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Local Correlation Functions for Mean-Field Dynamical Semigroups on C*-algebras

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Abstract. We study the dynamics of local correlation functions in dissipative mean-field systems. This is done by extending the abstract notion of a mean-field dynamical semigroup on a C*-algebra given in [1] , from an evolution on site-averaged observables, to one on a class of local observables. Conditions are established under which this generalized mean-field dynamics factorises, in the thermodynamic limit, into contributions from disjoint regions. Correspondingly, the nested correlation functions factorise into contributions for single site observables in this limit. We demonstrate that these conditions are satisfied for a large class of model systems.

1. Introduction.

In this paper we extend the theory of mean-field dynamical semigroups, as described in [1] , to include the mean-field dynamics of local observables. We formulate general conditions under which all nested local correlation functions factorise into contributions for disjoint regions in the thermodynamic limit. We show that these conditions are satisfied for a large class of model systems. First we recall the main features of mean-field dynamical systems, and summarize the results on which the present work relies.

Mean-field dynamical semigroups are used, implicitly or explicitly, to analyse the dynamics of dissipative quantum systems in the thermodynamic limit. Following the example of Hepp and Lieb **[2]** in their treatment of the dynamics of the laser, many other models have been treated by various authors. These include, for example, the BCS model **[3]** , H. Fröhlich's model of non-equilibrium boson condensation **[4,5]** , and the boson gas relaxing to thermal equilibrium **[6,7]** . Although these models differ in detail, they have the following common features: (a) a sequence of systems indexed by a volume parameter; (b) for each volume a dissipative quantum dynamics, and (c) a relationship between the generators of the dynamics for different volumes, which essentially specifies the mean-field nature of the model. At each volume, the dissipative quantum dynamics is obtained from the hamiltonian dynamics of a larger system (the system + thermal reservoirs) by isolation of the dynamics of the system variables through some limiting procedure (for example, the weak-coupling and long-time limit). A review of these matters can be found in **[8]** .

Building upon some of the original notions of **[2]** , and their abstract generalization in **[9]** , a general theory of mean-field dynamical limits has been obtained in **[1]** , as we shall describe shortly. First we outline the mathematical description of intensive observables, as given in **[9]** (but using the notation of **[1]**). The sequence of systems is labelled by the positive integers. The n^{th} system comprises n sites, at each which sits a copy of some C*-algebra $\mathcal{A}$ with identity $\mathbb{1}$. The observable algebra of the n^{th} system is $\mathcal{A}^n$: the tensor product of n copies of $\mathcal{A}$ (completed in the minimal C*-cross-norm). If $X_m \in \mathcal{A}^m$ is an m-site observable for some $m \in \mathbb{N}$, then for each $n > m$ we can form the observable X_n by averaging X_m over all n sites i.e. by averaging $X_m \otimes \mathbb{1}_{n-m}$ over all automorphisms of $\mathcal{A}^n$ induced by permutations of the n sites. We shall denote this process by the operator j_{nm}, so that in the above case $X_n = j_{nm}X_m$. To illustrate, if $X_1 \in \mathcal{A}$ then $j_{21}X_1 = \frac{1}{2}(X_1 \otimes \mathbb{1} + \mathbb{1} \otimes X_1) \in \mathcal{A}^2$.

The sequence of resymmetrized observables $(j_{nm}X_m)_{n>m}$ will be called strictly symmetric. From the mean-field point of view, we can say that an arbitrary sequence $n \mapsto Y_n$, with each Y_n in $\mathcal{A}^n$, represents an intensive observable if it can be approximated uniformly in n by strictly symmetric sequences, i.e. if for all $\varepsilon > 0$, there exists a strictly symmetric sequence $n \mapsto X_n^{\varepsilon}$ and an $n_{\varepsilon} \in \mathbb{N}$ such that $\|Y_n - X_n^{\varepsilon}\| < \varepsilon$ for all $n \geq n_{\varepsilon}$. Such sequences will be called approximately symmetric. Of course, all strictly symmetric sequences are also approximately symmetric.

From the thermodynamic point of view one expects that multiplication of intensive observables should be commutative in the thermodynamic limit. Correspondingly, it is a combinatorial result from [9] that for two approximately symmetric sequences $X.$ and $Y.$,

$$\lim_{n\to\infty} \|X_n Y_n - Y_n X_n\|_{\mathcal{A}^n} = 0 \quad .$$

This is a reflection of the fact that for approximately symmetric sequences, most of the factors in the tensor product are occupied by simply the identity element $\mathbb{1}$. Now, we can consider limits of approximately symmetric sequences as follows. In [9] the existence of the limit $X_\infty(\rho) \equiv \lim_{n\to\infty}\langle \rho^n, X_n\rangle$ is shown for all approximately symmetric sequences $X.$ and states ρ in space $K(\mathcal{A})$ of states on $\mathcal{A}$. (Here, ρ^n denotes the n-fold tensor product state $\rho \otimes \ldots \rho$ on $\mathcal{A}^n$, and $\langle\cdot,\cdot\rangle$ denotes the canonical bilinear form between a C*-algebra and its dual). If we consider the set of approximately symmetric sequences as an algebra with n-wise addition and multiplication, the the map $X. \mapsto X_\infty$ becomes a homomorphism from the set of approximately symmetric sequences onto the algebra $\mathcal{C}(K(\mathcal{A}))$ of weak*-continuous functions on the state space of $\mathcal{A}$. That this latter algebra is commutative is simply a reflection of the limiting commutativity of approximately symmetric sequences.

Having set up the general framework for intensive observables, we now turn to the question of dynamics, as described in [1] . Suppose that for each $n \in \mathbb{N}$ as strongly continuous semigroup of completely positive maps $(T_{t,n})_{t\geq 0}$ on $\mathcal{A}^n$ is given. We naturally say that the sequence $T_{t,\cdot}$ has good mean-field properties if it preserves the set of intensive observables i.e. if it maps the set of approximately symmetric sequences into itself. It is natural then to attempt to define a limiting semigroup on $\mathcal{C}(K(\mathcal{A}))$ via $(T_{t,\infty})_{t\geq 0}$ via the formula $T_{t,\infty}X_\infty = (T_{t,\cdot}X.)_\infty$. In [1] it is shown that this can be done when $T_{t,\cdot}$ satisfies a reasonable continuity condition. Moreover, a complete theory of mean-field dynamical semigroups on C*-algebras is obtained, paralleling the theory of contraction semigroups on Banach spaces.

In certain cases the limiting evolution is implemented by a continuous flow $(F_t)_{t\geq 0}$ on the state space: $(T_{t,\infty}X_\infty)(\rho) = X_\infty(F_t\rho)$. (The non-linear differential equation for the flow is just the Hartree equation). So it follows that the limiting evolution is a homomorphism: $(T_{t,\infty}(X_\infty Y_\infty))(\rho) = (T_{t,\infty}X_\infty)(\rho)(T_{t,\infty}Y_\infty)(\rho)$. In this case the correlation functions for intensive observables in product states factorise in the limit $n \to \infty$. For example, for approximately symmetric sequences $X.^1, \ldots X.^k$,

$$\begin{aligned}
&\lim_{n\to\infty}\langle \rho^n, T_{t_1,n}(X_n^1 T_{t_2,n}(X_n^2 \cdots T_{t_k,n}X_n^k)\ldots))\rangle \qquad (1.1)\\
&= \left(T_{t_1,\infty}(X_\infty^1 T_{t_2,\infty}(X_\infty^2 \cdots T_{t_k,\infty}X_\infty^k)\ldots))\right)(\rho)\\
&= (T_{t_1,\infty}X_\infty^1)(\rho)\,(T_{t_1+t_2,\infty}X_\infty^2)(\rho)\ldots(T_{t_1+t_2+\ldots+t_k,\infty}X_\infty^k)(\rho) \quad .
\end{aligned}$$

We remark that in all the physical examples mentioned, the generators G_n of the semigroups $T_{t,n}$ are polynomial in the sense that each G_n is obtained by symmetrization over n-sites of some fixed generator acting on the algebra of a finite number of sites. General theorems establishing a limiting dynamics obeying the factorisation conditions (1.1) had been given previously only for classes of models in which the generators are polynomial in the sense described, and bounded (possibly with the addition of an unbounded hamiltonian one-site term), **[10,11]**, sometimes with the requirement that $\mathcal{A}$ be finite dimensional **[12]**, or that evolution be hamiltonian **[13,14,15]**. As is shown in **[1]**, the class of models for which the limiting evolution is implemented by a flow is somewhat wider, and includes models in which the generator is not polynomial. Generators of this class occur in lattice systems for which the interaction does not link just finite numbers of sites. We emphasize, however, that mean-field dynamical limits need not be implemented by flows: in **[1]** an example is given in which the limit is diffusive.

We turn now to the central matter in the present paper, namely local correlation functions. By this, we mean that we want to consider expressions like eq. (1.1), but when the $X.^j$ are *not* approximately symmetric. In certain cases, and for certain choices of the X^j, the existing framework suffices. For example, if $T_{t,n}$ is itself invariant under permutation automorphisms, then we can estimate the correlation functions for two single site observables in a product state as follows:

$$\langle \rho^n, (A \otimes \mathbb{1}_{n-1}) T_{t,n} (\mathbb{1} \otimes B \otimes \mathbb{1}_{n-2}) \rangle = \langle \rho^n, (j_{n1} A) T_{t,n} (j_{n1} B) \rangle + O(n^{-1}) \quad . \tag{1.2}$$

Thus we could estimate the RHS of eq. (1.2) by using expressions of the form in eq. (1.1). The estimate relies on the fact that A and B lie in the algebras of different sites in the tensor product $\mathcal{A}^n$. To treat arbitrary correlation functions is an awkward combinatorial problem. What we do in this paper is to introduce a mean-field dynamical formalism which takes care of the combinatorics for us. It has the added advantage that we are able to treat states which are not simply products (or linear combinations of such states): we may also consider suitably local perturbations of such states. In this way we are extending the formalism of **[1]** which was designed to treat only the dynamics of approximately symmetric sequences. We finish the introduction by outlining the contents of the remainder of the paper.

