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Scattering theory in external fields slowly decaying in time

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Abstract. The scattering theory associated with the Dirac equation for a class of time-dependent external fields is analyzed, to show that the conditions required for second quantization of the electron-positron field are satisfied; the necessary and sufficient condition for this is that the off-diagonal parts S_{+-}, S_{-+} of the one-particle S-operator are Hilbert–Schmidt. The potentials considered are solutions of the homogeneous Maxwell equations. They decay slowly in time in such a way that previous methods do not work.

First, the existence of the wave operators is proven by means of Cook's method and it is shown that for weak enough potentials the corresponding perturbation series converge. Then the scattering operator is investigated and it is shown that every term in the perturbation series for S_{+-}, S_{-+} is indeed a HS-operator. The HS-convergence of the series cannot be proven by the methods of this work, but strong indications for this are obtained.

1. Introduction

The initial motivation of this work is to be found in a paper by Scharf [1], where a nonperturbative approach to quantum electrodynamics is described. Following the original ideas of Feynman [2], an expression for the S-matrix element

$$(\phi, S_{\text{full}}\psi) \tag{1.1}$$

is obtained where S_{full} means the full S-matrix of QED and both the incoming and the outgoing states are coherent states of the radiation field:

$$\phi = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} [a^+(f)]^n \Omega_0,$$

$$\psi = e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^{\infty} \frac{\beta^m}{m!} [a^+(g)]^m \Omega_0.$$

Here $a^+(g)$, $a^+(f)$ are the creation operators for an incoming respectively outgoing photon with normalized wave function

$$\begin{aligned} g^\mu(x) &= (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2|\mathbf{k}|}} \sum_{\lambda=1}^{2} \varepsilon^\mu(\mathbf{k}, \lambda)\{g(\mathbf{k}, \lambda)e^{-ikx} + g^*(\mathbf{k}, \lambda)e^{ikx}\} \\ &= g_+^\mu(x) + g_-^\mu(x) \end{aligned} \tag{1.2}$$

with

$$k_0 = |\mathbf{k}|, \qquad \varepsilon^\mu(\mathbf{k}, \lambda): \qquad \text{transverse polarisation vector,}$$

$$\alpha, \beta \in \mathbb{C}, \qquad \Omega_0: \qquad \text{vacuum state,}$$

$$\int d^3k \sum_{\lambda=1}^{2} |g(\mathbf{k}, \lambda)|^2 = 1: \quad \text{normalisation}$$

and similarly for $f^\mu(x)$.

According to the paper by Scharf, the S-matrix element (1.1) can be written as the following functional integral over the C-number potential A:

$$(\phi, S_{\text{full}}\psi) = e^{-\frac{1}{2}|\alpha|^2} e^{-\frac{1}{2}|\beta|^2} e^{(\alpha f, \beta g)} \cdot \frac{1}{C} \int e^{iW_0[A]} (\Omega, \mathbb{S}[A + \alpha^* f^- + \beta g^+]\Omega)[dA] \tag{1.3}$$

where $W_0[A]$ is the free action of the radiation field. $\mathbb{S}[A]$ is the scattering operator in Fock space for a quantized electron-positron field in the fixed C-number electromagnetic potential A and C is a normalization constant.

Looking at formula (1.3), the problem arises, whether for an external potential of the form

$$A = \alpha^* f^- + \beta g^+ \tag{1.4}$$

the S-operator $\mathbb{S}$ exists. A necessary and sufficient condition for this is that S_{+-} and S_{-+} are Hilbert–Schmidt. Here

$$S_{+-} = P_+ S P_-, \qquad S_{-+} = P_- S P_+, \tag{1.5}$$

S is the one particle S-operator (4.1) and P_+, P_- denote the projectors corresponding to the positive and negative part of the spectrum of the free Dirac–Hamiltonian H_0, respectively.

The potentials (1.4) are solution of the free Maxwell equations. Apart from the connection with full QED, the question arises whether such external fields allow second quantization of the electron-positron field. For this to be true, the Hilbert–Schmidt property of (1.5) are necessary and sufficient.

In the present work we analyze the scattering theory for a class of external fields like (1.4) to prove the properties necessary for second quantization. These fields decay slowly in time in such a way that previous methods [3] do not work.

After some preliminaries we first consider the wave operators and show existence and convergence of the perturbation series. The scattering operator is discussed in section 4, in particular the off-diagonal part (1.5) is investigated in detail. It is shown that every term in the perturbation series is indeed a Hilbert–Schmidt operator. The Hilbert–Schmidt convergence of the series cannot be proven by the methods of this work, but we obtain strong indications for this. Some complicated technical tools are given in the appendices.

Notation

Some comments on the notation used in the sequel will now be given.

4-vectors: $a^\mu = (a_0, \mathbf{a})$, $\quad a^\mu b_\mu = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}$.

The γ-matrices satisfy: $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$.

The α- and β-matrices: $\boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}$, $\beta = \gamma_0$.

We write: $\not{a} = \gamma^\mu a_\mu$, $\quad \Box = \dfrac{\partial^2}{\partial t^2} - \Delta$.

The Hilbert space: $X = L^2(\mathbb{R}^2, \mathbb{C}^4, d^3x)$ is the space of the Dirac four-spinors.

Concerning operators: let $V(t, \mathbf{x})$ denote a (4×4)-matrix valued function of $t, \mathbf{x}$. Then $V(t)$ represents the corresponding operator in X, defined as

$(V(t)\varphi)(\mathbf{x}) = V(t, \mathbf{x})\varphi(\mathbf{x}), \qquad \varphi \in X.$

$|V(t, \mathbf{x})|$ denotes the norm of the 4×4-matrix $V(t, \mathbf{x})$.

$\| V(t\ \cdot) \|_2 = \{\int d^3x |V(t, \mathbf{x})|^2\}^{1/2}$.

$\| V(t) \|$ is the norm of the operator $V(t)$.

If

$$\varphi \in X, \qquad \|\varphi\|_\infty = \text{ess} - \sup_{\mathbf{x} \in \mathbb{R}^3} |\varphi(\mathbf{x})|,$$

where

$|\varphi(\mathbf{x})| = (\varphi^*(\mathbf{x})\varphi(\mathbf{x}))^{1/2}$, is the norm of the 4-spinor,

$\|\varphi\| = \{\int d^3x \varphi^*(\mathbf{x})\varphi(\mathbf{x})\}^{1/2}$, is the L^2-norm or φ.

2. Preliminaries

The Dirac equation with time-dependent external field A^μ, has the following form

$$i \frac{\partial f}{\partial t} = H(t) f, \tag{2.1}$$

where

$$\begin{aligned} H(t) &= H_0 + e\gamma^0 \not{A}(t, \mathbf{x}), \\ H_0 &= -i\boldsymbol{\alpha} \cdot \nabla + m\beta, \\ \not{A}(t, \mathbf{x}) &= \gamma^\mu A_\mu(t, \mathbf{x}). \end{aligned} \tag{2.2}$$

Some known results, concerning existence and uniqueness of the solution of the Dirac equation (2.1), are not stated for the case of a real valued A^μ. They can be found in [4] and [5].

Under the following conditions for the potential A^μ:

(i) for all fixed t, the functions $A^\mu(t, \mathbf{x})$ are continuous and bounded on $\mathbb{R}^3$,
(ii) the maps $t \to A^\mu(t, \cdot)$ are continuous, for all $t \in \mathbb{R}$, with respect to the supremum norm,
(iii) the functions $(\partial/\partial x_i)A^\mu(t, \mathbf{x})$ are continuous and bounded on $\mathbb{R} \times \mathbb{R}^3$,

there exists a unique family of unitary operators $U(t, s)$ on $X = L^2(\mathbb{R}^3, \mathbb{C}^4, d^3x)$, defined for all s, t in $\mathbb{R}$, with the following properties:

(a) $U(t, s)$ is jointly strongly continuous in s and t (in the norm of X).
(b) $U(t, t) = I$.
(c) $U(t, r) = U(t, s)U(s, r)$.
(d) $U(t, s)Y \subset Y$.
(e) $(d/ds)U(t, s)y = iU(t, s)H(s)y$, $y \in Y$.
(f) For each fixed $y \in Y$ and s, $(d/dt)U(t, s)y$ exists, equals $-iH(t)U(t, s)y$ and is strongly continuous in t (in the norm of X).

