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# Statistics in the Propositional Formulation of Quantum Mechanics

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*Abstract.* We give a definition for the notion of statistics in the lattice-theoretical (or propositional) formulation of quantum mechanics of Birkhoff, von Neumann and Piron. We show that this formalism is compatible only with two types of statistics: Bose-Einstein and Fermi-Dirac. Some comments are made about the connection between this result and the existence of exotic statistics (para-statistics, infinite statistics, braid statistics).

## 1 Introduction

The lattice-theoretical formulation of quantum physics (Birkhoff-von Neumann-Piron) seems to be extremely well suited for the treatement of many problems connected with the logical foundations of a physical theory [1], [2]. The basic idea of this formulation is that all elementary ("yes-no") statements which can be made about a physical system can be organized in a lattice structure $\mathcal{L}$. For a pure quantum system the corresponding lattice $\mathcal{L}$ is made up of all orthogonal projectors in a given vector space of Hilbertian type $\mathcal{H}$: $\mathcal{L} = \mathcal{P}(\mathcal{H})$. A fundamental result asserts that this case is, essentially, generic. Namely, the most general physical system is described by a direct union of pure quantum lattices [1].

Among other things, the lattice-theoretical formulation of quantum mechanics affords an answer to the question why two (or many) quantum systems are usually described in the tensor product Hilbert space of the individual Hilbert spaces of the corresponding subsystems [3], [4]. This structure is a consequence of the so-called "weak coupling" condition. Essentially, this condition requires that the subsystems of the composite system do not loose their individuality. Mathematically, if $\mathcal{L}_1, ..., \mathcal{L}_n$ are the lattices of the individual subsystems and $\mathcal{L}_0$ is the lattice of the composite system, one requires the existence of a map $h : \mathcal{L}_1 \times ... \times \mathcal{L}_n \rightarrow \mathcal{L}_0$ with the following significance: if $a_1, ..., a_n$ are properties of the subsystems $\mathcal{L}_1, ..., \mathcal{L}_n$ respectively, then $h(a_1, ..., a_n) \in \mathcal{L}_0$ corresponds to the property: "the subsystem 1 has the property $a_1$,..., the subsystem $n$ has the property $a_n$". As we have said before, if $\mathcal{L}_i = \mathcal{L}(\mathcal{H}_i)$ with $dim(\mathcal{H}_i) \geq 3$ $(i = 0, 1, ..., n)$ and the map $h$ has some reasonable properties, then one can discover that in many cases of physical interest $\mathcal{H}_0$ has some tensorial nature.

An interesting problem is if this type of result can be extended for systems of identical particles. An attempt in this direction is announced in [5] where one finds the rather strange result that for a system of identical particles only Fermi statistics is allowed. We should note here that there are other abstract definitions of the notion of statistics in the framework of algebraic quantum theory [6].

The purpose of this paper is to give an alternative analysis for systems of identical particles in the framework of the lattice-theoretical formulation. We will give a reasonable "weak coupling" condition for a system of identical particles and we will be able to prove that, in quite general conditions, there are only two possible statistics: Bose and Fermi.

The idea of the proof is suggested already by [3], which uses as an auxilliary result, a certain generalization of Wigner theorem. So, the idea is to look for the "simplest" proof of Wigner theorem and try to apply it to our situation. We have found it profitable to use in such a way Uhlhorn proof of Wigner theorem [7]. Using the idea of this proof (which will be briefly presented) we will be able to give an alternative (and simpler in our opinion) proof of the result of [3], [4] for the case of a system composed of two different subsystems. In particular our proof shows that the conditions impsed in [3], [4] on the map $h : \mathcal{L}_1 \times ... \times \mathcal{L}_n \rightarrow \mathcal{L}_0$ can be relaxed. These topics are treated in Section 2.

In Section 3 we give our definition for a system of identical particles and derive the result concerning the possible statistics that was announced above. The proof will follow the same lines as the one in Section 2. In the end of Section 3 we will make some comments about the connection between our result and the existence in the litterature of other types of statistics as: parastatistics, infinite statistics [6] and braid statistics [8], [9].

