \|(1/r)G_0(\lambda + i\varepsilon)\| \leq \text{const}\, \varepsilon^{-1/2}$.

(b) Region $R \leq |\mathbf{r}| \leq \varepsilon^{-1/2}$. In this region, since $F(r)$ is decreasing, we have $F(r) \leq F(R)$. So $h(\mathbf{r}) \leq \text{const}\, r^{-1/2}$, and rh^2 is bounded. Hence, with $0 < \delta < \frac{1}{2}$,

$$\begin{aligned}\|E_{R<|\mathbf{r}|\leq\varepsilon^{-1/2}} rh^2 G_0(\lambda + i\varepsilon)\| &\leq \text{const}\, \|E_{R<|\mathbf{r}|\leq\varepsilon^{-1/2}} G_0(\lambda + i\varepsilon)\| \\ &= \text{const} \left\| r^{1/2} E_{R<|\mathbf{r}|\leq\varepsilon^{-1/2}} \frac{1}{r^{1/2}} G_0(\lambda + i\varepsilon) \right\| \\ &\leq \text{const}\, \|r^{1/2+\delta} E_{R<|\mathbf{r}|\leq\varepsilon^{-1/2}}\| \cdot \left\| \frac{1}{r^{1/2+\delta}} G_0(\lambda + i\varepsilon) \right\|\end{aligned}$$

Since $r^{1/2+\delta} \leq \varepsilon^{-1/4-\delta/2}$ in this region, and using (8) with $h = 1/r^{1/2+\delta}$, we have the estimate

$$\|E_{R<|\mathbf{r}|\leq\varepsilon^{-1/2}} rh^2 G_0(\lambda + i\varepsilon)\| \leq \text{const}\, \varepsilon^{-3/4-\delta/2}$$

(c) Region $|\mathbf{r}| > \varepsilon^{-1/2}$. Since $F(r)$ is decreasing, we have, here, $F(r) \leq F(\varepsilon^{-1/2})$. Hence

$$\begin{aligned}&\|E_{|\mathbf{r}|>\varepsilon^{-1/2}} rh^2 G_0(\lambda + i\varepsilon)\| \\ &\quad = \|E_{|\mathbf{r}|>\varepsilon^{-1/2}} F^2(r) G_0(\lambda + i\varepsilon)\| \leq F^2(\varepsilon^{-1/2})\, \|G_0(\lambda + i\varepsilon)\| \\ &\quad = \varepsilon^{-1} F^2(\varepsilon^{-1/2}).\end{aligned}$$

Combining the bounds for the three regions (a), (b), (c) above, we have

$$\|rh^2 G_0(\lambda + i\varepsilon)\| \leq \text{const}\, \{\varepsilon^{-3/4-\delta/2} + \varepsilon^{-1} F^2(\varepsilon^{-1/2})\}$$

It follows easily that

$$\varepsilon^{-1/2}\, \|rh^2 G_0(\lambda + i\varepsilon)\|^{1/2} \leq \text{const}\, \{\varepsilon^{-7/8-\delta/4} + \varepsilon^{-1} F(\varepsilon^{-1/2})\}.$$

This gives

$$\int_0^{1/R^2} d\varepsilon(\varepsilon)^{-1/2}\, \|rh^2 G_0(\lambda + i\varepsilon)\|^{1/2} \leq \text{const} + \text{const} \int_0^{1/R^2} d\varepsilon(\varepsilon)^{-1} F(\varepsilon^{-1/2}).$$

The integral on the r.h.s., on making the change of variable $\varepsilon = 1/r^2$, is just $2 \int_R^\infty r^{-1} F(r)\, dr$, and is convergent under the hypotheses of the Corollary. Hence also $\int_0^1 d\varepsilon(\varepsilon)^{-1/2}\, \|rh^2 G_0(\lambda + i\varepsilon)\|^{1/2} < \infty$, and the result is proved.

5. Proof of Theorem 1

(i) Let $h(\mathbf{r})$ satisfy $h(\mathbf{r}) \leq F(r)/r^{1/2}$, where $F(r)$ is non-increasing for $r \geq R$, and

$$\int_0^R rF^2(r)\, dr < \infty, \qquad \int_R^\infty \frac{F^2(r)}{r}\, dr < \infty.$$

We first estimate $\|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon)\|$ for large R. It will be convenient to suppose $R = 2^N$ for some positive integer N, which is taken to be large enough so that $F(r)$ is non-increasing for $r \geq 2^{N-1}$.

