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Mean Field Theories of the Quantum Heisenberg Model

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ABSTRACT: I review work done in collaboration with A. Auerbach on an SU(N) generalization of the SU(2) quantum Heisenberg model. For antiferromagnets on bipartite lattices, two distinct phases are identified: a Néel-ordered Bose condensate, and a featureless disordered spin insulator. The rotational invariance of the mean field theory and its applicability to finite temperature make it an attractive candidate for various extensions, such as the inclusion of holes.

1. Introduction and General Historical Prattle

The quantum Heisenberg model,

$$H = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j, \tag{1}$$

is the prototype of a broad class of interacting quantum systems. It is also blessed with physical relevance. What do we know about it?

Consider, for the moment, the classical Heisenberg model. On bipartite lattices with nearest neighbor interactions, which henceforth will be the only lattices considered, the transformation $\boldsymbol{S} \to -\boldsymbol{S}$ of spin variables on one sublattice takes J to $-J$, and thus the partition function is only a function of the magnitude $|J|$ – the thermodynamics of classical ferromagnets and antiferromagnets are identical. In two or fewer dimensions, the Mermin-Wagner theorem[1] precludes long range order at any finite temperature.

Since the commutation relations $[S^\alpha, S^\beta] = i\varepsilon_{\alpha\beta\gamma} S^\gamma$ for quantum spins are not preserved under $\boldsymbol{S} \to -\boldsymbol{S}$, there is no thermodynamic equivalence between J and $-J$ for quantum magnets. For the ferromagnet ($J < 0$), one can easily construct an exact ground state by maximally polarizing each spin in some fixed direction in internal space. Since this breaks the continuous SU(2) symmetry of the Hamiltonian, Goldstone's theorem guarantees the existence of gapless spin wave excitations. Conservation of the order parameter in ferromagnets means that the dynamics are diffusive, and the spin wave dispersion obeys $\omega \propto k^2$.

The nature of the ground state of quantum antiferromagnets ($J > 0$) is not easily determined. Expanding perturbatively about an ordered Néel state,[2] one finds gapless antiferromagnetic magnons with $\omega \propto k$, a consequence of the notionally broken symmetry. At finite temperature in low dimensions ($d \leq 2$), thermal fluctuations always destroy Néel order, and one must go beyond naïve spin wave theory if one is to obtain sensible finite T results.[3] A more interesting possibility is that the ground state itself is disordered due to quantum fluctuations. This is the case, for example, in one dimension, where a quantum analog of the Mermin-Wagner theorem exists. In such cases, the spectrum is *not* obliged to be gapless. Yet despite the general admonition that "that which is not forbidden is compulsory," it came as somewhat of a surprise to workers in the field when in 1983 Haldane[4] proposed that generic Heisenberg spin chains should possess a gap in their excitation spectra. Why? Haldane's reasoning relied on mapping the low-energy sector of the d-dimensional spin S quantum antiferromagnet to a $d+1$-dimensional classical nonlinear sigma model

$$\mathcal{S} = \frac{1}{g} \int d^{d+1}x \sum_{i=0}^{d} |\nabla^i \hat{\boldsymbol{\Omega}}|^2 \tag{2}$$

with coupling (temperature) $g \approx 2/S$. In hindsight, this seems perfectly reasonable – linear dispersion of the spin waves suggests a relativistic effective long-wavelength theory, and the nonlinear sigma model is a natural candidate. In two dimensions, the nonlinear sigma model exhibits 'dynamic mass generation' – the correlations decay exponentially – and from Lorentz invariance one expects a mass gap of order $\Delta \sim c/\xi$. What was troubling was that the excitation spectrum of the $S = \frac{1}{2}$ chain was demonstrably gapless,[5] with $S = 1$ excitations obeying a dispersion $\omega_k = \frac{1}{2}\pi J\,|\sin k|$ *identical* (aside from an overall factor of $\pi/2$) to that obtained via naïve spin wave theory.[2] Bethe's exact ground state[6] possessed no broken symmetry, yet something masquerading as a Goldstone mode, though ultimately quite different, was present in the spectrum. In addition, the Lieb-Schultz-Mattis theorem[7] and its generalization by Affleck and Lieb[8] rigorously require the existence of gapless excitations or a ground state degeneracy in half-odd integer S Heisenberg spin chains.