## 2 Many-Particle Systems

A. According to the lattice-theoretical philosophy one must describe any physical system by a lattice $(\mathcal{L}, <)$ which is complete, atomic, orthocomplemented, weakly modular and satisfies the covering law. Usually we will omit the order relation $<$ in writing the simbol of a lattice. Such a lattice is also called a *propositional system* [1] (see $2.1). As regards to the physical interpretation we mention only the following facts without bothering about a precise mathematical formulation:

- the elements of $\mathcal{L}$ are interpreted as elementary ("yes-no") assertions about the system (more precisely equivalence classes of "yes-no" questions)

- the order relation $<$ means logical implication

- the infimum operation $\wedge$ has the meaning of the logical "AND"

- the atoms of $\mathcal{L}$ are interpreted as the states of the system

- the minimal element 0 is interpreted as the property "the system does not exist" and the maximal element $I$ is interpreted as the property "the system exists".

The standard models of propositional systems are:

a) $\mathcal{P}(\Gamma)$; here $\Gamma$ is an arbitrary set and $\mathcal{P}(\Gamma)$ is the set of all subsets of $\Gamma$. The infimum is the intersection and the orthocomplementation is the usual complementation. This is the *pure clasical* case.

b) $\mathcal{L}(\mathcal{H})$; here $\mathcal{H}$ is an arbitrary vector space of Hilbertian type and $\mathcal{L}(\mathcal{H})$ is the set of all linear closed subspaces of $\mathcal{H}$. The infimum is the intersection and the orthocomplementation is the map associationg to any subspace of $\mathcal{H}$ its orthogonal suplement. This is the *pure quantum* case.

As asserted in [1] these cases are rather generic, in the sense that the most general situation is obtained by taking a direct union of pure quantum lattices; in this case the center of the lattice $\mathcal{L}$ is a pure classical lattice $I$: $\mathcal{L} = \vee_{\alpha \in I} \mathcal{L}(\mathcal{H}^\alpha)$.

B. We schetch briefly the proof of Wigner theorem from [7]. The idea of the proof will be afterwards adapted to the study of many-particle systems. The purpose of Wigner theorem is to classify symmetries of propositional systems. Such a symmetry is, by definition, a structure-preserving map between propositional systems. We adopt the following definition:

**Definition 1:** Let $\mathcal{L}_1, \mathcal{L}_2$ be two propositional systems. A map $h : \mathcal{L}_1 \to \mathcal{L}_2$ is called a *symmetry* if it verifies:

(a) if $p \in \mathcal{L}_1$ is an atom then $h(p) \in \mathcal{L}_2$ is an atom

(b) for any $a_i \in \mathcal{L}_1$ $i \in I$ an index set, one has:

$$\wedge_{i \in I} h(a_i) = h(\wedge_{i \in I} a_i)$$

(c) $h(I_1) = I_2$

(d) for any $a \in \mathcal{L}_1$ one has:

$$h(a)' = h(a')$$

**Remark 1:** Usually (a) and (c) are replaced with the condition of bijectivity, which is, in general, stronger.

Let us now suppose that the lattices $\mathcal{L}_1, \mathcal{L}_2$ are of the pure quantum type: $\mathcal{L}_i = \mathcal{L}(\mathcal{H}_i)$ where $\mathcal{H}_i$ is a vector space of Hilbertian type over the division ring $D_i$ with $dim(\mathcal{H}_i) \geq 3$ $(i = 1, 2)$. Then one can proceed to a rather exthaustive classification of maps $h$ verifying Definition 1. We provide the main steps below.

1. If $a_1, ..., a_n \in \mathcal{L}(\mathcal{H}_1)$ are atoms it is obvious what we mean when we say that $a_1, ..., a_n$ are linear independent: for any $x_i \in a_i \backslash \{0\}$ $(i = 1, ..., n)$, the vectors $x_1, ..., x_n \in \mathcal{H}_1$ are linear independent. We also use the following notation: if $x_1 \in \mathcal{H}_1 \backslash \{0\}$ then the atom containing $x_1$ is denoted by $D_1 \cdot x_1$.