Let $\mathscr{E}_n$ denote the region $2^n < |\mathbf{r}| \leq 2^{n+1}$ in position psace, and let E_n be the corresponding projection operator; E_n is the operator of multiplication by the characteristic function of the set $\mathscr{E}_n$.

For arbitrary $f \in L^2(\mathbb{R}^3)$, we have

$$\|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon) f\|^2 = \sum_{n=N}^{\infty} \|E_n h G_0(\lambda + i\varepsilon) f\|^2.$$

Hence

$$\|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon)\|^2 \leq \sum_{n=N}^{\infty} \|E_n h G_0(\lambda + i\varepsilon)\|^2, \tag{27}$$

provided the sum on the r.h.s. converges.

For $\mathbf{r} \in \mathscr{E}_n$, we have

$$h(\mathbf{r}) \leq \frac{F(r)}{r^{1/2}} \leq \frac{F(2^n)}{2^{n/2}},$$

so that

$$\|E_n h G_0(\lambda + i\varepsilon)\|^2 \leq \frac{F^2(2^n)}{2^n} \|E_n G_0(\lambda + i\varepsilon)\|^2.$$

However, from Lemma 2 with the projection $E_{|\mathbf{r}| \leq 2^{n+1}}$ in (14), we obtain, on taking the adjoint operator with $-\varepsilon$ for ε,

$$\|E_n G_0(\lambda + i\varepsilon)\|^2 \leq C_1^2(\lambda) 2^{n+1}/\varepsilon,$$

so that

$$\|E_n h G_0(\lambda + i\varepsilon)\|^2 \leq 2C^2(\lambda) F^2(2^n)/\varepsilon$$

Substituting this bound into (27) now leads to

$$\varepsilon \|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon)\|^2 \leq 2C_1^2(\lambda) \sum_{n=N}^{\infty} F^2(2^n) \tag{28}$$

In that case, for $n \geq N$, we know that $F(r)$ is decreasing over the *interval* $2^{n-1} < r \leq 2^n$ defined by $\mathscr{E}_{n-1}$. Hence, for $r \in \mathscr{E}_{n-1}$, we have

$$\frac{F^2(r)}{r} \geq \frac{F^2(2^n)}{2^n}.$$

It follows that

$$\int_{\mathscr{E}_{n-1}} \frac{F^2(r)}{r}\, dr \geq \int_{\mathscr{E}_{n-1}} \frac{F^2(2^n)}{2^n}\, dr = \tfrac{1}{2} F^2(2^n).$$

From (28) we have, then, with $R = 2^N$,

$$\varepsilon\ \|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon)\|^2 \le 4C_1^2(\lambda) \sum_{n=N}^{\infty} \int_{\mathscr{E}_{n-1}} \frac{F^2(r)}{r}\, dr$$

$$= 4C_1^2(\lambda) \int_{R/2}^{\infty} \frac{F^2(r)}{r}\, dr \tag{29}$$

We also have, from (21) on taking adjoints, with $E_{|\mathbf{r}|\le R}h$ for h,

$$\varepsilon\ \|E_{|\mathbf{r}|\le R} h G_0(\lambda + i\varepsilon)\|^2 \le \text{const} \int_{|\mathbf{r}|\le R} h^2(\mathbf{r})\, d^3\mathbf{r} \le \text{const} \int_0^R rF^2(r)\, dr.$$

Combining this result with (29), shows that in fact

$$\varepsilon\ \|hG_0(\lambda + i\varepsilon)\|^2 \le \text{const}, \tag{30}$$

for fixed $\lambda > 0$, in the limit $\varepsilon \to 0$. A further consequence of (29) is the result that

$$\lim_{R\to\infty} \varepsilon\ \|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon)\|^2 = 0, \tag{30$'$}$$

uniformly in ε for $0 < \varepsilon < 1$.