In section 2 we give the theory of partially symmetric sequences. Briefly, this is a follows. Let I be a finite subset of $\mathbb{N}$. For $n \geq m, \max_{i \in I}\{i\}$, we define the partial averaging operators j^I_{nm} on $\mathcal{A}^m$ by setting $j^I_{nm} X_m$ to be the average of $X_m \otimes \mathbb{1}_{n-m}$ over all automorphisms of $\mathcal{A}^n$ corresponding to permutations of $\{1, \ldots n\}$ which leave the set

I pointwise invariant. For example, if $I = \{1\}$, then for any $X_1 \in \mathcal{A}$, $j_{n1}^{\{1\}} X_1 = X_1 \otimes \mathbf{1}_{n-1}$. A sequence of observables $(j_{nm}^I X_m)_{n>m}$ will be called strictly I-symmetric, and we have a corresponding notion of approximate I-symmetry. Using some ideas from **[9]** , we show that the limiting objects of such sequences form the algebra $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$, of weak*-continuous functions on the state space of $\mathcal{A}$ which take values in $\mathcal{A}^I$. For the sequence $n \mapsto j_{n1}^{\{1\}} X_1$ this limit is $X_1 \otimes \mathbf{1}_{\mathcal{C}(K(\mathcal{A}))}$. Now the limit algebra is no longer abelian in general. However, we show when I and J are disjoint finite subsets of $\mathbb{N}$, and X and Y are approximately I-symmetric and J-symmetric, then their product, regarded as an $(I \cup J)$-symmetric sequence, has a limit in $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^{I \cup J})$ which takes the form $X_\infty^I \otimes Y_\infty^I$ for some $X_\infty^I \in \mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ and $Y_\infty^J \in \mathcal{C}(K(\mathcal{A}), \mathcal{A}^J)$.

In section 3 we turn to the question of dynamics of approximately I-symmetric sequences. We formulate the notion of a mean-field dynamical limit on $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ for each finite $I \subset \mathbb{N}$. We say that a family of semigroups $T_{t,\cdot}$ is approximate I-symmetry preserving if it maps approximately I-symmetric sequences onto approximately I-symmetric sequences. One constructs a general theory of these mean-field limits in a way entirely analogous to that used for the case $I = \emptyset$ in **[1]** . The limiting evolution, which we will denote by $T_{t,\infty}^I$, is a contraction semigroup on $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$. The analogue of the homomorphism property for the present case is the factorisation of the limiting evolution on for disjoint finite subsets of $\mathbb{N}$ i.e. $T_{t,\infty}^{I \cup J} = T_{t,\infty}^I \otimes T_{t,\infty}^J$ when $I \cap J = \emptyset$. In section 4 we show that when this factorisation holds, all multi-time and nested correlation functions factorise into contributions over single sites. Note that the present method does not give any easy method for calculating multi-time correlation functions for one site alone. However, there are certain systems for which these can be calculated for a sufficiently large subalgebra of one-site observables **[16]** .

In section 5 we show that that mean-field dynamical semigroups with bounded polynomial generators have the disjoint homomorphism property. In fact, it is possible to show the same for the class of approximately polynomial generators described in **[1]** .

In an appendix, we return to the subject of mean-field dynamical semigroup for fully symmetric sequences. We investigate the conditions under which limiting flows preserve the set of normal states when $\mathcal{A}$ is isomorphic with a von Neumann algebra.

Finally, we mention the fact that the present work is not limited in application to homogeneous mean-field models. Following **[17]** , one replaces the one site algebra $\mathcal{A}$ by the algebra $\mathcal{C}(X, \mathcal{A})$ of $\mathcal{A}$-valued continuous function on some compact space X. In this way one can treat (for example) lattice models with a spatially varying interaction. The thermodynamics of such models has been treated in [**18,19**] , while their (fully symmetric) mean-field dynamics is treated in [**20**] .

2. Sequences with partial symmetry.

We start this section by generalizing the notion of symmetric sequences, to sequences which are symmetric only under a subgroup of permutations. In what follows, for any C*-algebra $\mathcal{A}$ with identity $\mathbb{1}$, $\mathcal{A}^*$ denotes the dual of $\mathcal{A}$, $\langle\cdot,\cdot\rangle : \mathcal{A}^* \times \mathcal{A} \to \mathbb{C}$ denotes the canonical bilinear form between $\mathcal{A}^*$ and $\mathcal{A}$, $K(\mathcal{A}) = \{\rho \in \mathcal{A}^* \mid \rho \geq 0,\ \langle\rho, \mathbb{1}\rangle = 1\}$ is the state space of $\mathcal{A}$, and $\mathcal{A}_+$ denotes the set of positive elements in $\mathcal{A}$.

Let $\mathcal{A}$ be a C*-algebra with identity $\mathbb{1}$. Let us associate with each positive integer i a copy, $\mathcal{A}_{\{i\}}$, of $\mathcal{A}$. Let I be an finite subset of $\mathbb{N}$, $I = \{i_1, \ldots, i_{|I|}\}$, where $|I|$ denotes the cardinality of I. Denote by I_{max} the largest element of I: $I_{max} = \max_{k=1,\ldots,|I|}\{i_k\}$. All subsets of $\mathbb{N}$ specified henceforth will be taken to be finite, but we shall occasionally reiterate this. For every such set I, $\mathcal{A}^I$ will denote the tensor product of of the $(\mathcal{A}_{\{i\}})_{i\in I}$, completed in the minimal C*-cross-norm [**21**] . $\mathbb{1}_I$ will denote the unit in $\mathcal{A}^I$.

The choice of completion has important consequences for the continuity of certain linear functionals on the tensor products. For any finite $I \subset \mathbb{N}$ and any collection $\{\omega_1, \ldots \omega_{|I|}\} \subset K(\mathcal{A})$, the linear functional $\omega_1 \otimes \ldots \omega_{|I|}$ on the algebraic tensor product $\mathcal{A}^{\odot|I|}$ has an extension to $\mathcal{A}^I$ which is a state. If all ω_i are equal to some ω we will write the corresponding state on $\mathcal{A}^I$ as ω^I. It follows from Corollary 4.25 of [**21**] that for any finite $I \subset \mathbb{N}$ and $X \in \mathcal{A}^I$, the map $(K(\mathcal{A}))^{|I|} \ni (\omega_1, \ldots \omega_{|I|}) \mapsto (\omega_1 \otimes \ldots \omega_{|I|})(X)$ is weak*-continuous. In particular, this means that for any disjoint finite subsets I and J of $\mathbb{N}$ and state ω in $K(\mathcal{A}^I)$, then for any $B \in \mathcal{A}^{I\cup J}$ there is an $A \in \mathcal{A}^J$ such that $\langle\omega \otimes \sigma, B\rangle = \langle\sigma, A\rangle$ for all $\sigma \in K(\mathcal{A}^J)$. In the following, for a C*-algebra $\mathcal{B}$, $\mathcal{C}(K(\mathcal{A}), \mathcal{B})$ will denote the space of continuous functions on the state space of $\mathcal{A}$ (with the weak*-topology) taking values in $\mathcal{B}$ (with the norm topology). $\mathcal{C}(K(\mathcal{A}), \mathbb{C})$ will be denoted by $\mathcal{C}(K(\mathcal{A}))$.

We will adopt the convention on lower case subscripts and superscripts as follows: $\mathcal{A}^n$ denotes the algebra $\mathcal{A}^{\{1,\ldots n\}}$, ρ^n is the state $\rho^{\{1,\ldots n\}}$ on $\mathcal{A}^n$.

Let S_n be the set of permutations of the numbers $\{1, 2, \ldots, n\}$ and for each $\gamma \in S_n$ we define π_γ to be the automorphism of $\mathcal{A}^n$ induced by it. Thus for $A^1, \ldots, A^n \in \mathcal{A}$ we have $\pi_\gamma(A^1 \otimes \ldots \otimes A^n) = A^{\gamma(1)} \otimes \ldots \otimes A^{\gamma(n)}$. For I with $I_{max} \leq n$ we define $S_n(I)$ to be the set of permutations in S_n which leave I pointwise invariant.

For each n and each I with $I_{max} \leq n$ we define the injection $i_{nI} : \mathcal{A}^I \to \mathcal{A}^n$ by

$$i_{nI} X = X \otimes \mathbb{1}_{\{1,\ldots,n\}\setminus I} \quad ,$$

i.e. by tensoring with the identity on all $\mathcal{A}_{\{j\}}$ for $j \notin I$ with $j \leq n$. For $n \geq m$ and I with $I_{max} \leq n$ define $j^I_{nm} : \mathcal{A}^m \to \mathcal{A}^n$ by

$$j^I_{nm} X_m = \frac{1}{(n - |I|)!} \sum_{\gamma \in S_n(I)} \pi_\gamma i_{nm} X_m \quad .$$

Thus j^I_{nm} is the average of $i_{nm} X_m$ when operated on by the set of all permutation automorphisms of $\mathcal{A}^n$ which leave $\mathcal{A}^I$ invariant. The operators $j^\emptyset_{nm}$ are identical with the operators j_{nm} as defined in [1] for all $n, m \in \mathbb{N}$.