The first observation, following form Definition 1, is that $a_1, ..., a_n \in \mathcal{L}(\mathcal{H}_1)$ are linear independent atoms *iff* $h(a_1), ..., h(a_n) \in \mathcal{L}(\mathcal{H}_2)$ are linear independent atoms.

2. We define now a map $B : \mathcal{H}_1 \to \mathcal{H}_2$ as follows:

- $B(0) = 0$

- for any $x_1 \in \mathcal{H}_1 \backslash \{0\}$ we take $B(x_1)$ to be an arbitrary non-zero element in the atom $h(D_1 \cdot x_1)$.

In this way we have

$$D_2 \cdot B(x_1) = h(D_1 \cdot x_1)$$

for any $x_1 \in \mathcal{H}_1 \backslash \{0\}$. It is clear that, in general, the map $B$ will not be additive. However, one must observe that there is a phase factor arbitrariness in the definition of $B$. One takes advantage of this arbitrariness of $B$; namely one shows that by apropriately modifying $B$ one can make it an additive map.

3. Let $x_1, y_1 \in \mathcal{H}_1$, be linear independent vectors. Using step 1 one easily establish that one has:

$$B(x_1 + y_1) = \omega(x_1, x_1 + y_1)\ B(x_1) + \omega(y_1, x_1 + y_1)\ B(y_1) \tag{2.1}$$

where $\omega : \mathcal{H}_1 \times \mathcal{H}_1 \to D_2 \backslash \{0\}$ is defined for the moment only for linear independent vectors.

4. Let now $x_1, y_1, z_1 \in \mathcal{H}_1$ be linear independent vectors. If one writes $B(x_1 + y_1 + z_1)$ in two different ways with the help of (2.1) one easily discovers in this case a "cohomological" relationship:

$$\omega(y_1, z_1)\ \omega(x_1, y_1) = \omega(x_1, z_1). \tag{2.2}$$

5. An easy consequence of (2.2) is that for any $x_1, y_1 \in \mathcal{H}_1$ linear independent one has:

$$\omega(x_1, y_1)\ \omega(y_1, x_1) = 1. \tag{2.3}$$

6. We now define $\omega(x_1, y_1)$ for $x_1, y_1 \in \mathcal{H}_1 \backslash \{0\}$ linear dependent as follows. One takes $z_1 \in \mathcal{H}_1$ such that $x_1$ and $z_1$ are linear independent and tries to *define* $\omega(x_1, y_1)$ by:

$$\omega(x_1, y_1) \equiv \omega(z_1, y_1)\ \omega(x_1, z_1). \tag{2.4}$$

The right hand side does not depend on the choice of $z_1$ above, so this definition is consistent. We note that it is at this point that one needs the restriction $dim(\mathcal{H}_1) \geq 3$.

7. Using the extension of $\omega$ defined above, one shows easily that (2.3) is true for any $x_1, y_1 \in \mathcal{H}_1 \backslash \{0\}$.

8. Next, one shows that (2.2) above is true for any $x_1, y_1, z_1 \in \mathcal{H}_1 \backslash \{0\}$.

9. Finally, one extends (2.1) for any $x_1, y_1 \in \mathcal{H}_1\backslash\{0\}$.

10. We are ready to redefine the map $B$ such that it becomes additive. We take $x_1^0 \in \mathcal{H}_1\backslash\{0\}$ arbitrary but fixed and define for any $x_1 \in \mathcal{H}_1\backslash\{0\}$:

$$\widetilde{B}(x_1) \equiv \omega(x_1, x_1^0)\ B(x_1). \tag{2.5}$$

Then an easy computation shows that $\widetilde{B}$ verifies for any $x_1, y_1 \in \mathcal{H}_1\backslash\{0\}$:

$$\widetilde{B}(x_1 + y_1) = \widetilde{B}(x_1) + \widetilde{B}(y_1). \tag{2.6}$$

It is clear that (2.6) stays true even if $x_1$ or $y_1$ are zero. Because we still have $D_2 \cdot \widetilde{B}(x_1) = h(D_1 \cdot x_1)$ for any $x_1 \in \mathcal{H}_1\backslash\{0\}$ we might just well take instead of $B$ the new map $\widetilde{B}$.

