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Lyapunov Exponents of the Schrödinger Equation with Quasi-periodic Potential on a Strip

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Abstract. We decribe conditions under which all the nonnegative Lyapunov exponents of the quasi-periodic difference Schrödinger equation on a strip are strictly positive.

We shall study the Lyapunov exponents of the difference equation:

$$-y_{n+1} + Q_n y_n - y_{n-1} = 0 \qquad -\infty < n < +\infty \tag{1}$$

where $y_n \in \mathrm{R}^m$ and Q_n is a symmetric $m \times m$ matrix whose off-diagonal elements do not depend on n, and the diagonal elements are quasi-periodic functions

$$q_{ni}(\theta) = \lambda f_i(e^{2\pi i(\theta+n\alpha)}) - E$$

with $f_i(z)$ analytic on $\mathcal{A} \equiv \{z \mid r < |z| < 1/r\}$, taking values in $[-1,1]$ for $|z| = 1$, λ is a (large) parameter called coupling constant, E is the energy, and α is any irrational number. Without loss of generality we shall assume that $\max_{1\le i\le m} \sup_{|z|=1} f_i(z) = 1$ and $\min_{1\le i\le m} \inf_{|z|=1} f_i(z) = -1$.

Equation (1) becomes equivalent to the finite-difference Schrödinger equation when the off-diagonal elements are chosen properly. For example, the case $q_{ij} = -1$ for $|i-j| = 1$ and $q_{ij} = 0$ for $|i-j| > 1$ corresponds to Schrödinger operator on the strip $\mathbb{Z} \times \{1,\ldots,m\}$. Equation (1) can be written in the form

$$\begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix} = \begin{pmatrix} Q_n & -I \\ I & 0 \end{pmatrix} \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} \equiv A_n \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix} \tag{2}$$

and, thus, the asymptotic behavior of the solutions of equation (1) is determined by the asymptotic behavior of the product $S(n) := A_n \cdots A_1$.

Various problems in solid-state physics give rise to different classes of matrices Q_n. These classes are characterized by the level of randomness. The case of independent random Q_n was studied in [GM], where it was shown that under certain algebraic conditions on the support of the corresponding measure in the space of symmetric matrices all the

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Lyapunov exponents are different and, therefore, the smallest non-negative exponent is, in fact, positive. When Q_n's are non-deterministic at least some of the Lyapunov exponents are strictly positive [KS,S,K]. On the other hand, if Q_n's form a periodic sequence, it is easy to show that the "interesting" Lyapunov exponents vanish.

The case of quasi-periodic potentials exhibits mixed behavior. If the coupling constant λ is small and α is poorly approximated by rationals, it is known [BLT] that at least on part of the spectrum the Lyapunov exponent is zero when $m = 1$. When λ is large and $m = 1$ the Lyapunov exponent is positive [Si,CS,FSW,SS].

Here we study the case of $m > 1$ and large λ. We prove [GS] that all non-negative Lyapunov exponents are positive. The method we use is an extension of the one used in [SS], which, in turn, grew out of M. Herman's proof of positivity of the Lyapunov exponent in the case of $m = 1$, $f(z) = z + 1/z$, and $\lambda > 1$ [H].

To describe our result let us consider the following decomposition of $S(n)$:

$$S(n) = U(n)D(n)V(n)$$

where $U, V \in O(n)$, and $D(n) = \text{diag}(d_1^{(n)}, \ldots, d_{2m}^{(n)})$ with $d_1 \geq d_2 \geq \cdots \geq d_{2m} \geq 0$. Since $A_n \in Sp(m, \mathrm{R})$ for all n, we have $d_k = d_{2m-k+1}^{-1}$. The k-th Lyapunov exponent γ_k is defined by

$$\gamma_k := \lim_{n \to \infty} \frac{1}{n} \log d_k(n). \tag{3}$$

Clearly, $\gamma_k = -\gamma_{2m-k+1}$, and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m \geq 0$. Existence of the limit in (3) for almost every θ and its independence of θ are guaranteed by the Subadditive Ergodic Theorem [Ki] and ergodicity of the underlying dynamical system $\theta \mapsto \theta + \alpha$.

Our main result is the following

Theorem. *There exists λ_0 such that for all $\lambda > \lambda_0$ and all E, there exists a set $\Omega(E) \subset [0,1]$ of Lebesque measure 1, such that*

$$\gamma_m(E) = \gamma_m(E, \theta) > 0 \qquad \forall \theta \in \Omega(E).$$

This, together with Oseledeĉ's Multiplicative Ergodic Theorem [O,GM] implies existence of m solutions of equation (1), which decay exponentially as $n \to +\infty$ and grow exponentially as $n \to -\infty$, and m other solutions of equation (1), which decay exponentially as $n \to -\infty$ and grow exponentially as $n \to +\infty$. These $2m$ solutions are linearly independent and form a basis of all solutions of equation (1). There are two important consequences of this fact.

1) The Green's function of H decays exponentially for almost every energy E.

2) The spectrum of the operator H defined by the left hand side of equation (1) is singular.

Remarks. We should point out that the spectrum of H can actually be purely singular continuous if α is a Liouville number [CFKS].

The requirement that λ is large cannot be avoided, for when λ is small KAM theory is applicable and there is absolutely continuous spectrum [BLT].

We strongly use the non-triviality of f_i's in our proof. This condition is necessary, for there are examples in which the presence of constant f_i's leads to appearance of zero Lyapunov exponents.

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