The reason why half-odd integer S is somehow special was elucidated by Haldane,[4,9,10] who showed that a topological term

$$\mathcal{S}_{\rm top} = \frac{\theta}{4\pi}\int dx \int dt\, \hat{\boldsymbol{\Omega}} \cdot \frac{\partial \hat{\boldsymbol{\Omega}}}{\partial t} \times \frac{\partial \hat{\boldsymbol{\Omega}}}{\partial x}, \tag{3}$$

with $\theta = 2\pi S$, was present in the continuum field theory. This contribution to the action may be written as θQ_{xt}, where Q_{xt} is a topological integer, known as the Pontryagin index of the field $\hat{\boldsymbol{\Omega}}(x,t)$. The term $\exp(2\pi i S Q_{xt})$ is therefore 'invisible' for integer spins, but may lead to interference between configurations of differing Pontryagin index for half-odd integer S. The logical argument may now be stated as follows: We may not know how the $d = 2$ nonlinear sigma model behaves at $\theta = \pi$,[11] but we *do* know, based on Haldane's analysis, that aside from quantitative differences arising from the numerical value of the coupling g, all half-odd integer Heisenberg spin chains should behave alike. The $S = \frac{1}{2}$ chain is rigorously known to be gapless with low-lying excitations obeying $\omega \propto k$, therefore this behavior is also expected in the $S = \frac{3}{2}$, $S = \frac{5}{2}$, *etc.* chains. Integer S chains should exhibit a gap, because the $d = 2$ nonlinear sigma model at $\theta = 0$ is massive.[12]

Other evidence has arisen which corroborates this general picture. Faddeev and Takhtajan[13] showed that the $S = 1$ excitations of the Bethe chain found by des Cloizeaux and Pearson were not elementary, but rather composites of truly elementary $S = \frac{1}{2}$ excitations, and are thus quite different from conventional spin waves. Recently, a class of nonintegrable models with exactly solvable ground states was discovered by Affleck, Kennedy, Lieb, and Tasaki.[14,15] The AKLT models exist on *arbitrary* lattices in *any* dimension provided that the spin is an integer multiple of half the lattice coordination number. In $d = 1$, the $S = 1$ antiferromagnet

$$H_{\text{AKLT}} = \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + \tfrac{1}{3}(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1})^2, \tag{4}$$

which includes biquadratic exchange, rigorously is known to feature an excitation gap and exponentially decaying spin-spin correlations.[14] Another nonintegrable spin chain with a solvable ground state exists for $S = \frac{1}{2}$ with longer ranged $(1/r^2)$ exchange, as recently discovered both by Haldane and by Shastry.[16,17] Like Bethe's model, it features gapless elementary $S = \frac{1}{2}$ excitations, although a much richer structure to the spectrum, as yet not completely understood, also is present.[16]

For two-dimensional (2D) quantum antiferromagnets, there are few exact results. It is known, for instance, that the $S \geq 1$ Heisenberg model[18] on the square lattice possesses a Néel ordered ground state[19] – similar exact results exist for other two-dimensional lattices (*e.g.* there is order for $S \geq \frac{3}{2}$ on the honeycomb lattice[14]). There is considerable numerical evidence[20] suggesting that the $S = \frac{1}{2}$ model on the square lattice orders as well. The only exactly solvable ground states I know of are all associated with AKLT models. Recently, Haldane[9,10] has suggested a classification of the behavior of *disordered* 2D systems based on an examination of tunnelling between different instanton sectors of the associated continuum field theory. Interestingly, the interference from the topological term is sensitive only to the value of $2S$ modulo z, where z is the lattice coordination number. This interference vanishes when $2S$ modulo z is zero, and this is precisely the condition for the existence of an AKLT model.[21] A rather complete account of the physics of low-dimensional quantum antiferromagnets has now been given by Read and Sachdev.[22,23]

In this lecture, I will describe a simple mean field approach to the quantum Heisenberg model,[24] applicable to both ferromagnets and antiferromagnets at finite as well as zero temperature. The mean field approach is based on a generalization of the usual SU(2) Heisenberg model to a class of SU(N) models where a $1/N$ expansion yields exact results in the large-N limit. A much more general and insightful examination of SU(N) antiferromagnets, based in part on what I shall describe here, is to be found in the papers of Read and Sachdev.[22,23] In addition to deriving many new and important results, those authors have gone very far in providing a comprehensive picture of the physics which ties together virtually all extant theories.