In conclusion, if Definition 1 is true, one can find an additive map $B : \mathcal{H}_1 \to \mathcal{H}_2$:

$$B(x_1 + y_1) = B(x_1) + B(y_1) \quad (\forall x_1, y_1 \in \mathcal{H}_1) \tag{2.7}$$

such that:

$$D_2 \cdot B(x_1) = h(D_1 \cdot x_1) \quad (\forall x_1 \in \mathcal{H}_1\backslash\{0\}). \tag{2.8}$$

11. It is not difficult to show that $B$ also verifies:

$$Im(B) = \mathcal{H}_2. \tag{2.9}$$

It follows that $B$ is a bijective map.

12. Now it is rather easy to prove that there exists a map $\varphi : D_1 \to D_2$ such that:

$$B(\lambda_1 x_1) = \varphi(\lambda_1)\ B(x_1) \quad (\forall \lambda_1 \in D_1, \forall x_1 \in \mathcal{H}_1). \tag{2.10}$$

Moreover, the map $\varphi$ verifies for any $\lambda_1, \mu_1 \in D_1$:

$$\varphi(\lambda_1 + \mu_1) = \varphi(\lambda_1) + \varphi(\mu_1) \tag{2.11}$$

$$\varphi(\lambda_1 \mu_1) = \varphi(\lambda_1)\ \varphi(\mu_1) \tag{2.12}.$$

It is not hard to convince oneself that $\varphi$ is indeed a division ring isomorphism. So in fact one can take $D_1 = D_2 (\equiv D_0)$.

13. From Definition 1 it follows that the map $h$ preserves the orthogonality relationship. If the division ring $D_0$ is commutative, it follows that $\varphi$ also verifies:

$$\bar{\varphi}(\lambda) = \varphi(\bar{\lambda}) \tag{2.13}$$

where $\lambda \to \bar{\lambda}$ is the involution of $D_0$.

**Remark 2:** The usual cases $D_0 = R, C, H$ can be analysed in detail as in [2]. If $D_0 = R, H$ one finds that one can take $\varphi = id$ so $B$ becomes a linear map, and if $D_0 = C$ one has two cases: $\varphi(\lambda) = \lambda$ and $\varphi(\lambda) = \bar{\lambda}$ corresponding to $B$ linear and respectively antilinear. We have recovered the usual statement of Wigner theorem.

**Remark 3:** One can also show that there exists $\delta \in D_0\backslash\{0\}$ such that $\forall x_1, y_1 \in \mathcal{H}_1$ one has:

$$< B(x_1), B(y_1) >_{\mathcal{H}_2} = \delta\varphi(< x_1, y_1 >_{\mathcal{H}_1}) \tag{2.14}$$

with $\bar{\delta} = \delta$. The idea is to adopt (2.14) as the definition for $\delta$ as a function of $x_1, y_1, x_2, y_2$ and to show that in fact it is a constant.

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces over $R, C$ or $H$ one can show that in fact $\delta > 0$.

So, by a rescaling, $B$ can be made an isometry.

C. Now we come to the study of a composite system. We formulate [3], [4]:

**Definition 2:** Let $\{\mathcal{L}_i\}_{i=0}^{3}$ be three propositional systems with $\mathcal{L}_1 \neq \mathcal{L}_2$. We say that the system $\mathcal{L}_0$ is *composed of the subsystems* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ if there exists a map $h : \mathcal{L}_1 \times \mathcal{L}_2 \to \mathcal{L}_0$ verifying:

(a) if $p_1 \in \mathcal{L}_1$ and $p_2 \in \mathcal{L}_2$ are atoms then $h(p_1, p_2) \in \mathcal{L}_0$ is an atom