Here Y is the Sobolev space $H^1(\mathbb{R}^3, \mathbb{C}^4, d^3x)$ which is dense in X. Furthermore, $U(t, s)$ can explicitly be written as

$$U(t, s) = e^{-itH_0}\tilde{U}(t, s)e^{isH_0}, \tag{2.3}$$

where $\tilde{U}(t, s)$ is given by the Dyson series

$$\tilde{U}(t, s) = \sum_{n=0}^{\infty} (-ie)^n \int_s^t dt_1 \int_s^{t_1} dt_2 \cdots \int_s^{t_{n-1}} dt_n \tilde{V}(t_1) \cdots \tilde{V}(t_n), \tag{2.4}$$

and

$$\tilde{V}(t) = e^{itH_0}V(t)e^{-itH_0}, \qquad V(t) = \gamma^0 A\!\!\!/(t, \mathbf{x}), \tag{2.5}$$

The integrals in (2.4) are understood in the strong sense (of X) and the Dyson series is norm converging.

In the following, the external field A^μ is assumed to be solution of the homogeneous Maxwell equations and it fulfills the conditions (i)–(iii). For convenience we choose the Coulomb gauge, that is $A^0 = 0$ and $\mathbf{A}$ satisfies:

$$\Box \mathbf{A} = 0, \qquad \nabla \cdot \mathbf{A} = 0. \tag{2.6}$$

The associated propagator $U(t, s)$ then exists and has all the properties described above. From U the wave operators are then defined as the following strong limits:

$$\Omega^\pm = \underset{t \to \pm\infty}{\text{s-lim}}\ U(t, 0)^* e^{-itH_0} \tag{2.7}$$

In the next section the existence of these limits will be proved.

3. Wave Operators

The exixtence of the wave operators $\Omega^\pm$ can be proved by means of Cook's method. The proof given below follows the one given in [6] for a similar case.

Let $\psi \in Y$. By Cook's method it is sufficient to find a set D which is dense in $L^2(\mathbb{R}^3, \mathbb{C}^4, d^3x)$, so that for each $\psi \in D$

$$\int_{-\infty}^{\infty} \| V(\tau)e^{-iH_0\tau}\psi \| \, d\tau < \infty, \tag{3.1}$$

and the existence of $\Omega^{\pm}$ follows at once. With

$$\| V(t)e^{-iH_0t}\psi \| \leqslant \|\gamma^0\gamma^\mu\| \, \|A_\mu(t, \cdot)\|_2 \|e^{-iH_0t}\psi\|_\infty, \tag{3.2}$$

where Hölder's inequality was used, it remains the problem of estimating $\|A_\mu(t, \cdot)\|_2$ and $\|e^{-iH_0t}\psi\|_\infty$.

An estimate for the last quantity can be obtained by the stationary phase methods, described in [7]. Some results, obtained by these methods, and to be used in the following, are stated in Appendix I.

In order to obtain an estimate for $\|e^{-iH_0t}\psi\|_\infty$, it is sufficient to note that each component of a solution of the free Dirac equation is also solution of the Klein–Gordon equation (A.0) (where m is the electron mass).

On the other hand, if the Fourier transform $\hat{\psi}$ of ψ is C^∞ with compact support, then $\varphi = e^{-iH_0t}\psi$ (i.e. each component of φ) is a regular wave packet for the Klein–Gordon equation (see Appendix I). Theorem A1 can hence be applied and yields:

$$\|\varphi\|_\infty = \|e^{-iH_0t}\psi\|_\infty \leqslant d|t|^{-3/2}, \tag{3.3}$$

for some constant d and all ψ, for which the Fourier transform $\hat{\psi}$ is C^∞ with compact support.

At this point some further condition concerning the potential $A_\mu(t, \mathbf{x})$ is needed: in the following we assume $A_\mu(t, \mathbf{x})$ to be a regular wave packet for the wave equation (A.0) (see Appendix I)*).

For A_μ, satisfying this condition, it can be proved $\|A_\mu(t, \cdot)\|_2 \leqslant C$ where C is a constant (see Appendix I).

With (3.2) and (3.3) it follows that condition (3.1) is satisfied for all $\psi \in Y$, where $\hat{\psi} \in C^\infty$ with compact support. Since the set of these ψ's is dense in $L^2(\mathbb{R}^3, \mathbb{C}^4, d^3x)$, it follows that the wave operators $\Omega^{\pm}$ exist.

Representation of the wave operators $\Omega^{\pm}$ by a Dyson series

In this section it will be proved that for weak enough potentials A^μ the wave operators $\Omega^{\pm}$ can be represented by the following strongly convergent Dyson series:

$$\Omega^{\pm}\varphi = \mathbb{1} \cdot \varphi + \sum_{n=1}^{\infty} (-ie)^n \int_{\pm\infty}^{0} dt_1 \int_{\pm\infty}^{t_1} dt_2 \cdots \int_{\pm\infty}^{t_{n-1}} dt_n \tilde{V}(t_1) \cdots \tilde{V}(t_n)\varphi, \tag{3.4}$$

for all φ for which $\hat{\varphi}$ is C^∞ with compact support.

This statement will here be proved only for the case of Ω^-. For Ω^+ the proof is completely analogous.

---

*) The condition "$\mathbf{k} = 0$ must not be in the supports of $\widehat{A_\mu(0, \cdot)}$, $\widehat{A_{\mu_t}(0, \cdot)}$" (see Appendix I) is not really necessary for the results of this Section (see [6]), but will be needed in Section 4.

From (2.7) and (2.4) it can be seen that in order to prove equation (3.4), it is sufficient to show that the series

$$\sum_{n=1}^{\infty} e^n \int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \|\tilde{V}(t_1) \cdots \tilde{V}(t_n)\varphi\| \tag{3.5}$$

is convergent.

By Theorem A1

$$\|\tilde{V}(t)\| \leqslant \|V(t)\| \leqslant b(1+|t|)^{-1}.$$

Thus:

$$\|\tilde{V}(t_1)\tilde{V}(t_2) \cdots \tilde{V}(t_n)\varphi\|$$
$$\leqslant b^{n-1}(1+|t_1|)^{-1}(1+|t_2|)^{-1} \cdots (1+|t_{n-1}|)^{-1}\|\tilde{V}(t_n)\varphi\|.$$

From Hölder's inequality and Theorem A1 it follows that:

$$\|\tilde{V}(t_n)\varphi\| = \|V(t_n)e^{-iH_0t_n}\varphi\| \leqslant \|V(t_n,\cdot)\|_2\|e^{-iH_0t_n}\varphi\|_\infty$$
$$\leqslant C \cdot d(1+|t_n|)^{-3/2}$$

where $\|V(t,\cdot)\|_2 \leqslant C$ was used (see Appendix I). Thus the following estimate is obtained:

$$\|\tilde{V}(t_1)\tilde{V}(t_2) \cdots \tilde{V}(t_n)\varphi\|$$
$$\leqslant C \cdot b^{n-1} \cdot d \cdot (1+|t_1|)^{-1} \cdots (1+|t_{n-1}|)^{-1}(1+|t_n|)^{-3/2},$$

which in turn gives for the n-th term of the series (3.5) the estimate:

$$\int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \|\tilde{V}(t_1) \cdots \tilde{V}(t_n)\varphi\| \leqslant 2^n \cdot C \cdot d \cdot b^{n-1}.$$

Thus the following sufficient condition for the validity of equation (3.4) is obtained

$$b \cdot e \leqslant \tfrac{1}{2}.$$

This means that for weak enough potentials $A^\mu(t, \mathbf{x})$ the wave operators $\Omega^\pm$ can be represented by the Dyson series (3.4).

4. Scattering Operator

4.1. *Perturbative expressions*

From the wave operators $\Omega^\pm$ the S-operator is defined as follows:

$$S = (\Omega^+)^*\Omega^-. \tag{4.1}$$

Starting from the expansion (3.4) for the wave operators $\Omega^\pm$, the S-matrix (4.1) can be represented by the Dyson series:

$$S = \mathbb{1} + \sum_{n=1}^{\infty} (-ie)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \tilde{V}(t_1) \cdots \tilde{V}(t_n), \tag{4.2}$$

if the right side makes sense.

Now, while the results in Section 3 do not imply the unitarity of the S-operator and the strong convergence of (4.2), they do imply weak convergence of (4.2). Thus, this representation of the S-operator can be used to estimate the HS-norm of S_{+-} and S_{-+}.