SU(N) Quantum Magnets

The algebra of quantum spin is that of the group SU(2):

$$[S^\alpha, S^\beta] = i\varepsilon_{\alpha\beta\gamma} S^\gamma; \qquad \boldsymbol{S} \cdot \boldsymbol{S} = S(S+1). \tag{5}$$

Conventional spin wave theory makes use of the Holstein-Primakoff representation,

$$\begin{aligned} S^+ &= h^\dagger\sqrt{2S - h^\dagger h} \\ S^z &= h^\dagger h - S, \end{aligned} \tag{6}$$

in which each spin is represented by a single Bose oscillator subject to the anholonomic constraint that states in the physical sector must obey $0 \le n_h \le 2S$. In a functional integral approach, for example, constraints of this form are difficult to handle because they define boundaries in Hilbert space. Oftentimes, it proves convenient to eliminate the anholonomic constraint in favor of a holonomic constraint at the expense of introducing an additional 'slave' degree of freedom. For instance, we could introduce an additional 'slave' boson, g, subject to the condition $h^\dagger h + g^\dagger g = 2S$, and then take $S^+ = h^\dagger g$, $S^- = hg^\dagger$, and $S^z = \frac{1}{2}(h^\dagger h - g^\dagger g)$. In terms of the column vector $\boldsymbol{b} = (h, g)$, this is nothing but the Schwinger boson representation of spin: $\boldsymbol{S} = \frac{1}{2}\boldsymbol{b}^\dagger \cdot \boldsymbol{\sigma} \cdot \boldsymbol{b}$. There are several advantages to this formalism. First, whereas the vacuum for the Holstein-Primakoff system represents a broken symmetry state with maximally polarized spin, the Schwinger boson vacuum is trivial and featureless and does not presuppose any broken symmetry. Second, holonomic constraints are easily incorporated into a functional integral approach via delta functions.

Another feature of the Schwinger representation is its direct generalization to more than two species of boson. This is accomplished by defining the SU(N) generators S_α^β with α and β running from 1 to N via

$$S_\alpha^\beta = b_\alpha^\dagger b_\beta - \frac{n_c}{N}\delta_{\alpha\beta} \tag{7a}$$

$$\sum_{\alpha=1}^{N} b_\alpha^\dagger b_\alpha = n_c \ . \tag{7b}$$

The S_β^α thus satisfy

$$\begin{aligned} &\sum_{\alpha=1}^{N} S_\alpha^\alpha = 0 \\ &[S_\alpha^\beta, S_\gamma^\delta] = \delta_\gamma^\beta S_\alpha^\delta - \delta_\alpha^\delta S_\gamma^\beta, \end{aligned} \tag{8}$$

which are the defining relations for the generators of SU(N). For $N = 2$, the parameter n_c (adopted from the notation of Read and Sachdev[22]) simply is $2S$, and one directly recovers the original Schwinger representation of SU(2). In general, we may define an analog of the spin quantum number S by $S = n_c/N$ – this 'spin' is quantized in integer multiples of $1/N$. *The mean field theories described in this paper will be exact in the limit where N is large and S is held fixed.* The generalized SU(N) Heisenberg model may now be written:

$$H = \frac{J}{N}\sum_{\langle ij\rangle}\sum_{\alpha,\beta} S_\alpha^\beta(i)\, S_\beta^\alpha(j). \tag{9}$$

It should be stressed that while the Hamiltonian of Eq(9) together with the defining relations of Eq(8) are completely general, the representation of Eq(7) is but one of many possible for the group SU(N). Indeed, Eq(7) describes a totally symmetric representation of SU(N) corresponding to a Young tableau with a single row consisting of n_c boxes. As discussed by Sachdev,[23] this representation can also be described in terms of fermionic operators, and in addition other representations which are totally antisymmetric with m rows of one box each, or of mixed symmetry with m rows of n_c boxes each, may be derived.

