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PURE STATE QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS IN $\mathbb{C}^2$

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Abstract

We classify all the pure state valued stochastic differential equations in the two dimensional Hilbert space $\mathbb{C}^2$ that are such that the corresponding density matrix follows a closed evolution equation. In particular we show that pure state quantum stochastic differential equations exist for all possible evolutions of the density matrix, including the non completely positive ones.

1. Introduction

In recent years there has been quite a lot of interest for pure state valued stochastic differential equations which are such that the density operator ρ obtained by averaging over the "noise" follows a closed evolution equation (ie $d\rho/dt$ = function of ρ). We refer to such equations as *pure state quantum stochastic differential equation* (QSDE). For the motivations we refer to references [1] to [19].

In this article we present a method to find all the pure state QSDE with white noise in the two dimensional Hilbert space $\mathbb{C}^2$. We recover of course the well-known examples, but show also the existence of such equations for all possible evolutions of the density matrix, in particular pure state QSDE corresponding to non completely positive evolutions.

The problem treated in this article can also be considered in a completely different framework. Today's optical telecommunications will soon reach so high bandwidth that polarization mode dispersion will have to be taken into account. In standard single-mode optical fibers the state of polarization evolves stochastically along the fiber, due to randomly distributed local perturbations of the circular symmetry [20,21]. In this framework our problem consists in finding all the random evolutions with white noise of the state of polarization compatible with a linear and deterministic evolution of the Jones vector.

2. Classification of pure state QSDE in $\mathbb{C}^2$

In order to classify the pure state valued stochastic processes in $\mathbb{C}^2$, we remind the reader that the pure states can be parametrized by the points $\vec{m}\in\mathbb{R}^3$, $|\vec{m}|=1$, of the unit sphere (see for instance [22]). Let η denote the "height" of the point $\vec{m}$ of the sphere and φ the azimutal angle:

$$\vec{m} = (\sqrt{1-\eta^2}\,\cos(\varphi),\ \sqrt{1-\eta^2}\,\sin(\varphi),\ \eta).$$

The pure state and the corresponding one dimensional projectors, can then be written as:

$$\psi = \sqrt{1/2}\begin{pmatrix}\sqrt{1+\eta}\, e^{-i\varphi/2} \\ \sqrt{1-\eta}\, e^{i\varphi/2}\end{pmatrix} \qquad \mathbf{P}_\psi = \frac{1}{2}\begin{pmatrix}1+\eta & \sqrt{1-\eta^2}\, e^{-i\varphi} \\ \sqrt{1-\eta^2}\, e^{i\varphi} & 1-\eta\end{pmatrix} \tag{1}$$

with $\vec{m} = \langle\psi|\vec{\sigma}|\psi\rangle = \mathrm{Tr}(\vec{\sigma}\mathbf{P}_\psi)$, where $\vec{\sigma}$ are the usual Pauli matrices.

We are interested in the class of Itô stochastic differential equations [23,24]:

$$\begin{aligned} d\eta &= b(\eta,\varphi)\, dt + g(\eta,\varphi)\, d\xi, \\ d\varphi &= a(\eta,\varphi)\, dt + f(\eta,\varphi)\, dw, \\ (d\xi)^2 &= 2dt \quad (dw)^2 = 2dt \quad d\xi dw = 2\mu(\eta,\varphi)dt \quad (\mu^2 \le 1), \end{aligned} \tag{2}$$

where a, b, f, g are 4 functions[1] and ξ, w are 2 Wiener processes such that the corresponding density matrix ρ evolves in a closed form, ie $d\rho/dt$ = function of ρ. But if $\rho = x\rho_1 + (1-x)\rho_2$, where x describes my ignorance of the real state of the system, then clearly $d\rho/dt = xd\rho_1/dt + (1-x)d\rho_2/dt$, that is the function must be linear. In other words, inserting (2) into (1), the pure states satisfy an Itô stochastic differential equation:

$$d\mathbf{P}_\psi = \ldots d\xi + \ldots dw + \mathbf{L}\mathbf{P}_\psi dt, \tag{3}$$

with $\mathbf{L}$ a linear (super) operator acting on the space of trace class operators. Hence, the density matrix $\rho \equiv \langle\!\langle \mathbf{P}_\psi \rangle\!\rangle_{st}$, obtained by averaging $\mathbf{P}_\psi$ over the noises $d\xi$ and dw, satisfies the linear ordinary differential equation:

$$d\rho/dt = \mathbf{L}\rho \tag{4}$$

By construction of $\mathbf{L}$, this equation is an evolution equation for density matrices, ie it preserves the trace, the positivity and the hermiticity of ρ. Indeed, the construction of ρ provides for all times a decomposition of ρ into a normalized distribution of pure states.