In a first step it will be proved that each term in the Dyson series for S_{+-} (resp. S_{-+}) is Hilbert–Schmidt. This will be done starting with the expression for these operators in Fourier space.

Fourier transforms are introduced by:

$$\hat{f}(\mathbf{k}) = (2\pi)^{-3} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x}) \qquad f \in L^2(\mathbb{R}^3, \mathbb{C}^4, d^3x).$$

Then for the free Dirac Hamiltonian one has:

$$\widehat{H_0 f}(\mathbf{k}) = (\boldsymbol{\alpha}\cdot\mathbf{k} + m\beta)\hat{f}(\mathbf{k}) = H_0(\mathbf{k})\hat{f}(\mathbf{k}), \tag{4.3}$$

and for the projectors P_+ and P_-:

$$\widehat{P_\pm f}(\mathbf{k}) = \left(\frac{1}{2} \pm \frac{\boldsymbol{\alpha}\cdot\mathbf{k} + m\beta}{2E(\mathbf{k})}\right)\hat{f}(\mathbf{k}) = P_\pm(\mathbf{k})\hat{f}(\mathbf{k}), \tag{4.4}$$

where

$$E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}. \tag{4.5}$$

The free Dirac Hamiltonian and the projectors $P_\pm$ satisfy:

$$H_0(\mathbf{k}) = E(\mathbf{k})(P_+(\mathbf{k}) - P_-(\mathbf{k})), \tag{4.6}$$

and

$$H_0(\mathbf{k})P_\pm(\mathbf{k}) = P_\pm(\mathbf{k})H_0(\mathbf{k}) = \pm E(\mathbf{k})P_\pm(\mathbf{k}). \tag{4.7}$$

For the external field A^μ one has

$$(\widehat{A_\mu(t) f})(\mathbf{k}) = \int d^3k' \hat{A}_\mu(t, \mathbf{k} - \mathbf{k}')\hat{f}(\mathbf{k}'), \tag{4.8}$$

where

$$\hat{A}_\mu(t, \mathbf{k}) = (2\pi)^{-3} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}} A_\mu(t, \mathbf{x}).$$

Using the fact that A^μ is a real valued solution of the wave equation (2.6), $\hat{A}^\mu(t, \mathbf{k})$ can be written as:

$$\hat{\mathbf{A}}(t, \mathbf{k}) = \sum_{\lambda=1}^{2} (\boldsymbol{\varepsilon}(\mathbf{k}, \lambda) f(\mathbf{k}, \lambda) e^{-i|\mathbf{k}|t} + \boldsymbol{\varepsilon}(-\mathbf{k}, \lambda) f^*(-\mathbf{k}, \lambda) e^{i|\mathbf{k}|t}), \tag{4.9}$$

where $\boldsymbol{\varepsilon}(\mathbf{k}, \lambda)$ represent the transverse polarization vectors.

As a sequence

$$\hat{A\!\!\!/}(t, \mathbf{k}) = \gamma^\mu \hat{A}_\mu(t, \mathbf{k}) = \sum_{\rho=\pm 1} M_\rho(\mathbf{k}) e^{i\rho|\mathbf{k}|t}, \tag{4.10}$$

where $M_{+1}(\mathbf{k})$ and $M_{-1}(\mathbf{k})$ are C^∞, (4×4)-matrix valued functions of $\mathbf{k}$ with compact support which does not comprise $\mathbf{k}=0$.

With (4.2), the Dyson series for S_{+-} assumes the following form:

$$S_{+-}=\sum_{n=1}^{\infty}(-ie)^n S_{+-}^{(n)}$$
$$S_{+-}^{(n)}=\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n P_+\tilde{V}(t_1)\cdots\tilde{V}(t_n)P_-. \tag{4.11}$$

The operator $S_{+-}^{(n)}$ is represented in k-space as follows:

$$\widehat{S_{+-}^{(n)}f}(\mathbf{k})=\int d^3k' S_{+-}^{(n)}(\mathbf{k},\mathbf{k}')\hat{f}(\mathbf{k}'). \tag{4.12}$$

Using (2.5), (4.3), (4.4) and (4.8), the following expression is obtained:

$$\begin{aligned}S_{+-}^{(n)}(\mathbf{k},\mathbf{k}')=&\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{t_1}dt_2\cdots\int_{-\infty}^{t_{n-1}}dt_n\int d^3k_1\cdots d^3k_{n-1}\\&\cdot e^{ik^0t_1}P_+(\mathbf{k})\gamma^0\tilde{\mathcal{A}}(t_1,\mathbf{k}-\mathbf{k}_1)\\&\cdot e^{-iH_0(\mathbf{k}_1)t_1}e^{iH_0(\mathbf{k}_1)t_2}\gamma^0\tilde{\mathcal{A}}(t_2,\mathbf{k}_1-\mathbf{k}_2)\\&\cdot e^{-iH_0(\mathbf{k}_2)t_2}\cdots e^{iH_0(\mathbf{k}_{n-1})t_n}\gamma^0\tilde{\mathcal{A}}(t_n,\mathbf{k}_{n-1}-\mathbf{k}')P_-(\mathbf{k}')e^{-ik'^0t_n},\end{aligned} \tag{4.13}$$

where

$$k^0=+E(\mathbf{k})\qquad k'^0=-E(\mathbf{k}').$$

Since the dependence of $\tilde{\mathcal{A}}(t,\mathbf{k})$ as a function of t is known (4.10), the integrations over the time-variables in (4.13) can be carried out explicitly. From (4.6) and (4.7) it follows that:

$$e^{-iH_0(\mathbf{k})t}=\sum_{\sigma=\pm1}P_\sigma(\mathbf{k})e^{-i\sigma E(\mathbf{k})t}. \tag{4.14}$$

On the other hand

$$\begin{aligned}&\int_{-\infty}^{t_{j-1}}dt_j e^{iEt_j}\gamma^0\tilde{\mathcal{A}}(t_j,\mathbf{k}-\mathbf{k}')e^{-iE't_j}\\&\quad=\sum_{\rho=\pm1}\gamma^0M_\rho(\mathbf{k}-\mathbf{k}')i\left.\frac{e^{-i\omega t_{j-1}}}{\omega+i\varepsilon}\right|_{\omega=E'-E+\rho|\mathbf{k}-\mathbf{k}'|}.\end{aligned} \tag{4.15}$$

In the last equation (4.10) was used.

By inserting (4.10) and (4.14) into the expression (4.13) for $S_{+-}^{(n)}(\mathbf{k},\mathbf{k}')$ and integrating over the time variables $t_2,\ldots,t_n$ (using formula (4.15)) the following results:

$$
\begin{aligned}
S^{(n)}_{+-}(\mathbf{k},\mathbf{k}') = -2\pi i\gamma^0 \sum_{\rho_1\cdots\rho_n} \int & d^3k_1 \cdots d^3k_{n-1}\,\delta(\omega) P_+(\mathbf{k}) \\
& \cdot M_{\rho_1}(\mathbf{k}-\mathbf{k}_1) S_R(k_1^0,\mathbf{k}_1) \\
& \cdot M_{\rho_2}(\mathbf{k}_1-\mathbf{k}_2) S_R(k_2^0,\mathbf{k}_2) \cdots M_{\rho_{n-1}}(\mathbf{k}_{n-2}-\mathbf{k}_{n-1}) \\
& \cdot S_R(k_{n-1}^0,\mathbf{k}_{n-1}) M_{\rho_n}(\mathbf{k}_{n-1}-\mathbf{k}') P_-(\mathbf{k}'),
\end{aligned} \tag{4.16}
$$

where

$$k_j^0 = -E(\mathbf{k}') + \rho_{j+1}|\mathbf{k}_j - \mathbf{k}_{j+1}| + \rho_{j+2}|\mathbf{k}_{j+1} - \mathbf{k}_{j+2}| + \cdots + \rho_n|\mathbf{k}_{n-1} - \mathbf{k}'|, \quad j = 1, 2, \ldots, n-1, \tag{4.17}$$

$$\omega = -E(\mathbf{k}) - E(\mathbf{k}') + \rho_1|\mathbf{k} - \mathbf{k}_1| + \rho_2|\mathbf{k}_1 - \mathbf{k}_2| + \cdots + \rho_n|\mathbf{k}_{n-1} - \mathbf{k}'|, \tag{4.18}$$

and

$$S_R(k_j^0, \mathbf{k}_j) = \sum_{\sigma=\pm 1} \frac{P_\sigma(\mathbf{k}_j)\gamma^0}{k_j^0 - \sigma E(\mathbf{k}_j) + i\varepsilon} \tag{4.19}$$

is the retarded Green's function.