For each fixed finite I, the operators j^I_{nm} are consistent in that they satisfy

$$j^I_{nm} \circ j^I_{mr} = j^I_{nr}$$

for $r \leq m \leq n$ with $I_{max} \leq m$. Restricted to $j^I_{mm} \mathcal{A}^m$ each j^I_{nm} is injective. Thus we can consider abstractly the spaces $j^I_{nn} \mathcal{A}^n$ together with the maps j^I_{nm} as an inductive system of vector spaces. Since each j^I_{nm} is a contraction in the given norms on $\mathcal{A}^m$ and $\mathcal{A}^n$ the inductive limit carries a natural seminorm: for an arbitrary sequence $(X_n)_{n \in \mathbb{N}}$ with $X_n \in \mathcal{A}^n$ we write

$$\|X\| = \limsup_{n \to \infty} \|X_n\| \quad . \tag{2.1}$$

For each finite $I \subset \mathbb{N}$ we single out from the set of all sequences $(X_n)_{n \in \mathbb{N}}$ those in the inductive limit space: those sequences $X. : n \mapsto j^I_{nn} \mathcal{A}^n$ for which for some $m_0 \in \mathbb{N}$, $X_n = j^I_{nm_0} X_{m_0}$ for all $n \geq m_0, I_{max}$. Such sequences will be called **strictly I-symmetric**, and the number m_0 will be called the **degree** of the sequence X as defined above. The set of all such sequences will be denoted by $\mathcal{Y}^I$. Let $\rho \in K(\mathcal{A})$ and let σ be an arbitrary element of $(\mathcal{A}^I)^*$, the dual of $\mathcal{A}^I$. For $X_n \in \mathcal{A}^n$ we define $j^I_{\infty n} X_n \in \mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ by

$$\langle \sigma, (j^I_{\infty n} X_n)(\rho) \rangle = \langle \sigma \otimes \rho^{\{1,\ldots,n\}\setminus I}, X_n \rangle \quad ,$$

For strictly I-symmetric sequences this is independent of n for n sufficiently large. Trivially, $X^I_\infty \equiv \lim_{n \to \infty} j^I_{\infty n} X_n$ exists as an element of $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$.

A sequence $X. : n \mapsto \mathcal{A}^n$ is called **approximately I-symmetric** if for all $\varepsilon > 0$ there is an n_ε such that $\|X_n - j^I_{nm} X_m\| \leq \varepsilon$ for all $n \geq m \geq n_\varepsilon \geq I_{max}$. For the latter statement we also write

$$\lim_{n \geq m \to \infty} \|X_n - j^I_{nm} X_m\| = 0 \quad .$$

The set of approximately I-symmetric sequences will be denoted by $\tilde{\mathcal{Y}}^I$.

Now, for each strictly I-symmetric sequence X, we can think of X_n as a element of $\mathcal{A}^I \otimes j_{(n-|I|)(n-|I|)}\mathcal{A}^{n-|I|}$. This means that for each fixed I, the set of approximately I-symmetric sequences are approximately symmetric in the general sense of [**9**] . Having established this correspondence, we quote the following result from [**9**,§4] , but using the present terminology.

Theorem 2.1. [9] *Let I be a fixed finite subset of $\mathbb{N}$. Then*

(1) *For all $X \in \tilde{\mathcal{Y}}^I$, $\|X\| = \lim_{n\to\infty} \|X_n\|$ exists, and $\tilde{\mathcal{Y}}^I$ is the completion of $\mathcal{Y}^I$ in the seminorm (2.1). Furthermore, $\tilde{\mathcal{Y}}^I$ is closed within the set of all sequences $n \mapsto X^n \in \mathcal{A}^n$ in this seminorm.*

(2) *$\tilde{\mathcal{Y}}^I$ is an algebra with the operations of n-wise addition $(X., Y.) \mapsto X. + Y.$ and n-wise multiplication $(X., Y.) \mapsto X.Y.$. Furthermore, $\tilde{\mathcal{Y}} \equiv \tilde{\mathcal{Y}}^\emptyset$ is commutative under the seminorm (2.1) in the sense that*

$$\|XY - YX\| = \lim_{n\to\infty} \|X_n Y_n - Y_n X_n\| = 0 \quad ,$$

for all sequences $X.$ and $Y.$ in $\tilde{\mathcal{Y}}$.

(3) *For all $X \in \tilde{\mathcal{Y}}^I$, $X^I_\infty(\rho) = \lim_{n\to\infty}(j^I_{\infty\, n} X_n)(\rho)$ exists in the norm topology of $\mathcal{A}^I$, uniformly for $\rho \in K(\mathcal{A})$.*

(4) *The map $\tilde{\mathcal{Y}}^I \to \mathcal{C}(K(\mathcal{A}), \mathcal{A}^I) : X \to X^I_\infty$ is an isometric *-homomorphism from $\tilde{\mathcal{Y}}^I$ onto $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$.*

Note that in future we shall omit the label "$\emptyset$" when I is empty: in this case our notation becomes identical with that of [**1**] . We remark that the proof of (2) given in [**9**] is based on a decomposition of the product of two strictly symmetric sequences $X_n = j_{nx} X_x$ and $Y_n = j_{ny} Y_y$ as

$$X_n Y_n = \sum_r c_n(x, y; r) j_{n\,(x+y-r)}\big((X_x \otimes \mathbb{1}_{y-r})(\mathbb{1}_{x-r} \otimes Y_y)\big) \quad .$$

Here

$$c_n(x, y; r) = \frac{x!y!(n-x)!(n-y)!}{n!r!(x-r)!(y-r)!(n+r-x-y)!}$$

is the proportion of permutations γ of $\{1, \ldots n\}$ such that the intersection of $\{1, \ldots x\}$ and $\{\gamma(1), \ldots \gamma(y)\}$ has r elements. The size, r, of this intersection will be called the

overlap. The result (2) follows from the observation that except for $r = 0$ all c_n go to zero. One shows readily that $c_n(x,y;r) \sim n^{-r}x!y!/(r!(x-r)!(y-r)!)$. Entirely similar decompositions are made when dealing with I-symmetry.

For any $I \subset \mathbb{N}$ and $\mathcal{S} \subset \tilde{\mathcal{Y}}^I$ we shall define $\mathcal{S}^I_\infty \subset \mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$ to be the set $\{X^I_\infty \mid X \in \mathcal{S}\}$. A subset $\mathcal{D} \subset \tilde{\mathcal{Y}}^I$, will be called **dense**, if all elements of $\tilde{\mathcal{Y}}^I$ can be approximated in seminorm by elements of $\mathcal{D}$. This is equivalent to saying that $\mathcal{D}^I_\infty$ is dense in $\mathcal{C}(K(\mathcal{A}),A^I)$. We define $\mathcal{P}^I = \mathcal{Y}^I_\infty$. Clearly $\mathcal{P}^I$ is an algebra in $\mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$ and since $\mathcal{Y}^I$ is dense in $\tilde{\mathcal{Y}}^I$, $\mathcal{P}^I$ is dense in $\mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$. $\mathcal{P}^I$ can be regarded as a dense polynomial subalgebra of $\mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$.

Proposition 2.2. *Let $I \subset J \subset \mathbb{N}$, with J finite. Then*

(1) For $n \geq I_{max}$ and $m \geq J_{max}$

$$j^I_{nm} \circ j^J_{mr} = j^I_{nr} \quad .$$

(2) Let the sequence $X.$ be approximately J-symmetric. Then the sequence $n \mapsto j^I_{nn}X_n$ is approximately I-symmetric.

(3) Let the sequence $X.$ be approximately I-symmetric. Then $X.$ is also approximately J-symmetric, and $X^J_\infty = X^I_\infty \otimes \mathbb{1}_{J\setminus I}$.

Proof: (1)

$$\begin{aligned} j^I_{nm} \circ j^J_{mr} &= \frac{1}{(n-|I|)!(m-|J|)!} \sum_{\gamma\in S_n(I)} \sum_{\gamma'\in S_m(J)} \pi_\gamma i_{nm} \pi_{\gamma'} i_{mr} \\ &= \frac{1}{(n-|I|)!(m-|J|)!} \sum_{\gamma\in S_n(I)} \sum_{\gamma'\in S_n(J\cup\{m+1,\ldots,n\})} \pi_\gamma \pi_{\gamma'} i_{nr} \end{aligned}$$

Since $I \subset J \subset J\cup\{m+1,\ldots,n\}$, every permutation in S_n which leaves $J\cup\{m+1,\ldots,n\}$ pointwise invariant also leaves I pointwise invariant: $S_n(I) \supset S_n(J\cup\{m+1,\ldots,n\})$. Furthermore $S_n(I)$ is a subgroup of S_n, so that for any $\gamma' \in S_n(J\cup\{m+1,\ldots,n\})$ the set $\{\pi_\gamma\pi_{\gamma'} : \gamma \in S_n(I)\}$ has the same elements as the set $\{\pi_\gamma : \gamma \in S_n(I)\}$. Thus

$$j^I_{nm} \circ j^J_{mr} = \frac{1}{(n-|I|)!} \sum_{\gamma\in S_n(I)} \pi_\gamma i_{nr} = j^I_{nr}$$

(2) Let $X. \in \tilde{\mathcal{Y}}^J$. By Proposition 2.2(1) above, $j^I_{nn}X_n - j^I_{nm}j^I_{mm}X_m = j^I_{nn}(X_n - j^J_{nm}X_m)$. Thus, since j^I_{nn} is a contraction

$$\lim_{n\geq m\to\infty} \|j^I_{nn}X_n - j^I_{nm}X_m\| \leq \lim_{n\geq m\to\infty} \|X_n - j^J_{nm}X_m\| = 0$$

so that $n \mapsto j^I_{nn} X_n$ is approximately I-symmetric.