The following theorem states the restrictions imposed by the condition of closed evolution of the density operator on the admissible.functions a, b, f and g.

[1] We assume the 4 functions nice enough to ensure existence and uniqueness of the solution of equations (2), see reference [15].

Theorem:

The stochastic process (2) satisfies the condition (3) iff there are real numbers $\lambda_1, \lambda_2, \lambda_3, k$ and $\vec{\omega} \in \mathbb{R}^3$ such that

$$\lambda_3 \leq -k \leq 0 \quad \text{and} \quad \max(\lambda_1, \lambda_2) \leq \frac{1}{2}(\lambda_3 + \sqrt{\lambda_3^2 - k^2}) \tag{5}$$

and the 4 functions *a*, *b*, *f* and *g* satisfy conditions (6) to (8):

$$b(\eta, \varphi) = -\frac{\partial h(\eta, \varphi)}{\partial \varphi} + 2(k + \lambda_3 \eta) \tag{6}$$

$$g^2(\eta,\varphi)/(1-\eta^2) + (1-\eta^2) f^2(\eta,\varphi) = -\eta b(\eta,\varphi) - 2(1-\eta^2)\left[\lambda_1 \cos^2(\varphi) + \lambda_2 \sin^2(\varphi)\right] \tag{7}$$

$$(1-\eta^2) a(\eta,\varphi) = 2\eta\mu(\eta,\varphi) f(\eta,\varphi) g(\eta,\varphi) + (1-\eta^2)\left[\frac{\partial h(\eta,\varphi)}{\partial \eta} - (\lambda_1 - \lambda_2)\sin(2\varphi)\right] \tag{8}$$

where $h(\eta,\varphi) = \omega_1\sqrt{1-\eta^2}\cos(\varphi) + \omega_2\sqrt{1-\eta^2}\sin(\varphi) + \eta\omega_3 = \langle\psi|\vec{\omega}\,\vec{\sigma}|\psi\rangle$ corresponds to the usual hamiltonian [22].

Proof:

First, from (1) and (2) one deduces:

$$(\mathbf{L}P_\psi)_{11} = 1 - (\mathbf{L}P_\psi)_{22} = b(\eta,\varphi)/2, \tag{9}$$

$$(\mathbf{L}P_\psi)_{12} = (\mathbf{L}P_\psi)^*_{21} = -\frac{1}{2} e^{-i\varphi} (1-\eta^2)^{-1/2} \left[\eta . b + g^2/(1-\eta^2) + (1-\eta^2)(ia+f) - 2i\eta\mu f g\right] \tag{10}$$

Next, the most general super-operator **L** can be written, in an appropriate bases, as (see lemma 6 of ref. 25):

$$\mathbf{L}\rho = (\mathbf{R}\vec{M})\vec{\sigma} + \lambda_1 M_1 \sigma_1 + \lambda_2 M_2 \sigma_2 + (\lambda_3 M_3 + k)\sigma_3 \tag{11}$$

where σ_i are the usual Pauli matrices, $\vec{M} = \mathrm{Tr}(\vec{\sigma}\rho)$, **R** is a rotation operator acting in R^3. In ref. 25 Kossakowski has shown that positivity of the evolution generated by (11) is equivalent to the following condition on the λ_i and k:

$$\sup_{|\vec{m}|=1} [\lambda_1 m_1^2 + \lambda_2 m_2^2 + (\lambda_3 m_3^2 + k m_3)] \leq 0 \qquad (5b)$$

A direct computation shows that this is equivalent to (5).

Finally, the theorem follows from the identification of (9), (10) and (11). The vector $\vec{\omega}$ is determined by the rotation **R** [22].

QED

Note that the conditions (5) to (8), defined in the above theorem, hold in the basis defined by the super-operator **L** through the equality (11).