The reader should content himself by verifying that the Hamiltonian of Eq(9) reduces to the familiar form for the case $N = 2$. We shall choose *different* SU(N) generalizations of the ferromagnet and antiferromagnet. For the ferromagnet, we employ the same representation of SU(N) on each lattice site. For the antiferromagnet, once we pick an SU(N) representation for the A sublattice, we employ the *conjugate* representation $S^{\beta}_{\alpha} \rightarrow -S^{\alpha}_{\beta}$ on the B sublattice. The issue here is that all representations of SU(2) are self-conjugate, but this is not so for SU(N). By using conjugate SU(N) representations on alternate sublattices, the general tendency of antiferromagnets to form local singlets is preserved.

For the case of the ferromagnet, one can define a composite operator $\mathcal{F}_{ij}$ via

$$\mathcal{F}_{ij} = \sum_{\alpha=1}^{N} b^{\dagger}_{i,\alpha} b_{j,\alpha} \tag{10}$$

in terms of which the Hamiltonian of Eq(9) is

$$H_{\mathrm{F}} = -\frac{|J|}{N} \sum_{\langle ij \rangle} : \mathcal{F}^{\dagger}_{ij} \mathcal{F}_{ij} : - E_0. \tag{11}$$

where $:\mathcal{O}:$ is a normal-ordered operator, and $E_0 = -S^2 N_{\mathrm{bonds}}$ is a constant. Likewise, in the case of the antiferromagnet we may define

$$\mathcal{A}_{ij} = \sum_{\alpha=1}^{N} b_{i,\alpha} b_{j,\alpha} \tag{12}$$

and obtain

$$H_{\mathrm{A}} = -\frac{J}{N} \sum_{\langle ij \rangle} : \mathcal{A}^{\dagger}_{ij} \mathcal{A}_{ij} : - E_0. \tag{13}$$

(Here and henceforth we set $|J| = 1$ and measure energies in units of $|J|$.) This form of the Hamiltonian, $H = -N^{-1} \sum_{\langle ij \rangle} : \mathcal{Z}^{\dagger}_{ij} \mathcal{Z}_{ij} :$, with $\mathcal{Z}_{ij}$ a composite operator as in Eqs(10,12), motivates a Hubbard-Stratonovich decoupling via an auxiliary bond field Q_{ij}, yielding the following functional integral representation of the partition function:

$$Z = \int \mathcal{D}[b,\bar{b},Q,\bar{Q},\lambda]\,\exp(-\mathcal{S})$$

$$\mathcal{S} = \int_0^\beta d\tau \left\{ \tfrac{1}{2}\sum_{i,\alpha}(\bar{b}_{i,\alpha}\dot{b}_{i,\alpha} - \dot{\bar{b}}_{i,\alpha}b_{i,\alpha}) + \frac{N}{J}\sum_{\langle ij\rangle}\bar{Q}_{ij}Q_{ij} + \right. \tag{14}$$

$$\left. + \sum_{\langle ij\rangle}(\bar{Q}_{ij}\mathcal{Z}_{ij} + Q_{ij}\bar{\mathcal{Z}}_{ij}) + \sum_{i,\alpha}\lambda_i\,(\bar{b}_{i,\alpha}b_{i,\alpha} - S)\right\}.$$

Here, the λ integration contours stretch from $-i\infty$ to $+i\infty$; this enforces the fixed occupancy constraint of Eq(7b). The action possesses a local gauge invariance,

$$\begin{aligned} b_{i,\alpha}(\tau) &\rightarrow e^{i\phi_i(\tau)}b_{i,\alpha}(\tau) \\ \lambda_i(\tau) &\rightarrow \lambda_i(\tau) - i\frac{\partial}{\partial\tau}\phi_i(\tau) \\ Q_{ij}(\tau) &\rightarrow e^{i\phi_j(\tau)}e^{\mp i\phi_i(\tau)}\,Q_{ij}(\tau), \end{aligned} \tag{15}$$

where the minus (plus) sign in the last of the above equations refers to the case of the ferromagnet (antiferromagnet). In order to obtain a faithful representation of the partition function, one must fix a gauge in order to avoid integrating over the infinite copies of each gauge orbit; this may be accomplished by enforcing $d\lambda_i/dt = 0$ on all sites.[22] The utility of this representation of Z is that the action is now a quadratic form in terms of N 'flavors' of *decoupled* Schwinger bosons, and in principle the $b,\bar{b}$ fields may be integrated out, yielding an expression

$$Z = \int \mathcal{D}[Q,\bar{Q},\lambda]\,\exp(-NF[Q,\bar{Q},\lambda]), \tag{16}$$

where the free energy F is a function *only* of Q,$\bar{Q}$, and λ and *not* a function of N. Treating $1/N$ as a small parameter akin to $\hbar$ in conventional quantum theory, a systematic $1/N$ expansion for all thermodynamic properties and response functions can be derived.