(3) First note that it suffices to prove the assertion for all $X.$ which are strictly I-symmetric: for then any approximately I-symmetric sequence can be approximated uniformly for large enough n by sequences which are approximately J-symmetric, and is hence itself approximately J-symmetric.

For notational clarity we will set $a = |I|$ and $b = |J|$ Let $X.$ be strictly I-symmetric, so that $X_n = j^I_{nm} X_m$ for some m and all $n \geq m$. Clearly we are free to pick m to be not less than J_{max}. Then

$$\begin{aligned} X_n &= \frac{1}{(n-a)!} \sum_{\gamma \in S_n(I)} \pi_\gamma i_{nm} X_m \\ &= \mathcal{S}^{(1)}_n + \mathcal{S}^{(2)}_n \end{aligned}$$

where

$$\mathcal{S}^{(1)}_n = \frac{1}{(n-a)!} \sum_{\substack{\gamma \in S_n(I): \\ \gamma(J\setminus I) \subset \{m+1,\ldots,n\}}} \pi_\gamma i_{nm} X_m \quad ,$$

and

$$\mathcal{S}^{(2)}_n = \frac{1}{(n-a)!} \sum_{\substack{\gamma \in S_n(I): \\ \gamma(J\setminus I) \not\subset \{m+1,\ldots,n\}}} \pi_\gamma i_{nm} X_m \quad .$$

$\mathcal{S}^{(1)}_n$ derives from those $\gamma \in S_n(I)$ for which the $\mathcal{A}^{J\setminus I}$ component of $\pi_\gamma i_{nm} X_m$ is simply $\mathbf{1}_{J\setminus I}$. We will show that the sequence $n \mapsto \mathcal{S}^{(1)}_n$ is approximately J-symmetric, while $\lim_{n\to\infty} \mathcal{S}^{(2)}_n = 0$.

Now the number of terms the sum $\mathcal{S}^{(2)}_n$ is precisely $(n-a)!(1-c_{n-a}(b-a, m-a; 0)$. Since $\lim_{n\to\infty} c_{n-a}(b-a, m-a; 0) = 1$, we have that $\lim_{n\to\infty} \mathcal{S}^{(2)}_n = 0$.

Let γ' be any element of $S_{m+b-a}(I)$ for which $\gamma'(J \setminus I) = \{m+1, \ldots, m+b-a\}$. Then

$$\mathcal{S}^{(1)}_n = \frac{(n-m)!}{(n+a-m-b)!(n-a)!} \sum_{\gamma \in S_n(J)} \pi_\gamma i_{n\ (m+b-a)} \pi_{\gamma'} i_{(m+b-a)\ m} X_m \quad .$$

But $(n-b)!(n-m)!/(n+a-m-b)!(n-a)! = c_{n-a}(b-a, m-a; 0)$ which converges to 1 as $n \to \infty$. Comparing with the definition of j^J_{nm} we see that

$$X_n = c_{n-a}(b-a, m-a; 0) j^J_{n\ (m+b-a)} \pi_{\gamma'} i_{(m+b-a)\ m} X_m + \mathcal{S}^{(2)}_n \quad .$$

Thus

$$\lim_{n\to\infty} \|X_n - j^J_{n\ (m+b-a)} \pi_{\gamma'} i_{(m+b-a)\ m} X_m\| = 0 \quad ,$$

so that $X. \in \tilde{\mathcal{Y}}^J$. The particular form of X^J_∞ derives from the fact that in each $\mathcal{S}_n^{(1)}$ the $\mathcal{A}^{J\setminus I}$ factors are occupied by $\mathbb{1}_{J\setminus I}$.

■

Corollary 2.3. *Let $I_1 \ldots I_k$ be a finite collection of disjoint finite subsets of $\mathbb{N}$ and set $I = \cup_{k'=1}^k I_k$. Let $\ell \in \mathbb{N}$ and let α be any map of $\{1,\ldots,\ell\}$ into $\{1,\ldots,k\}$. Let $X.^{[1]},\ldots,X^{[\ell]}$ be a collection of sequences such that $X.^{[\ell']} \in \tilde{\mathcal{Y}}^{I_{\alpha(\ell')}}$ for all $\ell' \in \{1,\ldots,\ell\}$. Define the sequence $Y.$ by*

$$Y_n = X_n^{[1]} X_n^{[2]} \ldots X_n^{[\ell]} \quad . \tag{2.2}$$

For each $k' \in \{1,\ldots,k\}$ form the sequence $Z.^{[k']}$ from the right hand side of equation (2.2) by replacing all $X_n^{[\ell']}$ for which $\alpha(\ell') \neq k$ by the identity $\mathbb{1}_n$. Then $Y.$ lies in $\tilde{\mathcal{Y}}^I$, $Z.^{[k']}$ lies in $\tilde{\mathcal{Y}}^{I_{k'}}$, and

$$Y^I_\infty = Z^{[1]}{}^{I_1}_\infty \otimes \ldots \otimes Z^{[k]}{}^{I_k}_\infty \quad . \tag{2.3}$$

Proof: By Prop. 2.2(3) for each $\ell' \in \{1,\ldots,\ell]\}$ the sequence $X.^{[\ell']}$ is approximately I-symmetric, and as a product of approximately I-symmetric sequences, so is Y. By Proposition 2.2(3) each $X^{[\ell']}{}^I_\infty$ lies in $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^{I_{\alpha(\ell')}} \otimes \mathbb{1}_{I\setminus I_{\alpha(\ell')}})$. Since the $I_{k'}$ are disjoint $X^{[\ell']}{}^I_\infty$ and $X^{[\ell'']}{}^I_\infty$ commute when $\alpha(\ell') \neq \alpha(\ell'')$. For each k' we gather together the contributions to Y^I_∞ for all ℓ' such that $\alpha(\ell') = k'$ together as $Z^{[k']}{}^{I_{k'}}_\infty \otimes \mathbb{1}_{I\setminus I_{k'}}$. So Y^I_∞ factorises over the $\mathcal{A}^{I'_k}$ yielding equation (2.3).

■

3. Mean-field dynamical limits and the preservation of I-symmetry.

We continue the conventions of the previous section. For each $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$ let $T_{t,n} : \mathcal{A}^n \to \mathcal{A}^n$ be a completely positive, identity preserving contraction, such that for fixed n, $(T_{t,n} = e^{tG_n})_{t\geq 0}$ is a strongly continuous one-parameter semigroup on $\mathcal{A}^n$ with generator G_n.

From a physical point of view one can say that the family $(T_{t,n})_{n\in\mathbb{N}}$ has good mean-field properties if is maps approximately symmetric sequences into approximately symmetric sequences, i.e. if for all $X \in \tilde{\mathcal{Y}}$ and $t \geq 0$ the sequence $n \to T_{t,n}X_n$ lies in $\tilde{\mathcal{Y}}$. If this is the case, one naturally tries to define a limiting evolution $T_{t,\infty}$ on the limit space $\mathcal{C}(K(\mathcal{A}))$ by $T_{t,\infty}X_\infty = (T_{t,}.X.)_\infty$. It not *a priori* clear that $T_{t,\infty}$ is well defined as a strongly continuous contraction semigroup on $\mathcal{C}(K(\mathcal{A}))$. However, in **[1, Theorem 2.3]** , it shown that this is the case if and only if the set of sequences $\{X. \mid X_n \in \mathrm{Dom}(G_n) : \|G_nX_n\|$ uniformly bounded $\}$ is dense in $\tilde{\mathcal{Y}}$. We will not repeat the proof of this result: it occurs as a special case of the generalization which we will make.

We now seek to extend the general results of [1] to sequences of semigroups which for some finite subset I of $\mathbb{N}$ preserve approximate I-symmetry. We will say that a sequence $n \mapsto T_n$ of uniformly bounded linear maps on $\mathcal{A}^n$ is **approximate I-symmetry preserving** if for all $X \in \tilde{\mathcal{Y}}^I$, the sequence $n \mapsto T_nX_n$ is approximately I-symmetric.

Lemma 3.1. *Let I be a finite subset of $\mathbb{N}$ and let $T.$ be a sequence of approximate I-symmetry preserving maps. Then $T^I_\infty : X \mapsto (T.X.)^I_\infty$ is well defined.*

Proof: Let $X \in \tilde{\mathcal{Y}}^I$ with $X^I_\infty = 0$. Then

$$\|(T.X.)^I_\infty\| = \lim_{n\to\infty} \|T_nX_n\| \le \sup_{n\in\mathbb{N}} \|T_n\| \lim_{n\to\infty} \|X_n\| = 0 \quad .$$

■

As remarked in [1] , the existence of a limit for sequences of unbounded linear operators is not so clear. For each $n \in \mathbb{N}$ let P_n be an unbounded linear operator on $\mathcal{A}^n$ with domain $\mathrm{Dom}(P_n)$. For each finite subset I of $\mathbb{N}$, we denote by $\mathcal{D}om^I(P)$ the sequence space

$$\mathcal{D}om^I(P) = \{X \in \tilde{\mathcal{Y}}^I \mid X_n \in \mathrm{Dom}\, P_n \text{ for all } n \text{ and } P.X. \in \tilde{\mathcal{Y}}^I\} \quad .$$

In view of Proposition 2.2(4) we see that when $I \subset J$, $\mathcal{D}om^I(P) \subset \mathcal{D}om^J(P)$.