For the last integration (over t_1), the relation

$$\int_{-\infty}^{\infty} dt_1 e^{-i\omega t_1} = 2\pi\delta(\omega)$$

was used; $\delta(\cdot)$ denotes the Dirac δ-distribution.

Since $M_\rho(\mathbf{k})$ has compact support which does not comprise $\mathbf{k} = 0$ two positive real numbers δ, R can be found such that:

$$M_\rho(\mathbf{k}) = 0 \quad \text{for all } \mathbf{k} \text{ with } |\mathbf{k}| \geqslant R \text{ or } |k| \leqslant \delta \quad (\rho = \pm 1).$$

Thus in the expression (4.16) for $S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')$ the region of integration is bounded by the following conditions:

$$\delta \leqslant |\mathbf{k} - \mathbf{k}_1| \leqslant R, \qquad \delta \leqslant |\mathbf{k}_1 - \mathbf{k}_2| \leqslant R, \ldots, \qquad \delta \leqslant |\mathbf{k}_{n-1} - \mathbf{k}'| \leqslant R. \tag{4.20}$$

On the other hand, since in (4.16) ω is argument of the δ-distribution it must be

$$\omega = 0$$

Using the explicit expression (4.18) for ω and taking into account the conditions (4.20), the inequality

$$E(\mathbf{k}) + E(\mathbf{k}') \leqslant nR, \tag{4.21}$$

is obtained. This means that $S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')$ has also compact support (as a function of $\mathbf{k}, \mathbf{k}'$). The same is also true for $S^{(n)}_{-+}(\mathbf{k}, \mathbf{k}')$ and it is interesting to note that this is a consequence of the fact that in both cases, in the expression (4.18) for ω, $E(\mathbf{k})$ and $E(\mathbf{k}')$ have the same sign (minus for $S^{(n)}_{+-}$ and plus for $S^{(n)}_{-+}$).

The norm of the 4×4-matrix $S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')$ for each $\mathbf{k}$ and $\mathbf{k}'$ is denoted by $|S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')|$.

In the next section it will be proved that $|S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')|$ is bounded uniformly for all values of $\mathbf{k}, \mathbf{k}' \in \mathbb{R}^3$.

4.2. *Hilbert–Schmidt property of* $S^{(n)}_{+-}$

As can be seen from formula (4.16) the main problem in estimating $|S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')|$ arises from the singularities of the functions $S_R(k_j^0, \mathbf{k}_j)$.

In fact using the relation

$$\frac{1}{k_j^0 - \sigma_j E(\mathbf{k}_j) + i\varepsilon} = P\left(\frac{1}{k_j^0 - \sigma_j E(\mathbf{k}_j)}\right) - \pi i \delta(k_j^0 - \sigma_j E(\mathbf{k}_j)), \tag{4.22}$$

where P means principle part, it is at once clear that if $k_j^0 - \sigma_j E(\mathbf{k}_j) = 0$ the integrand in (4.16) diverges.

A possible way of handling this problem will now be discussed. We start by defining:

$$\begin{aligned} &T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; a) \\ &\quad = \int d^3k_1 \cdots d^3k_{n-1} \delta(\omega + a) M_{\rho_1}(\mathbf{k} - \mathbf{k}_1) \\ &\qquad \cdot S_{\sigma_1}(k_1^0, \mathbf{k}_1) M_{\rho_2}(\mathbf{k}_1 - \mathbf{k}_2) \cdots S_{\sigma_{n-1}}(k_{n-1}^0, \mathbf{k}_{n-1}) \cdot M_{\rho_n}(\mathbf{k}_{n-1} - \mathbf{k}'), \end{aligned} \tag{4.23}$$

where

$$k_j^0 = \sigma E(\mathbf{k}) - \rho_1 |\mathbf{k} - \mathbf{k}_1| - \cdots - \rho_j |\mathbf{k}_{j-1} - \mathbf{k}_j|, \qquad j = 1, \ldots, n-1, \tag{4.24}$$

$$\omega = \sigma' E(\mathbf{k}') - \sigma E(\mathbf{k}) + \rho_1 |\mathbf{k} - \mathbf{k}_1| + \cdots + \rho_n |\mathbf{k}_{n-1} - \mathbf{k}'|, \tag{4.25}$$

$$S_{\sigma_j}(k_j^0, \mathbf{k}_j) = \frac{P_{\sigma_j}(\mathbf{k}_j)\gamma^0}{k_j^0 - \sigma_j E(\mathbf{k}_j) + i\varepsilon}, \qquad \sigma_j = \pm 1, \tag{4.26}$$

and a is a real number.

With this definition we have:

$$\begin{aligned} S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}') = -2\pi i \gamma^0 \sum_{\rho_1 \cdots \rho_n} \sum_{\sigma_1 \cdots \sigma_{n-1}} P_+(\mathbf{k}) \\ T_{\rho_1 \cdots \rho_n, +1, \sigma_1 \cdots \sigma_{n-1}, -1}(\mathbf{k}, \mathbf{k}'; 0) P_-(\mathbf{k}'), \end{aligned} \tag{4.27}$$

and the following recursion formula holds:

$$\begin{aligned} &T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; a) \\ &\quad = \int d^3k_{n-1} T_{\rho_1 \cdots \rho_{n-1}, \sigma, \sigma_1, \ldots \sigma_{n-1}}(\mathbf{k}, \mathbf{k}_{n-1}; \rho_n |\mathbf{k}_{n-1} - \mathbf{k}'| - \sigma_{n-1} E(\mathbf{k}_{n-1}) \\ &\qquad + \sigma' E(\mathbf{k}') + a) S_{\sigma_{n-1}}(\sigma' E(\mathbf{k}') + \rho_n |\mathbf{k}_{n-1} - \mathbf{k}'| + a, \mathbf{k}_{n-1}) M_{\rho_n}(\mathbf{k}_{n-1} - \mathbf{k}'). \end{aligned} \tag{4.28}$$

In the following some transformations are introduced. The aim of these transformations is to bring the denominators in $S_{\sigma_j}(k_j^0, \mathbf{k}_j)$ in the elementary form $(x_j + i\varepsilon)$. Furthermore, the δ-distribution will be used to carry out one integration.

The last integral can be written in a (relatively) simpler form by introducing two substitutions.

With the substitution:

$$\mathbf{k}'_{n-1} = \mathbf{k}_{n-1} - \mathbf{k}', \tag{4.29}$$

formula (4.28) becomes

$$\begin{aligned} &T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; a) \\ &= \int d^3k_{n-1} T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1},}(\mathbf{k}, \mathbf{k}_{n-1} + \mathbf{k}'; f_{\sigma', \sigma_{n-1}, \rho_n}(\mathbf{k}_{n-1}, \mathbf{k}') + a) \\ &\quad \cdot \frac{P_{\sigma_{n-1}}(\mathbf{k}_{n-1} + \mathbf{k}')\gamma^0}{f_{\sigma', \sigma_{n-1}, \rho_n}(\mathbf{k}_{n-1}, \mathbf{k}') + a + i\varepsilon}, \end{aligned} \tag{4.30}$$

where for simplicity, after the substitution the prime $\mathbf{k}'_{n-1}$ was dropped again. Here equation (4.26) was used and the following notation:

$$f_{\sigma', \sigma_{n-1}, \rho_n}(\mathbf{k}_{n-1}, \mathbf{k}') = \sigma' E(\mathbf{k}') - \sigma_{n-1} E(\mathbf{k}_{n-1} + \mathbf{k}') + \rho_n |\mathbf{k}_{n-1}|. \tag{4.31}$$

The second substitution is as follows:

$$\mathbf{k}_{n-1} = k_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}')\hat{e}_{\vartheta, \varphi} \tag{4.32}$$

where $\hat{e}_{\vartheta, \varphi} = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$, $r, \vartheta\varphi$, are the new integration variables, and r is given by

$$f_{\sigma', \sigma_{n-1}, \rho_n}(\mathbf{k}_{n-1}, \mathbf{k}') = r. \tag{4.33}$$