Mean Field Theory of the Ferromagnet

Making the static assumption

$$\begin{aligned} Q_{ij}^{\mathrm{MF}}(\tau) &= Q \\ \lambda_i^{\mathrm{MF}}(\tau) &= \lambda, \end{aligned} \tag{17}$$

with Q and λ both real, the Schwinger bosons can be integrated out explicitly, resulting in a specific (per site) free energy per flavor of

$$F^{\mathrm{MF}} = \tfrac{1}{2}zQ^2 - S\lambda + \frac{1}{\beta}\int\frac{d^dk}{(2\pi)^d}\,\ln(1 - e^{-\beta\omega_{\mathbf{k}}}), \tag{18}$$

where z is the lattice coordination number, d is the number of spatial dimensions, and where the integral is performed over the first Brillouin zone. The dispersion $\omega_{\mathbf{k}}$ is defined by

$$\begin{aligned} \mu &\equiv \lambda - zQ \\ \gamma_{\mathbf{k}} &\equiv \frac{1}{z}\sum_{\mathbf{a}} e^{i\mathbf{k}\cdot\mathbf{a}} \\ \omega_{\mathbf{k}} &\equiv \mu + zQ(1-\gamma_{\mathbf{k}}), \end{aligned} \tag{19}$$

where $\mathbf{a}$ denotes a nearest neighbor displacement vector. The saddle point equations $\delta F/\delta Q = 0$ and $\delta F/\delta\lambda = 0$ are

$$S = \int\frac{d^dk}{(2\pi)^d}\, n_{\mathbf{k}} \tag{20a}$$

$$Q = S - \int\frac{d^dk}{(2\pi)^d}\,(1-\gamma_{\mathbf{k}})n_{\mathbf{k}}, \tag{20b}$$

with $n_{\mathbf{k}} = (e^{\beta\omega_{\mathbf{k}}}-1)^{-1}$. Remarkably, the above mean field equations are identical to those obtained earlier by Takahashi,[25] who employed a variational density matrix under a zero net magnetization constraint to study the finite temperature behavior of low-dimensional ferromagnets. Takahashi found his results for the free energy and magnetic susceptibility to be in remarkable agreement with exact thermodynamic Bethe *Ansatz* results. There are, however, differences in our theories. For instance, while the mean field equations coincide, our expression for the free energy (minus the classical contribution) is a factor of two larger than Takahashi's energy. This factor of two seems to be an artifact of the static assumption and is a generic consequence of approximations of this sort. The SU(N) theory is defined in terms of N bosons and one constraint per site. Taking $\lambda_i^{\rm MF} = \lambda$ constant amounts to ignoring the nonzero wavelength components of the constraint field, enforcing the local restriction $\sum_\alpha b_\alpha^\dagger b_\alpha = NS$ only on average, as in Eq(20a). Thus, at the mean field level, the number of independent degrees of freedom is overcounted by a factor $g = N/(N-1)$. This is partially corrected by the $\mathcal{O}(1/N)$ contribution $F^{(1/N)}$ arising from integration over the Gaussian fluctuations of the constraint field.[24] The main difference lies in the fact that our mean field theory preserves global rotational invariance, unlike the theory of Ref[25]. Our spin-spin correlation function and static susceptibility are given by

$$\begin{aligned} \langle S^\alpha_{\mathbf{R}} S^\beta_{\mathbf{R}'}\rangle &= \tfrac{1}{2}S\delta_{\alpha\beta}\,\delta_{\mathbf{R},\mathbf{R}'} + \tfrac{1}{2}\delta_{\alpha\beta}|f(\mathbf{R}-\boldsymbol{R}')|^2 \\ f(\mathbf{R}-\mathbf{R}') &= \int\frac{d^dk}{(2\pi)^d}\, e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')}\, n_{\mathbf{k}} \\ k_{\rm B}T\,\chi^{zz}(\mathbf{q}) &= \tfrac{1}{2}S + \tfrac{1}{2}\int\frac{d^dk}{(2\pi)^d}\, n_{\mathbf{k}}\, n_{\mathbf{k}+\mathbf{q}}. \end{aligned} \tag{21}$$