(b) $\forall a_i \in \mathcal{L}_1$ $(i \in I)$ and $\forall b_i \in \mathcal{L}_2 (i \in I)$ ($I$ is an index set), one has:

$$\wedge_{i \in I} h(a_i, b_i) = h(\wedge_{i \in I} a_i, \wedge_{i \in I} b_i)$$

(c) $h(I_1, I_2) = I_0$

(d) $\forall a \in \mathcal{L}_1$ and $\forall b \in \mathcal{L}_2$ one has:

$$h(a, b)' = h(a', b')$$

**Remark 4:** The existence of the map $h$ can be also called the *weak coupling condition* [3]. Indeed, when postulating the existence of such a map one implicitely asumes that the subsystems $\mathcal{L}_1$ and $\mathcal{L}_2$ do not loose their individuality. It is clear that the proposition $h(a_1, a_2)$ coresponds to the property: "the subsystem 1 has the property $a_1$ and the subsystem 2 has the property $a_2$".

**Remark 5:** The physical interpretation of the axioms (a)-(d) above is rather transparent [3], [4]. We note however that we did not include in this definition the condition: $h(a_1, I_2) \leftrightarrow h(I_1, a_2)$ $\forall a_1 \in \mathcal{L}_1$, $\forall a_2 \in \mathcal{L}_2$ which is explicitely admitted in [3], [4]. In fact, the analysis below will show that this condition is redundant.

We proceed now with the classification of maps $h$ in the case when $\mathcal{L}_i = \mathcal{L}(\mathcal{H}_i)$ where $\mathcal{H}_i$ is a vector space of Hilbertian type over the division ring $D_i$ (i = 0, 1, 2).

We also admit that $dim(\mathcal{H}_i) \geq 3 \quad (i = 0,1,2)$. We will follow closedly the steps 1-12 of part B.

1. One can show rather easily that the atoms $a_i \in \mathcal{L}_1 \quad (i \in I)$ are linear independent *and* the atoms $b_j \in \mathcal{L}_2 \quad (j \in J)$ are linear independent *iff* the atoms $h(a_i, b_j) \in \mathcal{L}_0 \quad (i \in I, j \in J)$ are linear independent.

2. We now define a map $B : \mathcal{H}_1 \times \mathcal{H}_2 \rightarrow \mathcal{H}_0$ as follows:

- $B(0, x_2) = 0, \quad B(x_1, 0) = 0 \quad (\forall x_1 \in \mathcal{H}_1, \forall x_2 \in \mathcal{H}_2)$

- for $x_1 \in \mathcal{H}_1 \backslash \{0\}, x_2 \in \mathcal{H}_2 \backslash \{0\}$ we take $B(x_1, x_2)$ to be an arbitrary non-zero vector in the atom $h(D_1 \cdot x_1, D_2 \cdot x_2)$. In this way we have $D_0 \cdot B(x_1, x_2) = h(D_1 \cdot x_1, D_2 \cdot x_2)$. Like in part B we will use the phase arbitrariness in the definition of $B$ to make it biadditive.

3. We fix the vector $x_2 \in \mathcal{H}_2 \backslash \{0\}$. Then we can repeat steps A3-A10 for the map $x_1 \rightarrow B(x_1, x_2)$ and we succeed to redefine $B$ such that it is additive in the first argument:

$$B(x_1 + y_1, x_2) = B(x_1, x_2) + B(y_1, x_2) \quad \forall x_1, y_1 \in \mathcal{H}_1, \ \forall x_2 \in \mathcal{H}_2 \tag{2.15}$$

4. Now we fix $x_1 \in \mathcal{H}_1 \backslash \{0\}$ and repeat steps A3-A9 for the map $x_2 \rightarrow B(x_1, x_2)$. We find that there exists a map: $\omega_{x_1} : (\mathcal{H}_2 \backslash \{0\}) \times (\mathcal{H}_2 \backslash \{0\}) \rightarrow D_0$ such that $\forall x_2, y_2 \in \mathcal{H}_2 \backslash \{0\}$:

$$B(x_1, x_2 + y_2) = \omega_{x_1}(x_2, x_2 + y_2) \ B(x_1, x_2) + \omega_{x_1}(y_2, x_2 + y_2) \ B(x_1, y_2). \tag{2.16}$$