By inserting (4.32) into the last equation and solving for $k_{\sigma', \sigma_{n-1}, \rho_n}$, we obtain:

$$k_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}') = \frac{(r - \sigma' E(\mathbf{k}'))^2 - E(\mathbf{k}')^2}{2(\rho_n(r - \sigma' E(\mathbf{k}')) + \mathbf{k}' \cdot \hat{e}_{\vartheta, \varphi})} \tag{4.34}$$

For the range of variation of the new integration variable r, two different cases must be considered;

$$\text{If} \quad -\sigma_{n-1} = \rho_n \quad \text{then} \quad E(\mathbf{k}') \leqslant \rho_n(r - \sigma' E\mathbf{k}')) \leqslant \infty. \tag{4.35'}$$

$$\text{If} \quad \sigma_{n-1} = \rho_n \quad \text{then} \quad -E(\mathbf{k}') \leqslant \rho_n(r - \sigma' E(\mathbf{k}')) \leqslant |\mathbf{k}'|. \tag{4.35''}$$

In the second case the following condition concerning the variable ϑ and φ must be added:

$$\mathbf{k}' \cdot \hat{e}_{\vartheta, \varphi} \leqslant -\rho_n(r - \sigma' E(\mathbf{k}')).$$

Let $D_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}')$ denote the Jacobian of the substitution (4.32). Then

$$D_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}') = [k_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}')]^2 \sin\vartheta \left| \frac{\partial k_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}')}{\partial r} \right| \tag{4.36}$$

and

$$\left|\frac{\partial k_{\sigma', \sigma_{n-1}, \rho_n}}{\partial r}\right| = \frac{(r - \sigma' E(\mathbf{k}'))^2 + 2\rho_n (r - \sigma' E(\mathbf{k}'))\mathbf{k}' \cdot \hat{e}_{\vartheta, \varphi} + E(\mathbf{k}')^2}{2(\rho_n (r - \sigma' E(\mathbf{k}')) + \mathbf{k}' \cdot \hat{e}_{\vartheta, \varphi})^2}. \tag{4.37}$$

By means of the second substitution the integral (4.30) assumes the following form:

$$\begin{aligned} &T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'} 11(\mathbf{k}, \mathbf{k}'; a) \\ &= \int_{-\infty}^{\infty} dr \int d\vartheta \, d\varphi D_{\sigma', \sigma_{n-1}, \rho_n}(r, \hat{e}_{\vartheta, \varphi}; \mathbf{k}') \\ &\cdot T_{\rho_1 \cdots \rho_{n-1}, \sigma, \sigma_1 \cdots \sigma_{n-1}}(\mathbf{k}, \mathbf{k}_{n-1} + \mathbf{k}'; r + a) \frac{P_{\sigma_{n-1}}(\mathbf{k}_{n-1} - \mathbf{k}')\gamma^0}{r + a + i\varepsilon} M_{\rho_n}(\mathbf{k}_{n-1}) \end{aligned} \tag{4.38}$$

Since the integrand in (4.38) has compact support the integration over r can be extended to the interval $(-\infty, \infty)$; (we set the integrand equal to 0 for those values of r that do not satisfy the conditions (4.35)).

Iterating (4.38) one gets:

$$\begin{aligned} &T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; 0) \\ &= \int_{-\infty}^{\infty} dr_{n-1} \int d\vartheta_{n-1} \, d\varphi_{n-1} \, D_{n-1} \int_{-\infty}^{\infty} dr_{n-2} \int d\vartheta_{n-2} \, d\varphi_{n-2} \\ &\cdot D_{n-2} \cdots \int_{-\infty}^{\infty} dr_2 \int d\vartheta_2 \, d\varphi_2 \, D_2 \\ &\cdot T_{\rho_1 \rho_2, \sigma, \sigma_1 \sigma_2}(\mathbf{k}, \mathbf{k}_2 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}'; r_2 + \cdots + r_{n-1}) \\ &\cdot \frac{P_{\sigma_2}(\mathbf{k}_2 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}')\gamma^0}{r_2 + r_3 + \cdots + r_{n-1} + i\varepsilon} \\ &\cdot M_{\rho_3}(\mathbf{k}_2) \cdots \frac{P_{\sigma_{n-2}}(\mathbf{k}_{n-2} + \mathbf{k}_{n-1} + \mathbf{k}')\gamma^0}{r_{n-2} + r_{n-1} + i\varepsilon} M_{\rho_{n-1}}(\mathbf{k}_{n-2}) \\ &\cdot \frac{P_{\sigma_{n-1}}(\mathbf{k}_{n-1} + \mathbf{k}')\gamma^0}{r_{n-1} + i\varepsilon} M_{\rho_n}(\mathbf{k}_{n-1}). \end{aligned} \tag{4.39}$$

where

$$\begin{aligned} &D_j = D_{\sigma_{j+1}, \sigma_j, \rho_{j+1}}(r_j, \hat{e}_j; \mathbf{k}_{j+1} + \cdots + \mathbf{k}_{n-1} + \mathbf{k}'), \\ &\mathbf{k}_{n-1} = k_{\sigma', \sigma_{n-1}, \rho_n}(r_{n-1}, \hat{e}_{n-1}; \mathbf{k}')\hat{e}_{n-1}, \\ &\mathbf{k}_j = k_{\sigma_{j+1}, \sigma_j, \rho_{j+1}}(r_j, \hat{e}_j; \mathbf{k}_{j+1} + \cdots + \mathbf{k}_{n-1} + \mathbf{k}')\hat{e}_j; \end{aligned}$$

and

$$\hat{e}_j = (\sin \vartheta_j \cos \varphi_j, \sin \vartheta_j \sin \varphi_j, \cos \vartheta_j), \qquad j = 2, \ldots, n-1.$$

Note that $\mathbf{k}_j$ is a function of $r_j, \hat{e}_j; r_{j+1}, \hat{e}_{j+1}; \ldots; r_{n-1}, \hat{e}_{n-1}$.

According to the definition (4.23), we have:

$$T_{\rho_1\rho_2,\sigma,\sigma_1\sigma_2}(\mathbf{k},\mathbf{K}';a) = \int d^3k_1\,\delta(\sigma_2E(\mathbf{K}') - \sigma E(\mathbf{k}) + \rho_1|\mathbf{k}-\mathbf{k}_1| + \rho_2|\mathbf{k}_1-\mathbf{K}'| + a)$$

$$\cdot\, M_{\rho_1}(\mathbf{k}-\mathbf{k}_1)\frac{P_{\sigma_1}(\mathbf{k}_1)\gamma^0}{\sigma E(\mathbf{k}) - \sigma_1 E(\mathbf{k}_1) - \rho_1|\mathbf{k}-\mathbf{k}_1| + i\varepsilon}M_{\rho_2}(\mathbf{k}_1-\mathbf{K}'). \tag{4.40}$$

In this integral the following substitution can be made:

$$\mathbf{k}_1 = k_1'\hat{e}_1 + \mathbf{K}',$$

where $\hat{e}_1 = (\sin\vartheta_1\cos\varphi_1, \sin\vartheta_1\sin\varphi_1, \cos\vartheta_1)$, k_1', ϑ_1, φ_1 are the new integration variables and $k_1'^2\sin\vartheta_1$ the corresponding Jacobian.

Because of the presence, of a δ-distribution in (4.40) it is at once possible to carry out the integration over k_1'. Setting the argument of the δ-function equal to zero

$$\sigma_2E(\mathbf{K}') - \sigma E(\mathbf{k}) + \rho_1|\mathbf{k}-\mathbf{K}'-k_1'\hat{e}_1| + \rho_2k_1' + a = 0$$

and solving the resulting equation with respect to k_1', one gets

$$k_1' = \frac{r^2-\kappa^2}{2(\rho_2r - \boldsymbol{\kappa}\cdot\hat{e}_1)}, \tag{4.41}$$

where $r \equiv \sigma E(\mathbf{k}) - \sigma_2E(\mathbf{K}') - a$ and $\boldsymbol{\kappa} \equiv \mathbf{k}-\mathbf{K}'$, and the following conditions must be satisfied:

$$\begin{aligned} &\text{for}\quad \rho_1 = \rho_2: &&\rho_2 r \geqslant |\boldsymbol{\kappa}| \\ &\text{for}\quad \rho_1 = -\rho_2: &&-|\boldsymbol{\kappa}| \leqslant \rho_2 r \leqslant |\boldsymbol{\kappa}|,\quad \rho_2 r - \boldsymbol{\kappa}\cdot\hat{e}_1 \leqslant 0. \end{aligned} \tag{4.42}$$

From the integration over the δ-distribution, finally the factor:

$$\frac{1}{\left|\dfrac{\partial}{\partial k_1'}(\rho_1|\boldsymbol{\kappa}-k_1'\cdot\hat{e}_1| + \rho_2k_1')\right|_{k_1'=\frac{r^2-\kappa^2}{2(\rho_2r-\boldsymbol{\kappa}\cdot\hat{e}_1)}}} = \frac{r^2+\kappa^2-2\rho_2r\boldsymbol{\kappa}\cdot\hat{e}_1}{2(\rho_2r-\boldsymbol{\kappa}\cdot\hat{e}_1)^2}$$

is obtained.