For a given evolution equation (11) of the density matrix, the conditions (6) and (8) are definitions of the functions $b(\eta,\varphi)$ and $a(\eta,\varphi)$, respectively. The condition (7), however, is stronger: its right hand side must be non negative for all $\eta \in [-1,1]$ and all φ. But, if one lets $m_1^2=(1-\eta^2)\cos\varphi$, $m_2^2=(1-\eta^2)\sin\varphi$, $m_3=\eta$, it is immediate that the non negativity of the right hand side of (7) is identical to condition (5b), which is itself equivalent to condition (5). This shows that for all possible evolutions of the density matrix there is at least one pure state QSDE. This result extends the known existence of pure state QSDE for all completely positive evolutions [10,11,13]. In section 5 we present an example of a non completely positive evolution of the density matrix that admits a corresponding pure state QSDE.

In order to get a better understanding of equation (11) let us rewrite $\mathbf{L}\rho$ in a form more suitable for our purpose:

$$\mathbf{L}\rho = -\frac{i}{2}[\vec{\omega}\vec{\sigma}, \rho] + \sum_{i=1}^{3} D_i(\sigma_i\rho\sigma_i - \rho) + k(\sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+, \rho\}) \qquad (12)$$

where $\{.,.\}$ denote the anticommutator, $\sigma_\pm=\frac{\sigma_1\pm\sigma_2}{2}$, $D_1=(\lambda_1-\lambda_2-\lambda_3-k)/2$, $D_2=(\lambda_2-\lambda_3-\lambda_1-k)/2$, $D_3=(\lambda_3-\lambda_1-\lambda_2)/2$ and $\vec{\omega}$ is the rotation axis (conversely: $\lambda_1=-D_2-D_3-k/2$, $\lambda_2=-D_1-D_3-k/2$, $\lambda_3=-D_1-D_2-k$).

The first term is the well-known hamiltonian term. The three terms in the sum describe an exponential damping of the off diagonal terms of ρ in the basis diagonalizing σ_i. The last term describes a relaxation towards the "up" state, ie towards $\eta=1$.

In the next sections we shall study two of the extreme case: $\vec{\omega} = k = D_1 = D_2 = 0$, $D_3 \neq 0$, ie diagonalization of ρ in a fixed basis (loss of coherence), and $\vec{\omega} = D_1 = D_2 = D_3 = 0$, $k \neq 0$, ie relaxation.

3. Case of loss of coherence

In this section we study the case:

$$L\rho = \frac{1}{2}(\sigma_z\rho\sigma_z - \rho), \tag{13}$$

ie $\vec{\omega} = \vec{0}$, $k = D_1 = D_2 = 0$, $D_3 = \frac{1}{2}$, hence $\lambda_3 = 0$, $\lambda_1 = \lambda_2 = -\frac{1}{2}$.

Let us first consider the subcase of independent noises $d\xi d\omega = \mu = 0$. The general conditions become:

$$a(\eta,\varphi) = 0$$
$$b(\eta,\varphi) = 0$$
$$f^2(\eta,\varphi) \leq 1$$
$$g^2(\eta,\varphi) = (1-\eta^2)^2\left[1-f^2(\eta,\varphi)\right]$$

The corresponding equation for the state vector reads, with $\circ$ the stratonovich product [23,24]:

$$\begin{aligned} d\psi &= -\frac{i}{2}\,\sigma_z\,\psi \circ d\varphi + \frac{1}{2}\,\frac{\sigma_z-\eta}{1-\eta^2}\,\psi \circ d\eta \\ &= -\frac{i}{2}\,\sigma_z\,\psi \circ f(\eta,\varphi)d\omega + \frac{1}{2}\sqrt{1-f^2}\,(\sigma_z - \eta)\,\psi\, d\xi \\ &\qquad - \frac{1}{4}(1-f^2)\,(\sigma_z - \eta)^2\,\psi\, dt \end{aligned} \tag{14}$$

We note that the case $f(\eta,\varphi) \equiv \pm 1$ describes a spin 1/2 in a magnetic field in the z-direction with a fluctuating amplitude [2,26,27]. The case $f(\eta,\varphi) \equiv 0$ has been already discussed in [2,3,4,7,9,10,11,12,13,18]. It corresponds to a measurement like situation: $\psi_t \longrightarrow |up\rangle$ or $|down\rangle$ with the quantum probabilities $\langle\psi|(1\pm\sigma_z)/2|\psi\rangle$. The other cases are new.