If $T_{t,\cdot}$ is approximate I-symmetry preserving, then by Lemma 3.1 the map $T^I_{t,\infty}$ on $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ is well defined. For $X \in \mathcal{D}om^I(G)$ we can try to define a limiting generator on $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ by $G^I_\infty : X^I_\infty \mapsto (G.X.)^I_\infty$ with domain $(\mathcal{D}om^I(G))^I_\infty$. We see in the following theorem, that under very reasonable conditions on the G_n, G^I_∞ is not only well-defined, but that $T^I_{t,\infty}$ is a strongly continuous contraction semigroup on $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ which has G^I_∞ as its generator.

Theorem 3.2. *For each $n \in \mathbb{N}$ let $(T_{t,n} = e^{tG_n})_{t\ge0}$ be a strongly continuous semigroup of completely positive contractions. Let I be a fixed finite subset of $\mathbb{N}$. Then the following conditions are equivalent:*

(1) For each t, $T_{t,\cdot}$ is approximate I-symmetry preserving, and the set of sequences X with $X_n \in \mathrm{Dom}(G_n)$ and $\|G_nX_n\|$ uniformly bounded is dense in $\tilde{\mathcal{Y}}^I$.

(2) *The operator* G_∞^I *with domain* $\mathrm{Dom}(G_\infty^I) = \left(\mathcal{D}om^I(G)\right)_\infty^I$ *is well-defined, closed, and generates a semigroup of contractions on* $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$.

Moreover, if these conditions are satisfied, $T_{t,\infty}^I = e^{tG_\infty^I}$, *and* $T_{t,\cdot}$ *will be said to have a* **mean-field limit on** $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$, *namely,* $T_{t,\infty}^I$.

We omit the proof of this theorem, since it can be obtained by repetition of the steps of the proof of Theorem 2.3 in [1] , using I-symmetry and I-symmetric sequences and operators instead of their symmetric counterparts. For $I = \emptyset$, the above result reduces to the equivalence (2)$\Longleftrightarrow$(5) of the theorem in [1] . Although the above statement is sufficient for our present purpose, note that all the statements in the theorem of [1] , and their proofs, generalize to the I-symmetric case.

For the case the ordinary mean-field limits on the commutative algebra $\mathcal{C}(K(\mathcal{A}))$, it was demonstrated in [1] that for certain classes, $T_{t,\infty}$ is implemented by a flow on $K(\mathcal{A})$. (Note, however, that there are mean-field limits which are implemented by diffusions, rather than flows). For such limits, the generator G_∞ is a derivation on its domain. If the G_n are sufficiently local (in the sense that they are approximately polynomial) this can be seen as a reflection of the combinatorial fact that commutators of approximately symmetric sequences are null in the seminorm (2.1). If $I \neq \emptyset$ this is no longer the case, and G_∞^I need not be a derivation even if $G_\infty \equiv G_\infty^\emptyset$ is. However, we shall see that some classes of generators do behave as derivations on certain sequences.

In the following definition we shall retain the usual assumptions that for each $n \in \mathbb{N}$, $t \mapsto T_{t,n} = e^{tG_n}$ is a strongly continuous one-parameter semigroup of contractions on $\mathcal{A}^n$ with generator G_n.

Definition 3.3. *For all finite* $I \subset \mathbb{N}$, *let* $T_{t,\cdot}$ *have a mean-field limit* $T_{t,\infty}^I$ *on* $\mathcal{C}(K(\mathcal{A}), \mathcal{A}^I)$ *with generator* G_∞^I. *We shall say that the family of generators* $\{G_\infty^I \mid I \subset \mathbb{N} : |I| < \infty\}$ *has the* **disjoint derivation property** *iff for all finite disjoint subsets* I *and* J *of* $\mathbb{N}$ *and for all* $X \in \mathcal{D}om^I(G)$, $Y \in \mathcal{D}om^J(G)$ *then* $X_\infty^I \otimes Y_\infty^J \in \mathrm{Dom}(G_\infty^{I\cup J})$ *and*

$$G_\infty^{I\cup J}(X_\infty^I \otimes Y_\infty^J) = G_\infty^I X_\infty^I \otimes Y_\infty^J + X_\infty^I \otimes G_\infty^J Y_\infty^J \quad . \tag{3.1}$$

The strength of the disjoint derivation property is that if it is satisfied, we are able to prove a factorisation property of correlation function for strictly local observables (i.e. those of the from $i_{nI}X$ for some fixed finite I and $X \in \mathcal{A}^I$), rather than just completely symmetrized observables (i.e. those of the form $j_{nI}X$). This property can be seen in its simplest form in the following proposition.

Proposition 3.4. *For all finite I and $t \geq 0$ let $T_{t,\cdot}$ have mean-field limits $T^I_{t,\infty}$ on $\mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$ and let the corresponding family of generators have the disjoint derivation property. Then for I and J disjoint, $X^I_\infty \in \mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$ and $Y^J_\infty \in \mathcal{C}(K(\mathcal{A}),\mathcal{A}^J)$ then*

$$T^{I\cup J}_{t,\infty}(X^I_\infty \otimes Y^J_\infty) = T^I_{t,\infty}X^I_\infty \otimes T^J_{t,\infty}Y^J_\infty \quad . \tag{3.2}$$

Proof:

$$\begin{aligned} &T^{I\cup J}_{t,\infty}(X^I_\infty \otimes Y^J_\infty) - T^I_{t,\infty}X^I_\infty \otimes T^J_{t,\infty}Y^J_\infty \\ &= \int_0^t ds \frac{d}{ds} T^{I\cup J}_{s,\infty}(T^I_{t-s,\infty}X^I_\infty \otimes T^J_{t-s,\infty}Y^J_\infty) \\ &= \int_0^t ds T^{I\cup J}_{s,\infty}\Big\{G^{I\cup J}_\infty(T^I_{t-s,\infty}X^I_\infty \otimes T^J_{t-s,\infty}Y^J_\infty) \\ &\qquad - G^I_\infty T^I_{t-s,\infty}X^I_\infty \otimes T^J_{t-s,\infty}Y^J_\infty - T^I_{t-s,\infty}X^I_\infty \otimes G^J_\infty T^J_{t-s,\infty}X^J_\infty\Big\} \end{aligned}$$

Now $\|T^{I\cup J}_{t,\infty}\|$ is bounded, and it follows from the general theory of contraction semigroups (e.g. **[22]**) that since X^I_∞ lies in $\mathrm{Dom}(G^I_\infty)$ (resp. Y^J_∞ in $\mathrm{Dom}(G^J_\infty)$) so does $T^I_{t,\infty}X^I_\infty$ (resp. $T^J_{t,\infty}X^J_\infty$). Thus the above expression is zero by virtue of equation (3.1).

■

We shall say that the semigroups $T_{t,\cdot}$ have the **disjoint homomorphism property** if for all finite $I \subset \mathbb{N}$ they have mean-field limits on $\mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$, and if furthermore equation (3.2) is satisfied for all $t \geq 0$, for all finite disjoint $I, J \subset \mathbb{N}$ and $X^I_\infty \in \mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$, $Y^J_\infty \in \mathcal{C}(K(\mathcal{A}),\mathcal{A}^J)$. A brief density argument shows that this is the case if and only if

$$T^{I\cup J}_{t,\infty} = T^I_{t,\infty} \otimes T^J_{t,\infty} \quad .$$

4. Correlation functionals.

In this section we shall show how, for a mean-field dynamical semigroup $(T_{t,n})_{n\in\mathbb{N}}$ with the disjoint derivation property, the evolutions of I-symmetric sequences for different disjoint subsets I of $\mathbb{N}$ become independent in the limit $n \to \infty$.

Proposition 4.1: Multi-time correlation functionals. *Let $T_{t,\cdot}$ be a mean-field dynamical semigroup which has the disjoint homomorphism property. Let $I_1 \dots I_k$ be a finite* *collection of disjoint finite subsets of* $\mathbb{N}$ *and set* $I = \cup_{k'=1}^{k} I_k$. *Let* $\ell \geq k$ *and let* α *be any map of* $\{1, \ldots, \ell\}$ *into* $\{1, \ldots, k\}$. *Let* $X^{[1]}, \ldots, X^{[\ell]}$ *be a collection of sequences such that* $X^{[\ell']} \in \tilde{\mathcal{Y}}^{I_{\alpha(\ell')}}$ *for all* $\ell' \in \{1, \ldots, \ell\}$. *Let* $(s_{\ell'})_{\ell'=1,\ldots,\ell}$ *be a collection of non-negative real numbers. Define the sequence* $Y.$ *by*

$$Y_n = \left(T_{s_1,n} X_n^{[1]}\right)\left(T_{s_2,n} X_n^{[2]}\right) \ldots \left(T_{s_\ell,n} X_n^{[\ell]}\right) \quad . \tag{4.1}$$

For each $k' \in \{1, \ldots, k\}$ *form the sequence* $Z.^{[k']}$ *from the right hand side of equation (4.1) by replacing all* $X_n^{[\ell']}$ *for which* $\alpha(\ell') \neq k$ *by the identity* $\mathbb{1}_n$. *Then* $Y.$ *lies in* $\tilde{\mathcal{Y}}^I$, $Z.^{[k']}$ *lies in* $\tilde{\mathcal{Y}}^{I_{k'}}$ *and*

$$Y_\infty^I = Z^{[1]}{}_\infty^{I_1} \otimes \ldots \otimes Z^{[k]}{}_\infty^{I_k} \quad .$$

Remark: Since the $T_{t,n}$ are identity preserving, replacing any $X_n^{[\ell']}$ by $\mathbb{1}_n$ in Proposition 4.1 amounts simply omitting the factor $T_{s_{\ell'},n} X_n^{[\ell']}$ from the definition of $X^{[\ell']}$.