At the mean field level, there is a temperature-dependent excitation gap μ which vanishes at the Curie transition. The transition temperature $T_{\rm C}$ is zero in two or fewer dimensions, in accordance with the Mermin-Wagner theorem, and is nonzero for $d > 2$.[24] The mean field critical exponents are determined by how μ vanishes as $T \to T_C^+$ and were derived in $d = 1, 2$ in Ref[24]. Recently, Sarkar, Jayaprakash, Krishnamurthy, and Ma,[26] examined the theory in $d > 2$ and obtained the following mean field exponents:

$$\begin{aligned} \mu &\sim t^{2/(d-2)} \\ \xi &\sim t^{-1/(d-2)} \\ \chi &\sim t^{-(4-d)/(d-2)} \end{aligned} \tag{22}$$

with $t = (T - T_C)/T_C$. The phase transition is thus a Bose condensation.[27]

Mean Field Theory of the Antiferromagnet

The static mean field approximation of Eq(17) leads to the following mean field Hamiltonian:

$$\begin{aligned} H_{\rm MF} = \tfrac{1}{2} N z Q^2 - N S \lambda + \tfrac{1}{2} \int \frac{d^dk}{(2\pi)^d} \sum_\alpha \Big[&\lambda (b^\dagger_{\mathbf{k},\alpha} b_{\mathbf{k},\alpha} + b^\dagger_{-\mathbf{k},\alpha} b_{-\mathbf{k},\alpha}) + \\ &+ zQ(\bar\gamma_{\mathbf{k}} b_{\mathbf{k},\alpha} b_{-\mathbf{k},\alpha} + \gamma_{\mathbf{k}} b^\dagger_{\mathbf{k},\alpha} b^\dagger_{-\mathbf{k},\alpha}) \Big] . \end{aligned} \tag{23}$$

It can easily be verified that the Hamiltonian does not break global rotational symmetry. $H^{\rm MF}$ is diagonalized by a Bogoliubov transformation to the quasiparticle operators $\beta_{\mathbf{k},\alpha} = \cosh\theta_{\mathbf{k}} b_{\mathbf{k},\alpha} + \sinh\theta_{\mathbf{k}} b^\dagger_{-\mathbf{k},\alpha}$ with $\tanh(2\theta_{\mathbf{k}}) = -zQ\gamma_{\mathbf{k}}/\lambda$. The mean field specific free energy per flavor is given by

$$\begin{aligned} F^{\rm MF} &= \tfrac{1}{2} z Q^2 - (S + \tfrac{1}{2})\lambda + \frac{1}{\beta} \int \frac{d^dk}{(2\pi)^d} \ln(2\sinh \tfrac{1}{2}\beta\omega_{\mathbf{k}}) \\ \omega_{\mathbf{k}} &= \sqrt{\lambda^2 - z^2 Q^2 \gamma^2_{\mathbf{k}}}. \end{aligned} \tag{24}$$

and the steepest descents equations are

$$\begin{aligned} \frac{dF^{\rm MF}}{d\lambda} &= \int \frac{d^dk}{(2\pi)^d} \cosh(2\theta_{\mathbf{k}}) (n_{\mathbf{k}} + \tfrac{1}{2}) - (S + \tfrac{1}{2}) = 0 \\ \frac{1}{Qz} \frac{dF^{\rm MF}}{dQ} &= 1 - \int \frac{d^dk}{(2\pi)^d} \gamma_{\mathbf{k}} \sinh(2\theta_{\mathbf{k}}) (n_{\mathbf{k}} + \tfrac{1}{2}) = 0. \end{aligned} \tag{25}$$

There is a $\mathbf{k} = 0$ energy gap $\Delta = \sqrt{\lambda^2 - z^2 Q^2}$, and when Δ is nonzero there is an exponential decay to the spin-spin correlations with $\xi \sim c/\Delta$. At finite temperature in two or fewer dimensions, $\Delta > 0$, and the system is disordered, again in conformity with the restrictions of the Mermin-Wagner theorem. For $d > 2$, there may be a nonzero Néel temperature $T_{\rm N}$ at which the spins freeze into a Schwinger boson condensate. For large S, Sarkar *et al.* have obtained the result $k_{\rm B} T_{\rm N} \simeq \frac{1}{4} z S^2$, which compares well in $d = 3$ with the cubic lattice numerical estimate of $1.45 S^2$.[26] The critical properties of the antiferromagnet derived in Ref[26] are identical to those of Eq(22) for the ferromagnet.