In reference [8] Ph Pearle has emphasized that a finite reduction time requires $g(\eta,\varphi) \cong (1-\eta^2)^r$ with $r < 1$. But this implies $\sqrt{1-f^2} > 1$. This shows again [8] that a finite reduction time is impossible within the present framework.

The equation (14) can easily be generalized to an arbitrary Hilbert space $\mathcal{H}$ by replacing σ_z with an arbitrary self-adjoint operator A, η with $\langle\psi|A|\psi\rangle/\langle\psi|\psi\rangle$, and f any function from $\mathcal{H}$ into the interval [-1,+1].

Finally we note that in the second subcase, $d\xi d\omega = 2\mu dt$ with $\mu\neq0$, the general conditions become:

$$\begin{aligned} a(\eta,\varphi) &= 2\eta\mu(\eta,\varphi)f(\eta,\varphi)\sqrt{1-f^2(\eta,\varphi)} \\ b(\eta,\varphi) &= 0 \\ f^2(\eta,\varphi) &\leq 1 \\ g^2(\eta,\varphi) &= (1-\eta^2)^2\left(1-f^2(\eta,\varphi)\right) \end{aligned}$$

4. Case of relaxation

In this section we study the case:

$$L\rho = \sigma_+\rho\sigma_- - \frac{1}{2}\{\sigma_-\sigma_+,\ \rho\}, \tag{15}$$

ie $\vec{\omega} = \vec{0}$, $D_1=D_2=D_3=0$, $k = 1$, hence $\lambda_1=\lambda_2=-\frac{1}{2}$, $\lambda_3=-1$. The general conditions become:

$$\begin{aligned} a(\eta,\varphi) &= 2\eta\mu(\eta,\varphi)f(\eta,\varphi)g(\eta,\varphi)/(1-\eta^2) \\ b(\eta,\varphi) &= 2(1-\eta) \\ f^2(\eta,\varphi) &\leq \frac{1-\eta}{1+\eta} \\ g^2(\eta,\varphi) &= (1-\eta^2)^2\left(\frac{1-\eta}{1+\eta} - f^2(\eta,\varphi)\right) \end{aligned} \tag{16}$$

The corresponding equation for the state vector reads, with $\circ$ the stratonovich product [23,24]:

$$d\psi = -\frac{i}{2}\,\sigma_z\,\psi \circ d\varphi + \frac{1}{2}\,\frac{\sigma_z-\eta}{1-\eta^2}\,\psi \circ d\eta \tag{17}$$

where $d\varphi$ and $d\eta$ are given by (2) and (16).

Note that if $f(\eta,\varphi)^2 = \frac{1-\eta}{1+\eta}$, then the evolution of η is deterministic.

The case $\mu\equiv1$, $f(\eta,\varphi) = \sqrt{\frac{1-\eta}{1+\eta}}\sin(\varphi)$ has been applied to spin relaxation in reference [11]; in this case the equation (17) simplifies drastically:

$$d\psi = (\sigma_+ - \langle\sigma_+\rangle)\psi \circ d\xi + 2\langle\sigma_x\rangle(\sigma_+ - \langle\sigma_+\rangle)\psi dt + \frac{1}{2}(\sigma_z - \langle\sigma_z\rangle)\psi dt$$
$$= (\sigma_+ - \langle\sigma_+\rangle)\psi d\xi - (\sigma_-\sigma_+ - 2\langle\sigma_-\rangle\sigma_+ + \langle\sigma_-\rangle\langle\sigma_+\rangle)\psi dt$$

where $\langle X\rangle \equiv \langle\psi|X|\psi\rangle/\langle\psi|\psi\rangle$. This case can straightforwardly be generalized to the case of arbitrary Hilbert spaces [11,12,13]:

$$d\psi = (B - b)\psi d\xi - (B^+B - 2b^*B + bb^*)\psi dt$$

where $b = \langle\psi|B|\psi\rangle/\langle\psi|\psi\rangle$.