Proof: This follows immediately from Corollary 2.3 upon noting that since $T_{t,\cdot}$ is approximate I-symmetry preserving for all I, the sequence $n \mapsto T_{s_{\ell'},n} X_n^{[\ell']}$ is in $\tilde{\mathcal{Y}}^{I_{\alpha(\ell')}}$ for all $\ell' \in \{1, \ldots, \ell\}$

■

Proposition 4.2: Nested correlation functionals. *Retain the assumptions of Proposition 4.1, but define instead*

$$Y_n = T_{s_1,n}\left(X_n^{[1]} T_{s_2,n}\left(X_n^{[2]} \ldots T_{s_\ell,n} X_n^{[\ell]}\right) \ldots\right)\Big) \quad . \tag{4.2}$$

For each $k' \in \{1, \ldots, k\}$ *form the sequence* $Z.^{[k']}$ *from the right hand side of equation (4.1) by replacing all* $X_n^{[\ell']}$ *for which* $\alpha(\ell') \neq k$ *by the identity* $\mathbb{1}_n$. *Then* $Y.$ *lies in* $\tilde{\mathcal{Y}}^I$, $Z.^{[k']}$ *lies in* $\tilde{\mathcal{Y}}^{I_{k'}}$ *and*

$$Y_\infty^I = Z^{[1]}{}_\infty^{I_1} \otimes \ldots \otimes Z^{[k]}{}_\infty^{I_k} \quad .$$

Proof: Let $U. \in \tilde{\mathcal{Y}}^{I_j}$ for some $j \in \{1, \ldots k\}$, and let $s \geq 0$. Assuming the truth of the proposition, then by Corollary 2.3 the sequence $n \mapsto U_n Y_n$ lies in $\tilde{\mathcal{Y}}^I$, and

$$(U.Y.)_\infty^I = Z^{[1]}{}_\infty^{I_1} \otimes \ldots \otimes U_\infty^{I_j} Z^{[j]}{}_\infty^{I_j} \otimes \ldots \otimes Z^{[k]}{}_\infty^{I_k}$$

Since $T_{t,\cdot}$ has the disjoint homomorphism property, we can use the factorisation of equation (3.2) to conclude that

$$T_{s,\infty}^I (U.Y.)_\infty^I = \left(T_{s,\infty}^{I_1} Z^{[1]}{}_\infty^{I_1}\right) \otimes \ldots \otimes \left(T_{s,\infty}^{I_j} U_\infty^{I_j} Z^{[j]}{}_\infty^{I_j}\right) \otimes \ldots \otimes \left(T_{s,\infty}^{I_k} Z^{[k]}{}_\infty^{I_k}\right)$$

Since $\mathbf{1}_I$ is trivially approximately I-symmetric, we can use this argument for m taking in turn the values $\ell, \ell-1, \ldots, 2$, using $T_{s_m,m}(X_n^{[m]} \ldots T_{s_\ell,n}X_n^{[\ell]}) \ldots)$ in place of Y_n, with $X_n^{[m-1]}$ in place of U_n and s_{m-1} in place of s, and hence conclude the statement of the proposition.

■

Example 4.3. Perhaps the simplest application of Proposition 4.3 is when $I_{k'} = \{k'\}$ for all k', and $X_n^{[\ell']} = i_{n\{k'\}}W^{[\ell']}$ for $k' = \alpha(\ell')$ and some $W^{[\ell']} \in \mathcal{A}_{\{k'\}}$. From Proposition 4.2 we obtain the factorisation of the nested one-site correlation functionals.

Now for any $\rho \in K(\mathcal{A})$ and $\sigma \in K(\mathcal{A}^I)$ we have

$$\lim_{n\to\infty} \langle \sigma \otimes \rho^{\{1,\ldots n\}\setminus I}, Y_n \rangle = \langle \sigma, Y_\infty^I(\rho) \rangle$$

for $Y.$ given in either Proposition 4.1 or Proposition 4.2. So the limiting correlation functions on sequences of products states on $\mathcal{A}^n$ with local perturbations on $\mathcal{A}^I$ may be calculated. Note that the results also extend to weakly convergent sequences of states as defined Def. III.2 of **[9]** . In our extended formalism, these become sequences of states φ_n on $\mathcal{A}^n$ such that for a given I and for all $X \in \tilde{\mathcal{Y}}^I$,

$$\lim_{n\to\infty} \langle \varphi_n, X_n \rangle = \int_{K(\mathcal{A})} d\mu(\rho) \langle \sigma_\rho, X_\infty^I(\rho) \rangle \quad ,$$

where μ is some probability measure on $K(\mathcal{A})$, and for each $\rho \in K(\mathcal{A})$, σ_ρ is some state on $\mathcal{A}^I$.

Although Propositions 4.1 and 4.2 give a factorisation of correlation functions over different sites, the mean-field formalism does not furnish any easy method for obtaining the one site correlation functions. However, there are systems for which these one-site correlation functions may be calculated for a sufficiently large subalgebra of one-site observables **[16]** .

5. Example: bounded polynomial generators.

In this section we give an example of a class of mean-field dynamical semigroups which have the disjoint homomorphism property. This is the class in which the generator G_n is obtained for all n larger than some fixed g, by resymmetrization of G_g, and multiplication by a scaling factor (n/g). G_g itself will be the generator of a norm-continuous semigroup of completely positive maps. These are discussed in **[23]** and **[24]** .

Definition 5.1. *A sequence of operators $G. = (G_n)_{n\geq g}$, with $G_n \in \mathcal{B}(\mathcal{A}^n)$ will be called* **a bounded polynomial generator** *of* **degree** *g if*

$$G_n = \frac{n}{g}\,\mathrm{Sym}_n\, G_g \quad ,$$

where G_g is the generator of a norm-continuous semigroup of completely positive unital maps on $\mathcal{A}^g$, and the symmetrization operator $\mathrm{Sym}_n : \bigcup_{m\leq n}\mathcal{B}(\mathcal{A}^m) \to \mathcal{B}(A^n)$ is defined by

$$\mathrm{Sym}_n\, G_m = \frac{1}{n!}\sum_{\gamma\in S_n} \pi_\gamma^{-1}(G_m \otimes \mathrm{id}_{n-m})\pi_\gamma$$

for all $m \leq n$ and $G_m \in \mathcal{B}(\mathcal{A}^m)$.

This type of generator occurs frequently in applications. However, the bounded polynomial generators are certainly not the broadest class of generators possessing the disjoint derivation property. Indeed , one can extend the results of this section to treat *approximately* polynomial generators, as defined in [1] . One can show (see [1]) that each G_n is also the generator of a norm-continuous semigroup of completely positive maps on $\mathcal{A}^n$.

We will fix a dense subset of each $\tilde{\mathcal{Y}}^I$ with which it will be convenient to work. For each finite $I \subset \mathbb{N}$ define $\hat{\mathcal{Y}}^I \subset \tilde{\mathcal{Y}}^I$ by

$$\hat{\mathcal{Y}}^I = \{X \in \tilde{\mathcal{Y}}^I : X_n = j^I_{nm}(X^n_m), n \geq m, \lim_{n\to\infty} X^n_m = \hat{X}_m \in \mathcal{A}^m\} \quad .$$

We shall call m the degree of $X \in \hat{\mathcal{Y}}^I$ and $\hat{X}_m$ its limiting element, as so defined. Clearly $\hat{\mathcal{Y}}^I_\infty = \mathcal{Y}^I_\infty$.

Proposition 5.2. *Let $G.$ be a bounded polynomial generator of degree g, and let I be any finite subset of $\mathbb{N}$. Then $\hat{\mathcal{Y}}^I \subset \mathcal{D}om^I(G)$ and $G\hat{\mathcal{Y}}^I \subset \hat{\mathcal{Y}}^I$. Let the sequence $X \in \hat{\mathcal{Y}}_I$ be of degree x. Then $G.X.$ is of degree $g + x - 1$, and $\|(G.X.)^I_\infty\| \leq x\|G_g\|\|\hat{X}_x\|$.*

Proof: For notational simplicity we take the case that $I = \{1, 2, \ldots, |I|\}$. This involves no loss of generality since we can map any I onto this set with a suitable permutation, the corresponding automorphism leaving the permutation symmetric operator G_n invariant.