The condition for an ordered ground state ($T = 0$) is

$$M_s = S + \tfrac{1}{2} - \tfrac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{\sqrt{1-\gamma_{\mathbf{k}}^2}} > 0, \tag{26}$$

with M_s the spontaneous staggered magnetization. For $d = 2$ this yields the condition $S = n_c/N \gtrsim 0.19$. Thus, for what it's worth, the mean field theory predicts the $S = \frac{1}{2}$ square lattice Heisenberg antiferromagnet to order at $T = 0$. Of course, small N extrapolations are not expected to be reliable. In $d = 1$, the integral in Eq(26) diverges, and thus the ground state is predicted to be disordered for all S, as is known to be the case. As shown by Read and Sachdev, interference effects from Berry phases unaccounted for at the mean field level lead to a qualitative alteration of the physics of the disordered phase when n_c is odd in $d = 1$, or whenever n_c is not an integer multiple of 4 on the square lattice in $d = 2$. In such cases, spin-Peierls order is generic.

It is instructive to analyze the mean field ground state,

$$|\,G\,\rangle = \exp\left(\sum_{\mathbf{R},\mathbf{R}',\alpha} g(\mathbf{R}-\mathbf{R}')\, b^\dagger_{\mathbf{R},\alpha} b^\dagger_{\mathbf{R}',\alpha}\right) |\,0\,\rangle \tag{27}$$

where

$$\begin{aligned} g(\mathbf{R}-\mathbf{R}') &= \int \frac{d^dk}{(2\pi)^d} \tanh\theta_{\mathbf{k}}\, e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} \\ \tanh\theta_{\mathbf{k}} &= \left(\sqrt{1-\eta^2\gamma_{\mathbf{k}}^2} - 1\right)/\eta\gamma_{\mathbf{k}} \end{aligned} \tag{28}$$

and $\eta = zQ/\lambda$. It is easy to see that $g(\mathbf{R}-\mathbf{R}')$ vanishes whenever $\mathbf{R}$ and $\mathbf{R}'$ lie in the same sublattice. Mindful of the fact that the SU(N) representations on alternate sublattices are conjugates, one sees that $|\,G\,\rangle$ describes a condensate of singlet pairs which, when projected onto the physical subspace in which there are n_c bosons per site, is an SU(N) version of the RVB states of Liang, Douçot, and Anderson.[28] We can also expand the function $g(\mathbf{R}-\mathbf{R}')$ in powers of η and find that the amplitude to connect two sites which are L links apart behaves as η^L. If g is restricted to connect only nearest neighbors (lowest order in η), one finds when $n_c = 0$ modulo z that the projected ground state is dominated by the associated AKLT state in the thermodynamic limit.

Quantitatively, the mean field theory seems to work well when compared with existing results. For instance, the mean field result[24] for the $d = 1$ Haldane gap is given (for S large) by $\Delta \sim Se^{-\pi S}$, which is precisely the result obtained from a 1-loop renormalization group calculation of the nonlinear sigma model with coupling $g = 2/S$. For the square lattice antiferromagnets, the low-temperature correlation length was found[24] to behave as $\xi \sim e^{A(S)/T}$, with $A(S = \frac{1}{2}) = 1.10$ and $A(S = 1) = 5.45$, which again compares well the sigma model results of Chakravarty, Halperin, and Nelson.[3] The renormalization constants Z_c and Z_χ for the spin wave velocity and magnetic susceptibility, respectively, are found[24] to be $Z_c = 1.16$ and $Z_\chi = 0.53$ for $S = \frac{1}{2}$, which nicely match the numerical results $Z_c = 1.175$, $Z_\chi = 0.529$ obtained by Singh and Huse.[29] Such quantitative agreement may well be spurious, given that the mean field theory is exact only in the $N \to \infty$ limit, however the qualitative physical picture makes good sense and I suspect that the techniques discussed here should prove serviceable if extended to models of the $t - J$ variety, where slave fermions can be employed to keep track of holes.[30,31]

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