The above discussion of (4.40) combined with (4.39) leads to the following result:

$$T_{\rho_1\cdots\rho_n,\sigma,\sigma_1\cdots\sigma_{n-1},\sigma'}(\mathbf{k},\mathbf{k}';0)$$

$$= \int_{-\infty}^{\infty}dr_{n-1}\int d\vartheta_{n-1}\,d\varphi_{n-1}\,D_{n-1}\cdots\int_{-\infty}^{\infty}dr_2\int d\vartheta_2\,d\varphi_2$$

$$\cdot\, D_2\int d\vartheta_1\,d\varphi_1\sin\vartheta_1\cdot k_1^2\frac{r^2+\kappa^2-2\rho_2r\boldsymbol{\kappa}\cdot\hat{e}_1}{2(\rho_2r-\boldsymbol{\kappa}\cdot\hat{e}_1)^2}M_{\rho_1}(\mathbf{k}-\mathbf{k}_1-\cdots-\mathbf{k}_{n-1}-\mathbf{k}')$$

$$\cdot\,\frac{P_{\sigma_1}(\mathbf{k}_1+\cdots+\mathbf{k}_{n-1}+\mathbf{k}')\gamma^0}{\sigma E(\mathbf{k}) - \sigma_1E(\mathbf{k}_1+\cdots+\mathbf{k}_{n-1}+\mathbf{k}') - \rho_1|\mathbf{k}-\mathbf{k}_1-\cdots-\mathbf{k}_{n-1}-\mathbf{k}'| + i\varepsilon}M_{\rho_2}(\mathbf{k}_1)$$

$$\cdot\,\frac{P_{\sigma_2}(\mathbf{k}_2+\cdots+\mathbf{k}_{n-1}+\mathbf{k}')\gamma^0}{r_2+\cdots+r_{n-1}+i\varepsilon}M_{\rho_3}(\mathbf{k}_2)\cdots\frac{P_{\sigma_{n-1}}(\mathbf{k}_{n-1}+\mathbf{k}')\gamma_0}{r_{n-1}+i\varepsilon}M_{\rho_n}(\mathbf{k}_{n-1}), \tag{4.43}$$

where

$$\boldsymbol{\kappa} \equiv \mathbf{k} - \mathbf{k}_2 - \cdots - \mathbf{k}_{n-1} - \mathbf{k}',$$

$$r \equiv \sigma E(\mathbf{k}) - \sigma_2 E(\mathbf{k}_2 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}') - r_2 - r_3 - \cdots - r_{n-1}, \quad \mathbf{k}_1 \equiv k_1 \hat{e}_1.$$

k_1 is just the same as in (4.41) (the prime of k_1' has been dropped) and the conditions (4.42) must be fulfilled.

By means of the substitution

$$x_j = \sum_{l=j}^{n-1} r_l \qquad j = 2, \ldots, n-1,$$

where the inverse transformation is given by

$$r_{n-1} = x_{n-1},$$

$$r_j = x_j - x_{j+1}, \qquad j = 2, \ldots, n-2,$$

and the Jacobian is equal to 1, the integral in (4.43) assumes the following form:

$$T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; 0) = \int_{-\infty}^{\infty} dx_{n-1} \int_{-\infty}^{\infty} dx_{n-2} \cdots \int_{-\infty}^{\infty} dx_2 \cdot \frac{F(x_2, \ldots, x_{n-1}; \mathbf{k}, \mathbf{k}')}{(x_2 + i\varepsilon)(x_3 + i\varepsilon) \cdots (x_{n-1} + i\varepsilon)} \tag{4.44}$$

The explicit expression for F can be deduced at once from (4.43);

$$F(x_2, \ldots, x_{n-1}; \mathbf{k}, \mathbf{k}') = \int d\vartheta_{n-1}\, d\varphi_{n-1}\, D_{n-1} \cdots \int d\vartheta_2\, d\varphi_2 D_2 \int d\vartheta_1\, d\varphi_1 \tilde{D}_1 \cdot M_{\rho_1}(\mathbf{k} - \mathbf{k}_1 - \cdots - \mathbf{k}_{n-1} - \mathbf{k}') \frac{P_{\sigma_1}(\mathbf{k}_1 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}')\gamma^0}{N_1 + i\varepsilon} M_{\rho_2}(\mathbf{k}_1) \cdot P_{\sigma_2}(\mathbf{k}_2 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}')\gamma^0 \cdot M_{\rho_3}(\mathbf{k}_2) \cdots P_{\sigma_{n-1}}(\mathbf{k}_{n-1} + \mathbf{k}')\gamma^0 M_{\rho_n}(\mathbf{k}_{n-1}), \tag{4.45}$$

where

$$\tilde{D}_1 \equiv \sin \vartheta_1 \cdot k_1^2 \frac{r^2 + \kappa^2 - 2\rho_2 r \boldsymbol{\kappa} \cdot \hat{e}_1}{2(\rho_2 r - \boldsymbol{\kappa} \cdot \hat{e}_1)^2} \tag{4.46}$$

$$N_1 \equiv \sigma E(\mathbf{k}) - \sigma_1 E(\mathbf{k}_1 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}') - \rho_1 |\mathbf{k} - \mathbf{k}_1 - \cdots - \mathbf{k}_{n-1} - \mathbf{k}'| \tag{4.47}$$

(For simplicity the indices $\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'$, of F were dropped). Note that F has compact support as a function of $x_2, \ldots, x_{n-1}$.

We will now make use of the following result, proved in Appendix II.

Let $G \in C^1(\mathbb{R}^m)$ have compact support. Then the following estimate holds:

$$\left| \int_{\mathbb{R}^m} d^m y \frac{G(y_1, \ldots, y_m)}{\prod_{j=1}^{m} (y_j + i\varepsilon)} \right| \leqslant c \cdot \left\{ \|G\|_\infty + \sum_{j=1}^{m} \left\| \frac{\partial G}{\partial y_j} \right\|_\infty \right\} \tag{4.48}$$

where c is a constant.

The advantage of this estimate is that it contains only first derivatives of G. Thus, according to (4.44), $|T_{\rho_1,\ldots,\sigma,\sigma_1}\cdots|$ can be estimated as:

$$|T_{\rho_1\cdots\rho_n,\sigma,\sigma_1\cdots\sigma_{n-1},\sigma'}(\mathbf{k},\mathbf{k}';0)| \leqslant c\cdot\left\{\|F\|_\infty + \sum_{j=2}^{n-1}\left\|\frac{\partial F}{\partial x_j}\right\|_\infty\right\} \tag{4.49}$$

Here

$$\|F\|_\infty = \sup_{\substack{\mathbf{k},\mathbf{k}'\in\mathbb{R}^3\\(x_2,\ldots,x_{n-1})\in\mathbb{R}^{n-2}}} |F(x_2,\ldots,x_{n-1};\mathbf{k},\mathbf{k}')| \tag{4.50}$$

and similarly for $\left\|\frac{\partial F}{\partial x_j}\right\|_\infty$.

With this result and (4.27) we are now able to prove the uniform boundedness of $|S^{(n)}_{+-}(\mathbf{k},\mathbf{k}')|$ in $\mathbf{k},\mathbf{k}'$. This can simply be done by showing that:

$$\|F\|_\infty < \infty, \qquad \left\|\frac{\partial F}{\partial x_j}\right\|_\infty < \infty, \qquad j = 2,\ldots,n-1. \tag{4.51}$$

which we are now going to prove.

Taking (4.21) into account, it is at once clear that F has compact support as a function of $x_2,\ldots,x_{n-1},\mathbf{k},\mathbf{k}'$. Thus, it suffices to show that F and $\partial F/\partial x_j$ $(j = 2,\ldots,n-1)$ are continuous with respect to these variables. Then (4.51) holds.