Since we are free to choose $x \geq I_{max}$,

$$\frac{n}{g}\,\mathrm{Sym}_n\, G_g j^I_{nm} X_x = \frac{n}{g} j^I_{nm}\,\mathrm{Sym}_n\, G_g i_{nx} X_x$$

We collect together all terms in G_nX_n with the same overlap between G_g and X_m. First note that since each $T_{t,n}$ is identity preserving, the terms of overlap zero vanish identically. Second, note that since $c_n(x,y;r) = O(n^{-r})$, the contribution to G_nX_n of terms of overlap between 2 and $\min\{g,x\} - 1$ is $O(n^{-1})$. This leaves the terms with overlap 1, and one calculates that

$$G_nX_n = \frac{n}{g}c_n(g,x,1)j^I_{n\ (g+x-1)}\frac{1}{x}\sum_{y=1}^{x}\pi^{-1}_{\gamma(y)}(\mathrm{id}_{x-1}\otimes G_g)\pi_{\gamma(y)}(\hat{X}^n_x \otimes \mathbb{1}_{g-1}) + O(n^{-1}) \quad ,$$

where $\gamma(y)$ is the element of S_{g+x-1} which exchanges y and x. ($\gamma(x)$ is just the identity). Since $\lim_{n\to\infty}(n/gx)c_n(g,x;1) = 1$, and since the terms of overlap greater than 1 have degree less than $g+x-1$, G_nX_n lies in $\hat{\mathcal{Y}}^I$, and is of degree $x+g-1$. Finally,

$$\|(G.X.)^I_\infty\| = \lim_{n\to\infty}\|G_nX_n\| \le x\|G_g\| \lim_{n\to\infty}\|X^n_x\| = x\|G_g\|\|\hat{X}_x\|.$$

■

Recall now that the set of polynomials $\mathcal{P}^I \subset \mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$ is defined to be $\mathcal{Y}^I_\infty$. The proof of the following Proposition is adapted from one in [**1**] .

Proposition 5.3. *Let G be a bounded polynomial generator with $T_{t,\cdot} = e^{tG\cdot}$ for all $t \ge 0$. Then*

(1) For all finite $I \subset \mathbb{N}$, $T_{t,\cdot}$ is approximate I-symmetry preserving.

(2) For all finite $I \subset \mathbb{N}$, $T_{t,\cdot}$ is a mean-field dynamical semigroup on $\mathcal{C}(K(\mathcal{A}),\mathcal{A}^I)$ with limit $T^I_{t,\infty} = e^{tG^I_\infty}$.

(3) $\mathcal{P}^I$ is a core for G^I_∞.

(4) The family of generators $\{G^I_\infty \mid I \subset \mathbb{N} : |I| < \infty\}$ has the disjoint derivation property.

Proof: (1) Let g be the degree of $G.$ and let $X \in \hat{\mathcal{Y}}^I$ be of degree x. Iterating the integral equation for $T_{t,n}$ we write

$$T_{t,n}X_n = X^{(m)}_{t,n} + R^{(m)}_{t,n} \quad ,$$

where

$$X^{(m)}_{t,n} = \sum_{p=0}^{m-1}\frac{t^p}{p!}(G_n)^pX_n \quad ,$$

and

$$R_{t,n}^{(m)} = \int_0^t ds_m \ldots \int_0^{s_2} ds_1 T_{s_1,n}(G_n)^m X_n \quad .$$

By Prop. 5.2 $(G.^m)X. \in \hat{\mathcal{Y}}^I$ for all $m \in \mathbb{N}$. $T_{t,n}$ is a contraction. Thus

$$\lim_{n\to\infty} \|T_{t,n}X_n - X_{t,n}^{(m)}\| \leq \frac{t^m}{m!} \lim_{n\to\infty} \|(G_n)^m X_n\|$$
$$\leq \frac{t^m}{m!}\|G_g\|^m \|\hat{X}_x\| \prod_{p=0}^{m-1}(x + p(g-1)) \quad .$$

Now for $a, b, m \in \mathbb{N}$

$$\frac{1}{m!}\prod_{p=0}^{m-1}(a+pb) \leq \frac{1}{m!}\prod_{p=0}^{m-1}(a+(m-1)b-p)$$
$$= \binom{a+(m-1)b}{m} \leq 2^{a+(m-1)b} \quad .$$

Thus

$$\lim_{n\to\infty} \|T_{t,n}X_n - X_{t,n}^{(m)}\| \leq 2^{x+g-1}(2^{g-1}t\|G_g\|)^m \|\hat{X}_x\| \quad . \tag{5.1}$$

For $t < \tau = (2^{g-1}\|G_g\|)^{-1}$, we can take the limit $m \to \infty$ and conclude that $T_{t,n}X_n$ can be approximated uniformly for large n by approximately I-symmetric sequences and so $(T_{t,n}X_n) \in \tilde{\mathcal{Y}}^I$. Note that τ is independent of X.

We extend to the whole of $\tilde{\mathcal{Y}}^I$ by continuity, and finally for all $t \in \mathbb{R}^+$ by joining together the solutions on successive intervals of length less that τ.

(2) From Proposition 5.2, $\|G_nX_n\|$ is uniformly bounded in n for each $X \in \hat{\mathcal{Y}}^I$. Since $\hat{\mathcal{Y}}^I$ is dense in $\tilde{\mathcal{Y}}^I$, the conclusion follows from (1) above and the implication (1)$\Longrightarrow$(2) of Theorem 3.2.

(3) By taking the limits n and then $m \to \infty$ in equation (5.1) we conclude from the power series approximation that all polynomials $p \in \mathcal{P}^I$ are analytic for G_∞^I when $t < \tau$, in the sense that $T_{t,\infty}^I p$ can be expressed as the convergent power series $\sum_{r=0}^\infty (r!)^{-1}(tG_\infty^I)^r p$ when $t < \tau$. Each term in this sum is itself a polynomial (i.e. in $\mathcal{P}^I$), so the partial sums of the series are polynomials approximating $T_{t,\infty}^I p$. Replacing p with the polynomial $G_\infty^I p$ in the series, we we find that

$$G_\infty^I T_{t,\infty}^I p = \lim_{m\to\infty} \sum_{r=0}^\infty \frac{t^r}{r!}(G_\infty^I)^{r+1} p = \lim_{m\to\infty} G_\infty^I \sum_{r=0}^\infty \frac{t^r}{r!}(G_\infty^I)^r p \quad .$$

We conclude that for $t < \tau$, $p \in \mathcal{P}^I$ and $\varepsilon > 0$ we can find $p^\varepsilon \in \mathcal{P}^I$ such that

$$\|T_{t,\infty}^I p - p^\varepsilon\| < \varepsilon \qquad \text{and} \qquad \|G_\infty^I(T_{t,\infty}^I p - p^\varepsilon)\| < \varepsilon \quad . \tag{5.2}$$

In fact, this conclusion holds for all $t \geq 0$. We demonstrate for $t < 2\tau$: the argument may be iterated for $t < 4\tau$, $t < 8\tau$ and so on. Let $s, t < \tau$. Given $p \in \mathcal{P}^I$ we can choose $p^\varepsilon \in \mathcal{P}^I$ satisfying equation (5.2). Since $p^\varepsilon \in \mathcal{P}^I$, then for all $\delta > 0$ we may pick $q^\delta \in \mathcal{P}^I$ such that

$$\|T^I_{s,\infty}p^\varepsilon - q^\delta\| < \delta \qquad \text{and} \qquad \|G^I_\infty(T^I_{s,\infty}p^\varepsilon - q^\delta)\| < \delta \quad .$$

Thus

$$\|T^I_{t+s,\infty}p - q^\delta\| \leq \|T^I_{s,\infty}(T^I_{t,\infty}p - p^\varepsilon)\| + \|T^I_{s,\infty}p^\varepsilon - q^\delta\| \leq \varepsilon + \delta$$

while

$$\|G^I_\infty T^I_{t+s,\infty}p - G^I_\infty q^\delta\| \leq \|T^I_{s,\infty}G^I_\infty(T^I_{t,\infty}p - p^\varepsilon)\| + \|G^I_\infty(T^I_{s,\infty}p^\varepsilon - q^\delta)\| \leq \varepsilon + \delta \quad .$$

where we have used the fact that $T^I_{t,\infty}$ is a contraction. By choosing ε and then δ sufficiently small, we see that the desired approximation is possible.

The set $\Omega = \bigcup_{t\geq 0} T^I_{t,\infty}\mathcal{P}^I$ is a dense $T^I_{t,\infty}$-invariant subset of $\mathrm{Dom}(G^I_\infty)$, and is hence a core for G^I_∞. From this and the above argument, we conclude that the subset $\mathcal{P}^I$ of Ω is also a core for G^I_∞.

(4) Since, by Proposition 5.2, the action on $G.$ is determined by terms of overlap 1, it follows that

$$\begin{aligned} G^{I\cup J}_\infty(j^I_{\infty x}X_x \otimes j^J_{\infty y}Y_y) &= G^{I\cup J}_\infty j^{I\cup J}_{\infty\, x+y}(X_x \otimes Y_y) \\ &= (G^I_\infty j^I_{\infty\, x}X_x) \otimes j^J_{\infty\, y}Y_y + j^I_{\infty\, x}X_x \otimes (G^J_\infty j^J_{\infty\, y}Y_y) \quad . \end{aligned} \tag{5.3}$$

Thus there are no mixed terms involving $G.$ acting on both $X.$ and $Y.$. By (3) above, for all finite $I \subset \mathbb{N}$, $\mathcal{P}^I$ is a core for G^I_∞. So, we can approximate any sequences X in $\mathcal{D}om^I(G)$ and Y in $\mathcal{D}om^J(G)$ by polynomials, then take the limit of the expression (5.3) to conclude that $X^I_\infty \otimes Y^J_\infty \in \mathrm{Dom}(G^{I\cup J}_\infty)$ and that equation (3.1) holds.