The other cases are unknown. Their complexity is surprising, specially the case $f(\eta,\varphi)\equiv 0$.

5. Positivity versus complete positivity

Complete positivity is a condition about the evolution of density operators which is stronger than positivity (ie mapping the set of positive trace class operator into itself). Let us recall that a map $\rho_0 \in \mathcal{T}(\mathcal{H}) \longmapsto \rho_t \in \mathcal{T}(\mathcal{H})$ is completely positive [28,29,30] if and only if its natural extension to $\mathcal{T}(\mathcal{H}\otimes\mathbb{C}^n)$ is positive for all n. Considering the physical meaning of the tensor product, complete positivity is a natural condition. In the case of two dimensional Hilbert space $\mathbb{C}^2$, the conditions on the D_i and k of eq. (12) for complete positivity are well-known [25,30]. For simplicity we recall them only in the case $D_x=D_y\equiv D_\perp$:

$$\text{complete positivity} \iff D_\perp \geq 0 \text{ and } D_z \geq 0$$
$$\text{existence of pure state QSDE} \iff D_\perp \geq 0 \text{ and } D_z \geq -\sqrt{D_\perp^2 + D_\perp k} \iff \text{positivity}$$

Hence, as is well known, there are positive, but non completely positive evolutions. There are thus pure state QSDE corresponding to non completely positive evolutions. This may be surprising, since one could try to extend a pure state QSDE on a Hilbert space $\mathcal{H}$ to $\mathcal{H}\otimes\mathbb{C}^n$ by replacing all operators X by $X\otimes\mathbf{1}$. But this extension is not unique (recall that $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, $\langle\sigma_z\rangle$ are not independent: $\sum\langle\sigma_i\rangle^2=1$). Moreover, such extended stochastic differential equations do not necessarily correspond to close evolution for the density matrix.

To illustrate to above discussion, let us consider the following non completely positive evolution:

$$d\rho/dt = \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y - \sigma_z\rho\sigma_z - \rho \tag{18}$$

ie $D_x=D_y=-D_z=1$, k=0. A corresponding pure state QSDE is given by (2) with:

$$a(\eta,\varphi) = g(\eta,\varphi) = 0, \ b(\eta,\varphi) = -4\eta, \text{ and } f(\eta,\varphi) = \frac{4\eta^2}{1-\eta^2}.$$

Let **P** be the projector onto the normalized vector ψ: $\mathbf{P} \equiv |\psi\rangle\langle\psi|$. After some computation one gets:

$$dP = \ldots dw - \frac{2\eta}{1-\eta^2} (\{\sigma_z, P\} - \eta P - \eta\sigma_z P\sigma_z)\ dt \tag{19}$$

where the dots stand for some unimportant factor multiplying the "noise" dw. Using the following equality:

$$\eta P + \eta\sigma_z P\sigma_z - \{\sigma_z, P\} = -2(1-\eta^2)\sigma_z \tag{20}$$

one checks that eq. (19) corresponds indeed to the evolution (18).

The natural extension of eq. (19) to $\mathcal{H}\otimes\mathbb{C}^n$ is obtained by replacing σ_z by $\sigma_z\otimes\mathbf{1}$ and η by $Tr(\sigma_z\otimes\mathbf{1})$ where **1** is the identity operator in $\mathbb{C}^n$. But the extension of equality (20) holds if and only if n=1 ! Hence the natural extension of (19) is a pure state QSDE if and only if n=1.

6. Conclusion

We have reduced the problem of finding pure state QSDE in $\mathbb{C}^2$ to the choice of functions satisfying the general conditions (5) to (8). This shows the existence of infinitely many pure state QSDE, covering all the possible (ie positive, but not necessarily completely positive) evolution equations of the density matrix. This raises the question of the criterions for "good" candidates. The assumption that there are no arbitrarily fast communication implies that the averaged density operator should follow a closed evolution (this is the main result of ref. 11, see also ref. 4,7,16 and 19). The use of Gaussian processes with white noises is natural as a first trial. Then simplicity leads to equation (14) with $f(\eta,\varphi) = 0$, see references [2,3,4,7,9,10,11,12,13,18]. Maybe that the main result of this article is the emphasis of the importance for new good physical criterions to restrict the "interesting" pure state quantum stochastic differential equations.

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