The function $\omega_{x_1}$ also verifies $\forall x_2, y_2, z_2 \in \mathcal{H}_2 \backslash \{0\}$:

$$\omega_{x_1}(y_2, z_2) \ \omega_{x_1}(x_2, y_2) = \omega_{x_1}(x_2, z_2) \tag{2.17}$$

and:

$$\omega_{x_1}(x_2, y_2) \ \omega_{x_1}(y_2, x_2) = 1. \tag{2.18}$$

5. We want to apply the trick A10 to get additivity in the second argument of $B$ without ruining the same property in the first argument. For this we have to show first that in fact $\omega_{x_1}(\cdot, \cdot)$ does not depend on $x_1$. This is rather simple. One takes $x_i, y_i \in \mathcal{H}_i \ (i = 1, 2)$ linear independent and makes in (2.16) $x_1 \rightarrow x_1 + y_1$. If we use now the additivity (2.15) we arrive quite easily at the following relationship:

$$\omega_{x_1}(x_2, y_2) = \omega_{y_1}(x_2, y_2). \tag{2.19}$$

If $x_1, y_1 \in \mathcal{H}_1 \backslash \{0\}$ verify $D_1 \cdot x_1 = D_1 \cdot y_1$ one takes $z_1 \in \mathcal{H}_1 \backslash \{0\}$ such that $D_1 \cdot x_1 \neq D_1 \cdot z_1$ and has from (2.19):

$$\omega_{x_1}(x_2, y_2) = \omega_{z_1}(x_2, y_2) = \omega_{y_1}(x_2, y_2)$$

In this way one extends (2.19) for all $x_1, y_1 \in \mathcal{H}_1 \backslash \{0\}$. We have only the restriction $D_2 \cdot x_2 \neq D_2 \cdot y_2$. But if $D_2 \cdot x_2 = D_2 \cdot y_2$, then $\omega_{x_1}(x_2, y_2)$ is defined according to A6 as follows:

$$\omega_{x_1}(x_2, y_2) = \omega_{x_1}(z_2, y_2) \ \omega_{x_1}(x_2, z_2)$$

where $z_2 \in \mathcal{H}_2\backslash\{0\}$ verifies $D_2 \cdot x_2 \neq D_2 \cdot y_2$. (see (2.4)). The right hand side of this relation does not depend on $x_1$ according to what has been proved so far . So we have succeeded to extend (2.19) to all $x_i, y_i \in \mathcal{H}_i\backslash\{0\}$ $(i = 1, 2)$.

6. It follows that in fact in (2.16)-(2.18) $\omega_{x_1}(\cdot,\cdot)$ does not depend on $x_1$ so we have for any $x_2, y_2 \in \mathcal{H}_2\backslash\{0\}$:

$$B(x_1, x_2 + y_2) = \omega(x_2, x_2 + y_2)\ B(x_1, x_2) + \omega(y_2, x_2 + y_2)\ B(x_1, y_2). \tag{2.20}$$

where the function $\omega$ also verifies

$$\omega(y_2, z_2)\ \omega(x_2, y_2) = \omega(x_2, z_2) \tag{2.21}$$

and:

$$\omega(x_2, y_2)\ \omega(y_2, x_2) = 1. \tag{2.22}$$

Now we apply the trick A10 for the map $x_2 \mapsto B(x_1, x_2)$ and we succeed to make it additive in the second argument, preserving in the mean time the same property in the first argument. So, beside (2.15) we have:

$$B(x_1, x_2 + y_2) = B(x_1, x_2) + B(x_1, y_2) \tag{2.23}$$

and:

$$D_0 \cdot B(x_1, x_2) = h(D_1 \cdot x_1, D_2 \cdot x_2). \tag{2.24}$$

7. Like in part A it is not difficult to show that B also verifies:

$$Im(B) = \mathcal{H}_0. \tag{2.25}$$

8. Step A12 goes now practically unchanged. One proves the existence of a map $\varphi : D_1 \times D_2 \to D_0$ such that:

$$B(\lambda_1 x_1, \lambda_2 x_2) = \varphi(\lambda_1, \lambda_2)\ B(x_1, x_2). \tag{2.26}$$