■

By Propositions 3.4 and 5.3 we see that the hypotheses of Propositions 4.1 and 4.2 are satisfied mean-field dynamical semigroups with bounded polynomial generators.

Appendix: Invariance of the predual under limiting flows.

An interesting special case of the theory of mean-field dynamical semigroups, which turns out to be important for applications, occurs when $\mathcal{A}$ is *-isomorphic with a von Neumann algebra. In this case will not distinguish notationally between $\mathcal{A}$ and the von Neumann algebra to which it is *-isomorphic. According to [**25**] , the *-isomorphism exists if and only if $\mathcal{A}$ is the dual of a Banach space. If this is the case, we will denote this predual of $\mathcal{A}$ by $\mathcal{A}_*$. We will view $\mathcal{A}_*$ canonically as a closed subspace of $\mathcal{A}^*$.

For certain classes of mean-field dynamical semigroups, the mean-field dynamical limit is implemented by a flow: for $X \in \tilde{\mathcal{Y}}$ and all $\rho \in K(\mathcal{A})$,

$$\lim_{n\to\infty} \langle \rho^n, T_{t,n} X_n \rangle = (T_{t,\infty} X_\infty)(\rho) = X_\infty(F_t \rho) \quad ,$$

for some continuous flow $(F_t)_{t\geq 0}$ on $K(\mathcal{A})$. The following main result of this section establishes conditions under which the set of **normal states** $K(\mathcal{A}) \cap \mathcal{A}_*$ is invariant under the limiting flows of mean-field dynamical semigroups.

In the following, we will denote the dual of any $T \in \mathcal{B}(\mathcal{A})$ by T^* i.e. T^* is the element of $\mathcal{B}(\mathcal{A}^*)$ such that $\langle T^* \omega, A \rangle = \langle \omega, TA \rangle$ for all $\omega \in \mathcal{A}^*$ and $A \in \mathcal{A}$.

Proposition A.1. *Let $\mathcal{A}$ be a C-* algebra with predual $\mathcal{A}_*$. Let $T_{t,\cdot}$ be a mean-field dynamical semigroup such that $T_{t,\infty}$ is implemented by a flow $(F_t)_{t\geq 0}$. For each $n \in \mathbb{N}$ let $T_{t,n}$ have the property that its dual $T^*_{t,n}$ action on $(\mathcal{A}^n)^*$ leaves the closed subspace $(\mathcal{A}^n)_*$ invariant. Furthermore assume the following continuity condition on $T_{t,\cdot}$: that for all $\varepsilon > 0$ there exists an n_ε such that for all $n \geq m \geq n_\varepsilon$ and $A \in \mathcal{A}$*

$$\|T_{t,n} \jmath_{n1} A - \jmath_{nm} T_{t,m} \jmath_{m1} A\| < \varepsilon \|A\| \quad . \tag{A.1}$$

Then the set $K(\mathcal{A}) \cap \mathcal{A}_*$ of normal states on $\mathcal{A}$ is invariant under each F_t.

Before we proceed with the proof we will note that the same conclusion was reached in [**10**] for the class of (in our terminology) mean-field dynamical semigroups with bounded polynomial generators (with the possible addition of an unbounded polynomial hamiltonian generator of degree 1). As is well known [**23**] , these have the property that each $T_{t,n}$ preserves $(A^n)_*$. Indeed, the treatment in [**10**] was carried out entirely in the pre-dual spaces. We will see that the mean-field dynamical semigroups of bounded polynomial generators satisfy the continuity condition of Proposition A.1.

To prove our result we will need the following technical result, in which we quote without proof some results from [**26, p77**] .

Proposition A.2. *Let $\mathcal{A}$ be a C^*-algebra with predual $\mathcal{A}_*$. Let $\omega \in K(\mathcal{A})$. Then ω is normal if and only if $\omega(A) = \sup_\alpha \omega(A_\alpha)$ for each increasing net (A_α) in $\mathcal{A}_+$ with least upper bound A.*

Proof of Proposition A.1: Let (A_α) be an increasing net in $\mathcal{A}_+$ with least upper bound $A \in \mathcal{A}$. Since the net is increasing we have that $\|\mathcal{A}_\alpha\| \leq \|A\|$ for all α. By the continuity assumption in equation (A.1), then for all $\varepsilon > 0$ we can find an n_ε such that for all $n \geq n_\varepsilon$, $B \in \mathcal{A}$ and $\rho \in \mathcal{A}_*$

$$|\langle F_t\rho, B\rangle - \langle T^*_{t,n}\rho^n, j_{n1}B\rangle| < \varepsilon\|B\|$$

Thus for $n \geq n_\varepsilon$

$$\begin{aligned} 0 \leq \langle F_t\rho, A - A_\alpha\rangle &\leq 2\varepsilon\|A\| + \langle T^*_{t,n}\rho^n, j_{n1}A - j_{n1}A_\alpha\rangle \\ &= 2\varepsilon\|A\| + \langle j^*_{nn}T^*_{t,n}\rho^n, (A - A_\alpha)\otimes \mathbf{1}_{n-1}\rangle \quad . \end{aligned}$$

Since $T^*_{t,n}\rho^n$ is normal, the restriction of its symmetrization $j^*_{nn}T^*_{t,n}\rho^n$ to the subalgebra $\mathcal{A}\otimes \mathbf{1}_{n-1}$ of $\mathcal{A}^n$ is also normal. Thus we can take the supremum over α on both sides and use the "only if" part of Proposition A.2 to conclude that

$$0 \leq \langle F_t\rho, A\rangle - \sup_\alpha \langle F_t\rho, A_\alpha\rangle \leq 2\varepsilon\|A\| \quad .$$

Since ε is arbitrary we conclude again by the "if" part of Proposition A.2 that $F_t\rho$ is normal.

■

Proposition A.3. *Mean-field dynamical semigroups with bounded polynomial generators satisfy equation (A.1) for all t in some compact interval. Hence if $\mathcal{A}$ is $*$-isomorphic with a von Neumann algebra, the limiting flows preserve the set of normal states.*

Proof: Let $T_{t,n} = e^{tG_n}$, with $G_n = (n/g)\,\mathrm{Sym}_n\, G_g$ for some g and for all $n \geq g$, and let F_t be the continuous flow implementing $T_{t,\infty}$. From equation (5.1) we see that equation (A.1) is satisfied for all positive $t < \tau = (2^{g-1}\|G_g\|)^{-1}$. If $\mathcal{A}$ has predual $\mathcal{A}_*$, then by Proposition A.1 the flow F_t preserves the set of normal states for all $t < \tau$, and hence for all $t \geq 0$ by composition.

■

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References.

[1] N.G. Duffield & R.F. Werner, Preprint DIAS-STP-90-13

[2] K. Hepp & E.H. Lieb, *Helv. Phys. Acta.* **46**(1973) 573-603

[3] E. Buffet & Ph.A. Martin, *J. Stat. Phys.* **18**(1978) 585-603

[4] N.G. Duffield, *Helv. Phys. Acta.* **61**(1988) 363-378

[5] N.G. Duffield, *J. Phys. A: Math. Gen.* **21**(1988) 625-641

[6] E. Buffet, Ph. de Smedt & J.V. Pulè, *Ann. of Phys.* **155**(1984) 269-304

[7] E. Buffet, Ph. de Smedt & J.V. Pulè, *Ann. Inst. H. Poincaré: Analyse non-lineaire* **1**(1984) 413-451

[8] H. Spohn, *Rev. Mod. Phys.* **53**(1980) 569-615

[9] G.A. Raggio & R.F. Werner, *Helv. Phys. Acta.* **62**(1989) 980-1003

[10] R. Alicki & J. Messer, *J. Stat. Phys.* **32**(1983) 299-312

[11] E. Duffner, *Physica* **133A**(1985) 187-212

[12] T. Unnerstall, *Lett. Math. Phys.* **20**(1990) 183-187

[13] P. Bóna, *J. Math. Phys.* **29**(1988) 2223-2235

[14] P. Bóna, *J. Math. Phys.* **30**(1989) 2994-3007

[15] T. Unnerstall, *J. Math. Phys.* **31**(1990) 680-688

[16] N.G. Duffield, In preparation

[17] A. Blobel & J. Messer, *J.Math.Phys.* **26**(1985) 1049-1056

[18] G.A. Raggio & R.F. Werner, *Europhys. Lett.* **9**(1989) 633-637

[19] G.A. Raggio & R.F. Werner, Preprint DIAS-STP-89-23

[20] N.G. Duffield, H. Roos, & R.F. Werner, *Ann. Inst. H. Poincaré: Phys. Th.* to appear.

[21] M. Takesaki, *Theory of Operator Algebras,* Volume 1; Springer, New York, Heidelberg, Berlin, 1979

[22] E.B. Davies, *One parameter semigroups,* Academic Press, London, 1980

[23] G. Lindblad, *Commun. Math. Phys.* **48**(1976) 119-130

[24] E. Christensen & D.E. Evans, *J. Lon. Math. Soc.* **20**(1979) 358-368

[25] S. Sakai, *C*-algebras and W*-algebras,* Springer, New York, Heidelberg, Berlin, 1971

[26] O. Bratteli & D.W. Robinson, *Operator Algebras and statistical mechanics,* Volume 1; Springer, New York, Heidelberg, Berlin, 1979