Looking at the expression (4.45) for F, we see that concerning continuity, the only difficulties could arise from the factors D_j $(j = 2,\ldots,n-1)$, $\tilde{D}_1$ and $1/N_1$, when the denominators in the corresponding expressions (4.36), (4.46) (4.47) become 0. But in fact, it can be shown that all these factors are bounded. In order to see this, further detailed considerations are required.

The denominator N_1 has the form

$$\sigma E(\mathbf{k}) - \sigma_1 E(\mathbf{K}') - \rho_1|\mathbf{k} - \mathbf{K}'|$$

with

$$\mathbf{K}' = \mathbf{k}_1 + \cdots + \mathbf{k}_{n-1} + \mathbf{k}'.$$

Since

$$\delta \leqslant |\mathbf{k} - \mathbf{K}'| \leqslant R, \qquad \delta \leqslant |\mathbf{k}_j| \leqslant R \qquad (j = 1,\ldots,n-1)$$

and

$$|\mathbf{k}| \leqslant nR, \qquad |\mathbf{K}'| \leqslant (n+1)R \qquad (\text{cf. (4.21)})$$

it is easy to see that (for all σ,σ_1,ρ_1)$|1/N_1|$ is bounded.

The denominator in the expression for D_j $(j = 2,\ldots,n-1)$ (cf. (4.36)) is the same as the one in the expression for k_j (cf. (4.34)). If we write

$$k_j = \frac{f_j(r,\hat{e}_{\vartheta,\varphi};\mathbf{k}')}{g_j(r,\hat{e}_{\vartheta,\varphi};\mathbf{k}')}$$

and take into account that

$$\delta \leqslant k_j \leqslant R$$

we get

$$\frac{1}{R} \inf_{r, \vartheta, \varphi, \mathbf{k}'} |f_j| \leqslant \frac{1}{|g_j|} \leqslant \frac{1}{\delta} \sup_{r, \vartheta, \varphi, \mathbf{k}'} |f_j|.$$

Since $\sup |f_j| < \infty$, we see that $1/|g_j|$ is bounded and so is $|D_j|$.

In the same way it can be seen that $|\tilde{D}_1|$ is bounded. Thus, the proof of the uniform boundedness of $|S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')|$ is completed. With

$$\|S^{(n)}_{+-}\|^2_{\mathrm{HS}} = \mathrm{tr} \int d^3k\, d^3k' [S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')]^* [S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')],$$

where tr denotes the trace in $\mathbb{C}^4$, and since $S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')$ has compact support in $\mathbf{k}, \mathbf{k}'$, we finally conclude that $S^{(n)}_{+-}$ is Hilbert-Schmidt. The same holds for $S^{(n)}_{-+}$ because $S^{(n)}_{-+}(\mathbf{k}, \mathbf{k}')$ has also compact support in $\mathbf{k}, \mathbf{k}'$.

For $S^{(n)}_{++}(\mathbf{k}, \mathbf{k}')$ and $S^{(n)}_{--}(\mathbf{k}, \mathbf{k}')$ the argument of the compact support does not work anymore. But from the above considerations it follows that (for $n \neq 1$) $S^{(n)}_{++}(\mathbf{k}, \mathbf{k}')$, $S^{(n)}_{--}(\mathbf{k}, \mathbf{k}')$ are locally bounded.

4.3. *Hilbert–Schmidt property for operators* S_{+-}, S_{-+}

The method described in the foregoing section to estimate $\|S^{(n)}_{+-}\|_{\mathrm{HS}}$, could now be used to prove that the operator S_{+-} (respectively S_{-+}) is Hilbert–Schmidt. This would require control of the n-dependence of the right side in (4.49), in order to show that the series

$$\sum_{n=1}^{\infty} \|S^{(n)}_{+-}\|_{\mathrm{HS}} \tag{4.52}$$

which is an estimate of $\|S_{+-}\|_{\mathrm{HS}}$, is convergent. With the present method this is rather a complicated task, which includes estimating the area of the $(3(n-1)-1)$-dimensional surface defined by equation $\omega = 0$ (cf. (4.16), (4.18)) and similar problems. Furthermore, in the end, the obtained n-dependence could prove to be too bad to grant the convergence of (4.52). This is because the estimate (4.49) of each single term, in expression (4.27) for $S^{(n)}_{+-}(\mathbf{k}, \mathbf{k}')$, neglects the sign of $T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; 0)$ and thus ruins the estimate for $\|S^{(n)}_{+-}\|_{\mathrm{HS}}$. We will show in an example below that cancellations are indeed very important in our problem.

However, these difficulties arise only for such terms

$$T_{\rho_1 \cdots \rho_n, \sigma, \sigma_1 \cdots \sigma_{n-1}, \sigma'}(\mathbf{k}, \mathbf{k}'; 0)$$

in the sum (4.27), for which the number of denominators (in $S_{\sigma_j}(k_j^0, \mathbf{k}_j)$) that may simultaneously become zero is of order n

Let us now consider an n-fold t-integral as in equation (4.13)

$$I_n = \int_t^{t'} dt_1 \int_t^{t_1} dt_2 \cdots \int_t^{t_{n-1}} dt_n e^{i(\alpha_1 t_1 + \alpha_2 t_2 + \cdots + \alpha_n t_n)}. \tag{4.53}$$

Since the domain of integration is a simplex, we have the simple estimate:

$$|I_n| \leqslant \frac{|t' - t|}{n!} \tag{4.54}$$

which has a good n-dependence to prove the convergence of $\Sigma_{n=0}^{\infty} I_n$. On the other hand, we may carry out the t-integrations as we have done in section 4.1. This results in a sum of 2^n exponentials terms.

$$\pm \frac{e^{if}}{\prod_{j=1}^{n} \sum_{l_j} \alpha_{l_j}} \tag{4.55}$$

where f is linear in α_j and t, t'. In this exact expression the n-dependence is completely hidden. It comes about by cancellations between the terms (4.55) such that (4.54) holds.

From this example we learn that the n-dependence requires a complementary method. But because of the simplex structure of the original expression (4.13) there is strong indication that the n-dependence is all right. Then the sum (4.11) is Hilbert–Schmidt-convergent and S_{+-} is Hilbert–Schmidt.

At last it is worth mentioning that there are methods to prove the HS-property of S_{+-} without using the Dyson expansion (see Refs [8], [9]). Hopefully they are powerful enough to solve completely the problem posed in the present paper.

Appendix I

In this Appendix, first some results obtained with the stationary phase methods are stated in the form of a theorem. For a proof, see Ref [7].

Then, an unequality used in Section 3, concerning the electromagnetic potential A_μ, will be proved. First we need the following definition:

A solution of the Klein–Gordon equation

$$\varphi_{tt} = (\Delta - m^2)\varphi \tag{A.0}$$

is called a *regular wave packet*, if the Fourier transforms of the initial data $f = \widehat{\varphi(\cdot, 0)}$ and $g = \widehat{\varphi_t(\cdot, 0)}$ are C^∞ and have compact support; in addition, if $m = 0$, $\mathbf{k} = 0$ must not be in this support.

Theorem I

1. *Let φ be a regular wave packet for the Klein–Gordon equation (A.0) for $m \neq 0$. Then, there is a constant d such that:*

   $|\varphi(\mathbf{x}, t)| \leqslant d(1 + |t|)^{-3/2}$ *for all values of* $\mathbf{x}, t$.

2. *Let φ be a regular wave packet for the wave equation (A.0) with $m = 0$. Then*

   $|\varphi(\mathbf{x}, t)| \leqslant b(1 + |t|)^{-1}$ *for all values of* $\mathbf{x}, t$

   and some constant b.

We will now prove that if the potential A^{μ} is a regular wave packet for the wave equation (2.6), *there is a constant C such that*:

$$\|A_\mu(t,\cdot)\|_2 \leqslant C, \qquad \|V(t,\cdot)\|_2 \leqslant C \qquad \textit{for all } t \in \mathbb{R}. \tag{A.1}$$

Here

$$V(t,\cdot) = \gamma^0\gamma^\mu A_\mu(t,\cdot) \qquad (\textit{see } (2.5)).$$

Proof. According to (2.6) $A_0 = 0$. Since $\mathbf{A}$ is a real valued solution of the wave equation (2.6), the Fourier transform $\hat{\mathbf{A}}(t, \mathbf{k})$ can be written as:

$$\hat{\mathbf{A}}(t,\mathbf{k}) = \sum_{\lambda=1}^{2} \{\boldsymbol{\varepsilon}(\mathbf{k},\lambda) f(\mathbf{k},\lambda) e^{-i|\mathbf{k}|t} + \boldsymbol{\varepsilon}(-\mathbf{k},\lambda) f^*(-\mathbf{k},\lambda) e^{i|\mathbf{k}|t}\}$$

cf. (4.9)) where $f(\mathbf{k}, \lambda)$ has compact support as a function of $\mathbf{k}$ (cf. definition of regular wave packet).