Given any $\theta > 0$, (30) and (30)′ imply that we can find R sufficiently large that

$$\varepsilon^2\ \|hG_0(\lambda + i\varepsilon)\|^2 \cdot \|E_{|\mathbf{r}|>R} h G_0(\lambda + i\varepsilon)\|^2 < \theta^2/2^7, \tag{31}$$

for all ε in the interval $0 < \varepsilon < 1$. Having fixed R such that (31) is satisfied, let $E^{(n)}$ be the projection operator in position space for the region $\mathscr{E}^{(n)}$ defined by

$$\mathscr{E}^{(n)} = \{\mathbf{r} \in \mathbb{R}^3;\ |\mathbf{r}| \le R,\ h(\mathbf{r}) > n/r\}.$$

I.e. $E^{(n)}$ is the operator of multiplication by the characteristic function of the set $\mathscr{E}^{(n)}$. Noting that $h(\mathbf{r})$ is square integrable over the region $|\mathbf{r}| \le R$, we have, from (21) with $E^{(n)}h$ for h,

$$\varepsilon\ \|E^{(n)} h G_0(\lambda + i\varepsilon)\|^2 \le \text{const} \int_{\mathscr{E}^{(n)}} h^2(\mathbf{r})\, d^3\mathbf{r} \to 0 \quad \text{as} \quad n \to \infty,$$

by the Lebesgue Dominated-Convergence Theorem. Using (30) we can therefore find n sufficiently large that

$$\varepsilon^2\ \|hG_0(\lambda + i\varepsilon)\|^2 \cdot \|E^{(n)} h G_0(\lambda + i\varepsilon)\|^2 < \theta^2/2^7 \tag{31$'$}$$

Let E be the projection operator in position space for the region $\mathscr{E}$ defined to be

$$\mathscr{E} = \mathscr{E}^{(n)} \cup \{\mathbf{r} \in \mathbb{R}^3;\ |\mathbf{r}| > R\},$$

and $E^\perp = 1 - E$ be the projection operator for the complement $\mathscr{E}^c$ of this region. For $\mathbf{r} \in \mathscr{E}^c$, we have $h(\mathbf{r}) \le n/r$. By the first part of Lemma 2 this implies the convergence of $E^\perp h G_0(\lambda \pm i\varepsilon) E^\perp h$ in operator norm in the limit as $\varepsilon \to 0+$. Hence also the norm convergence of $E^\perp M_1(\lambda, \varepsilon) E^\perp$, where $M_1(\lambda, \varepsilon)$ is given by

(4). This allows us to assert that

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0+} E^{\perp}\{M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2)\}E^{\perp} = 0. \tag{32}$$

Summing inequalities (31), (31)′ gives

$$\varepsilon^2 \|hG_0(\lambda + i\varepsilon)\|^2 \|EhG_0(\lambda + i\varepsilon)\|^2 < \theta^2/64 \tag{33}$$

Let us now write

$$\begin{aligned} &M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2) \\ &\quad = E^{\perp}\{M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2)\}E^{\perp} \\ &\qquad + E\{M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2)\}E^{\perp} + \{M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2)\}E \end{aligned} \tag{34}$$

From equation (5) we obtain

$$\begin{aligned} &\|EM_1(\lambda, \varepsilon)E^{\perp}\| \le 2\varepsilon \|EhG_0(\lambda + i\varepsilon)\| \cdot \|hG_0(\lambda + i\varepsilon)\| \\ &\qquad < \theta/4, \text{ by (33), and similarly } \|M_1(\lambda, \varepsilon)E\| < \theta/4. \end{aligned}$$

Applying these bounds, with $\varepsilon = \varepsilon_1$, ε_2 respectively on the r.h.s. of (34), and using (32), gives

$$\limsup_{\varepsilon_1, \varepsilon_2 \to 0+} \|M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2)\| \le 4 \cdot \theta/4 = \theta.$$

But $\theta > 0$ was arbitrary,and it follows that $M_1(\lambda, \varepsilon_1) - M_1(\lambda, \varepsilon_2)$ converges to zero in operator norm. By completeness of the Banach space of bounded linear operators on $\mathscr{H}$, $M_1(\lambda, \varepsilon)$ converges in operator norm, and the proof of the Theorem follows, under hypothesis (i).

(ii) Suppose that (ii) of the Theorem holds, and as before set $R = 2^N$ for some integer N such that $F(r)$ is non-increasing for $r \ge 2^{N-1}$.