Moreover, the map $\varphi$ verifies:

$$\varphi(\lambda_1 + \mu_1, \lambda_2 + \mu_2) = \varphi(\lambda_1, \lambda_2) + \varphi(\lambda_1, \mu_2) + \varphi(\mu_1, \lambda_2) + \varphi(\mu_1, \mu_2) \tag{2.27}$$

$$\varphi(\lambda_1 \mu_1, \lambda_2 \mu_2) = \varphi(\lambda_1, \lambda_2)\ \varphi(\mu_1, \mu_2). \tag{2.28}$$

9. Like at A13 one can show that if $D_0$ is commutative $\varphi$ also satisfies:

$$\bar{\varphi}(\lambda_1, \lambda_2) = \varphi(\bar{\lambda}_1, \bar{\lambda}_2) \tag{2.29}$$

where the bar denotes the corresponding involutions of $D_i$ $(i = 1, 2)$.

**Remark 6:** If $D_0 = D_1 = D_2 = R$ we have $\varphi(\lambda_1, \lambda_2) = \lambda_1\lambda_2$ and if $D_0 = D_1 = D_2 = C$ we have four possibilities:

$$\varphi(\lambda_1, \lambda_2) = \lambda_1\lambda_2, \quad \varphi(\lambda_1, \lambda_2) = \bar{\lambda}_1\bar{\lambda}_2$$

$$\varphi(\lambda_1, \lambda_2) = \bar{\lambda}_1\lambda_2, \quad \varphi(\lambda_1, \lambda_2) = \lambda_1\bar{\lambda}_2.$$

On the contrary if $D_0 = D_1 = D_2 = H$ one can prove that (2.27)+(2.28) have no solution. It is quite possible that, in general, the weak coupling problem does not admit solutions if $D_1$ and $D_2$ are non-commutative division rings. (See however in connection with this problem [10].)

10. Taking into account the Remark above it is intersting to consider the particular case when $D_0 = D_1 = D_2$ is a *commutative* division ring and $\varphi : D_0 \times D_0 \to D_0$ is:

$$\varphi(\lambda_1, \lambda_2) = \lambda_1\lambda_2. \tag{2.30}$$

We have:

**Theorem 1:** In the conditions above one can take $\mathcal{H}_0 = \mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, if $\mathcal{H}_i$ are Hilbert spaces and we identify $\mathcal{L}(\mathcal{H}_i)$ with the lattice of the orthogonal projectors in $\mathcal{H}_i$, then the map $h$ is:

$$h(a_1, a_2) = a_1 \otimes a_2. \tag{2.31}$$

**Proof:** One has to check that the map $B$ has the universality properties defining the tensor product (see e.g. [11]).

$\otimes_1$: We have $Im(B) = \mathcal{H}_0$ according to step 7.

$\otimes_2$: Let $\mathcal{H}$ be vector space over the division ring $D_0$ and $g : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}$ be a bilinear map. Our purpose is to identify a linear map $f : \mathcal{H}_0 \to \mathcal{H}$ such that:

$$f \circ B = g. \tag{2.32}$$

We procced as follows. First, let us consider $x_i, y_i \in \mathcal{H}_i$ $(i = 1, 2)$ such:

$$B(x_1, x_2) = B(y_1, y_2). \tag{2.33}$$

Then, one easily establish that $y_i = \lambda_i x_i$ $(i = 1, 2)$ with $\lambda_1, \lambda_2 \in D_0$ verifying $\lambda_1\lambda_2 = 1$. It follows that

$$g(y_1, y_2) = g(x_1, x_2). \tag{2.34}$$

So, taking into account $\otimes_1$ we define $f : \mathcal{H}_0 \to \mathcal{H}$ as follows:

- for elements of the form $B(x_1, x_2)$:

$$f(B(x_1, x_2)) = g(x_1, x_2) \tag{2.35}$$

(and this is consistent because of the implication (2.33) $\Rightarrow$ (2.34)).