Thus:

$$\|A_i(t,\cdot)\|_2 = (2\pi)^{3/2}\|\hat{A}_i(t,\cdot)\|_2 \leqslant 2(2\pi)^{3/2} \sum_{\lambda=1}^{2} \left\{\int d^3k |f(\mathbf{k},\lambda)|^2\right\}^{1/2} \tag{A.2}$$

where $|\boldsymbol{\varepsilon}(\mathbf{k},\lambda)| = 1$ was used, $i = 1, 2, 3$.

Since $f(\mathbf{k}, \lambda)$ has compact support, the last expression on the right side of (A.2) is finite. Thus (A.1) follows.

Appendix II

In this appendix the inequality (4.48) will be proved. We begin by proving the following result: For $G \in C^1(\mathbb{R}^m)$ with compact support, we define $I_{m,L}[G]$ as:

$$I_{m,L}[G] = P\int_{-L}^{L} dy_1 \int_{-L}^{L} dy_2 \cdots \int_{-L}^{L} dy_m \frac{G(y_1,\ldots,y_m)}{y_1\cdot\ldots\cdot y_m} \tag{A.3}$$

Then the following holds:

$$I_{m,L}[G] = R_{m,L}[G] + H_{m,L}[G] \tag{A.4}$$

where

$$R_{m,L}[G] = (\log L)^m \sum_{\sigma_1\cdots\sigma_m} G(\sigma_1 L, \sigma_2 L, \ldots, \sigma_m L) \tag{A.5}$$

$$|H_{m,L}[G]| \leqslant C_m(L) \sum_{j=1}^{m} \left\|\frac{\partial G}{\partial y_j}\right\|_\infty \tag{A.6}$$

and

$$\sigma_j = \pm 1 \qquad (j = 1, \ldots, m).$$

Proof. The above result will be proved by means of mathematical induction. For $m = 1$:

$$I_{1,L} = P\int_{-L}^{L} dy_1 \frac{G(y_1)}{y_1} = \log|y_1|G(y_1)|_{-L}^{L} - \int_{-L}^{L} \log|y_1| \frac{dG}{dy_1} dy_1.$$

Thus for $m = 1$, (A.4)–(A.6) are fulfilled.

Now suppose that for $m \in \mathbb{N}$, (A.4)–(A.6) are satisfied. Then one may write $I_{m+1,L}$ as:

$$I_{m+1,L} = \sum_{j=1}^{m+1} P\int_0^L d\lambda \frac{1}{\lambda}\int_{-\lambda}^{\lambda} dy_1 \cdots \int_{-\lambda}^{\lambda} dy_{j-1}\int_{-\lambda}^{\lambda} dy_{j+1}\cdots\int_{-\lambda}^{\lambda} dy_{m+1}$$

$$\cdot \frac{G(y_1,\ldots,y_{j-1},\lambda,y_{j+1},\ldots,y_{m+1}) - G(y_1,\ldots,y_{j-1},-\lambda,y_{j+1},\ldots,y_{m+1})}{y_1\cdot\ldots\cdot y_{j-1}y_{j+1}\cdot\ldots\cdot y_{m+1}}$$

$$= \sum_{j=1}^{m+1}\int_0^L d\lambda \frac{1}{\lambda}\{I_{m,\lambda}[G(y_1,\ldots,y_{j-1},\lambda,y_j,\ldots,y_m)]$$
$$- I_{m,\lambda}[G(y_1,\ldots,y_{j-1},-\lambda,y_j,\ldots,y_m)]\}.$$

Since for m, (A.4)–(A.6) are fulfilled one has:

$$I_{m+1,L} = \sum_{j=1}^{m+1}\int_0^L d\lambda\frac{1}{\lambda}\Bigg\{(\log\lambda)^m \sum_{\sigma_1\cdots\sigma_m}\sigma_1\cdots\sigma_m$$
$$\cdot[G(\sigma_1\lambda,\ldots,\sigma_{j-1}\lambda,\lambda,\sigma_j\lambda,\ldots,\sigma_m\lambda)$$
$$- G(\sigma_1\lambda,\ldots,\sigma_{j-1}\lambda,-\lambda,\sigma_j\lambda,\ldots,\sigma_m\lambda)]$$
$$+ H_{m,\lambda}[G(y_1,\ldots,y_{j-1},\lambda,y_j,\ldots,y_m)]$$
$$- H_{m,\lambda}[G(y_1,\ldots,y_{j-1},-\lambda,y_j,\ldots,y_m)]\Bigg\}.$$

Integrating by parts the contributions of $R_{m,\lambda}$ we get:

$$I_{m+1,L} = (\log\lambda)^{m+1}\sum_{\sigma_1\cdots\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}G(\sigma_1\lambda,\ldots,\sigma_{m+1}\lambda)|_0^L$$
$$-\int_0^L d\lambda(\log\lambda)^{m+1}\frac{d}{d\lambda}\sum_{\sigma_1\cdots\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}G(\sigma_1\lambda,\ldots,\sigma_{m+1}\lambda)$$
$$+\int_0^L d\lambda\frac{1}{\lambda}\sum_{j=1}^{m+1}\{H_{m,\lambda}[G(y_1,\ldots,y_{j-1},\lambda,y_j,\ldots,y_m)]$$
$$- H_{m,\lambda}[G(y_1,\ldots,y_{j-1},-\lambda,y_j,\ldots,y_m)]\}.$$

Since

$$\lim_{\lambda\to 0}(\log\lambda)^{m+1}\sum_{\sigma_1\cdots\sigma_{m+1}}\sigma_1\cdots\sigma_{m+1}G(\sigma_1\lambda,\ldots,\sigma_{m+1}\lambda) = 0$$

and $H_{m,\lambda}$ can be estimated according to (A.6), we have that:

$$I_{m+1,L}[G] = R_{m+1,L}[G] + H_{m+1,L}[G]$$

where

$$|H_{m+1,L}[G]| \leqslant \left\{2^{m+1}\int_0^L |\log\lambda|^{m+1}\,d\lambda + 2m\int_0^L d\lambda\,\frac{1}{\lambda}\,C_m(\lambda)\right\}\sum_{j=1}^{m+1}\left\|\frac{\partial G}{\partial y_j}\right\|_\infty.$$

Thus an iteration formula for $C_m(L)$ is obtained:

$$C_{m+1}(L) = 2^{m+1}\int_0^L |\log\lambda|^{m+1}\,d\lambda + 2m\int_0^L d\lambda\,\frac{1}{\lambda}\,C_m(\lambda)$$

with

$$C_1(L) = 2\int_0^L |\log\lambda|\,d\lambda.$$

It can easily be seen that $C_m(L)$ is finite for all $m \in \mathbb{N}$. The proof of (A.4)–(A.6) is thus completed.

From (A.4)–(A.6) it follows that for $G \in C^1(\mathbb{R}^m)$ with compact support, there is an $L < \infty$ such that:

$$\left|P\int_{\mathbb{R}^m} d^m y\,\frac{G(y_1,\ldots,y_m)}{y_1\cdots y_m}\right| \leqslant C_m(L)\sum_{j=1}^{m}\left\|\frac{\partial G}{\partial y_j}\right\|_\infty$$

(since G has compact support, $L < \infty$ exists with $R_{m,L}[G] \equiv 0$).

By means of the relation

$$\frac{1}{y_j + i\varepsilon} = P\left(\frac{1}{y_j}\right) - \pi i\delta(y_j)$$

we finally obtain that:

$$\left|\int_{\mathbb{R}^m} d^m y\,\frac{G(y_1,\ldots,y_m)}{\prod_{j=1}^{m}(y_j + i\varepsilon)}\right| \leqslant c\left\{\|G\|_\infty + \sum_{j=1}^{m}\left\|\frac{\partial G}{\partial y_j}\right\|_\infty\right\}$$

where c is a constant.

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