Let T be a bounded linear operator on a Hilbert Space $\mathscr{H}$, and let $\{F_j\}$ be a family of (orthogonal) projection operators satisfying $F_iF_j = 0$, for $i \ne j$, and $\sum_i F_i = 1$. For arbitrary $f \in \mathscr{H}$, we have

$$\|T^*f\|^2 = \sum_j \|F_jT^*f\|^2,$$

so that $\|T\|^2 = \|T^*\|^2 \le \sum_j \|F_jT^*\|^2 = \sum_j \|TF_j\|^2$, on taking adjoints of those operators within the norm. On repeating the argument, $\|TF_j\|^2 \le \sum_i \|F_iTF_j\|^2$, so that, finally,

$$\|T\|^2 \le \sum_{i,j} \|F_iTF_j\|^2. \tag{35}$$

Let us apply inequality (35) to the operator

$$T = T(\varepsilon_1, \varepsilon_2) = h[G_0(\lambda + i\varepsilon_1) - G_0(\lambda + i\varepsilon_2)]h, \tag{36}$$

and take the family $\{F_i\}$ of projections to consist of the single projection $E_{|\mathbf{r}|<R}$, together with the sequence $\{E_n\}$, $n = N, N+1, N+2, \ldots$ defined previously.

Thus

$$\|T(\varepsilon_1, \varepsilon_2)\|^2 \le \|E_{|\mathbf{r}|<R}T(\varepsilon_1, \varepsilon_2)E_{|\mathbf{r}|<R}\|^2$$
$$+ \sum_{n=N}^{\infty} \|E_{|\mathbf{r}|<R}T(\varepsilon_1, \varepsilon_2)E_n\|^2$$
$$+ \sum_{n=N}^{\infty} \|E_nT(\varepsilon_1, \varepsilon_2)E_{|\mathbf{r}|<R}\|^2 + \sum_{n,n'=N}^{\infty} \|E_nT(\varepsilon_1, \varepsilon_2)E_{n'}\|^2 \tag{37}$$

To estimate the final sum on the r.h.s. of (37), note that, by the hypotheses of the Theorem,

$$h(\mathbf{r}) \le \frac{F(r)}{r^{1/2}} \le \frac{F(2^n)}{2^{n/2}} \quad \text{for} \quad \mathbf{r} \in \mathscr{E}_n,$$

and a similar bound holds for $\mathbf{r} \in \mathscr{E}_{n'}$. With T given by (36), lemma 2 then gives

$$\|E_nT(\varepsilon_1, \varepsilon_2)E_{n'}\|^2 \le 4(C_2(\lambda))^2\{2^{n+1}2^{n'+1}\} \frac{F^2(2^n)F^2(2^{n'})}{2^n \cdot 2^{n'}}$$
$$= 16(C_2(\lambda))^2F^2(2^n)F^2(2^{n'}), \tag{38}$$

where we have used (14)′ with $R = 2^{n+1}$, $R' = 2^{n'+1}$ and $\varepsilon = \varepsilon_1, \varepsilon_2$ respectively. Proceeding as for equation (29), the terms of the final sum in (37) are bounded uniformly in $\varepsilon_1, \varepsilon_2$, and for this sum we have a bound

$$2^2 \cdot 16(C_2(\lambda))^2\left(\int_{R/2}^{\infty} \frac{F^2(r)}{r}\,dr\right)^2 < \infty.$$

Since, by hypothesis, $h(\mathbf{r}) \le \text{const}/r$ for $|\mathbf{r}| < R$, we may use (22) to estimate the third sum on the r.h.s of (37). For fixed R, we obtain for this sum a bound of the form

$$\text{const} \int_{R/2}^{\infty} \frac{F^2(r)}{r}\,dr,$$

and a similar result holds for the second sum on the r.h.s. of (37). The first term on the r.h.s. is bounded uniformly in $\varepsilon_1, \varepsilon_2$ as $\varepsilon_1, \varepsilon_2 \to 0+$, according to the first conclusion of Lemma 2. Note, also, that each individual term of the sums on the r.h.s. of (37) converges to zero as $\varepsilon_1, \varepsilon_2 \to 0+$. This is because $(1/r)G_0(\lambda + i\varepsilon)(1/r)$ converges in norm to a limit as $\varepsilon \to 0$, and moreover $h(\mathbf{r}) \le \text{const}/r$ in each of the regions $|\mathbf{r}| < R$, $2^n < |\mathbf{r}| \le 2^{n+1}$ (the constant in the bound depending, however, on the value of n). An application of the Lebesgue Dominated-Convergence theorem as applied to series now allows us to conclude, from (37), that $T(\varepsilon_1, \varepsilon_2)$ converges in norm to zero as $\varepsilon_1, \varepsilon_2 \to 0+$. Given the explicit form (36) of T, we have proved the norm convergence of $hG_0(\lambda + i\varepsilon)h$, and with it the remainder of Theorem 1.

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