- for all the other elements we extend $f$ by linearity:

$$f(\sum_{i=1}^{n} B(x_i, y_i)) = \sum_{i=1}^{n} g(x_i, y_i) \tag{2.36}$$

(and again one can prove the consistency of this definition). But (2.35) and (2.36) gives (2.32). Q. E. D.

**Remark 7:** The theorem above takes care of the case $D_0 = R$. In the case $D_0 = C$ we have the four possibilities from Remark 6. ( We note that in this case $\mathcal{H}_i$ are Hilbert spaces according to Amemiya-Araki theorem [1], [2]). We still have to analyse the last three of them. We define the antilinear maps $\alpha_i : \mathcal{H}_i \to (\mathcal{H}_i)^*$ $(i = 1,2)$ by:

$$(\alpha_i(x_i), y_i)_{\mathcal{H}_i} = < x_i, y_i >_{\mathcal{H}_i} . \tag{2.37}$$

Here $<,>_{\mathcal{H}_i}$ is the scalar product on $\mathcal{H}_i$ and $(,)_{\mathcal{H}_i}$ is the duality form between $\mathcal{H}_i$ and $(\mathcal{H}_i)^*$. For the case $\varphi(\lambda_1, \lambda_2) = \bar{\lambda}_1 \bar{\lambda}_2$ we define: $B_{12} : (\mathcal{H}_1)^* \times (\mathcal{H}_2)^* \to \mathcal{H}_0$ by:

$$B_{12}(x_1, x_2) = B(\alpha_1^{-1}(x_1), \alpha_2^{-1}(x_2))$$

and note that $B_{12}$ is bilinear so applying the theorem above we can take $\mathcal{H}_0 = (\mathcal{H}_1)^* \otimes (\mathcal{H}_2)^* \cong \mathcal{H}_1 \otimes \mathcal{H}_2$. In the last two cases one procceeds similarly and finds out that it is possible to take $\mathcal{H}_0 = (\mathcal{H}_1)^* \otimes \mathcal{H}_2 \cong \mathcal{H}_1 \otimes (\mathcal{H}_2)^*$.

**Remark 8:** A formula of the type (2.14) can be also proved in this case, namely:

$$< B(x_1, x_2), B(y_1, y_2) >_{\mathcal{H}_0} = \delta\varphi(< x_1, y_1 >_{\mathcal{H}_1}, < x_2, y_2 >_{\mathcal{H}_2}) \tag{2.38}$$

for some $\delta \in D_1 \backslash \{0\}$.

**Remark 9:** One can extend the results obtained up till now for the more general case when $\mathcal{L}_0, \mathcal{L}_1$ and $\mathcal{L}_2$ are systems with superselection rules i.e. $\mathcal{L}_i = \vee_{\alpha_i \in I_i} \mathcal{L}(\mathcal{H}_i^{(\alpha_i)})$ $(i = 0, 1, 2)$. Here $I_0, I_1$ and $I_2$ are some index sets. We proceed in analogy to [1] \$3-2. First we note that the map $h$ preserves the relationship of compatibility, namely if $a_1$ is in the center of $\mathcal{L}_1$ and $a_2$ is in the center of $\mathcal{L}_2$, then $h(a_1, a_2)$ is in the center of $\mathcal{L}_0$. It follows that the map $h$ induces a map $\hat{h} : \mathcal{P}(I_1) \times \mathcal{P}(I_2) \to \mathcal{P}(I_0)$ where $\mathcal{P}(I_i)$ are classical propositional systems (see the begining of part A). The map $\hat{h}$ also verifies the axioms of Definition 1. In this case one can easily discover that $I_0 \cong I_1 \times I_2$ [12] so in fact we have $\mathcal{L}_0 \cong \vee_{\alpha_1 \in I_1, \alpha_2 \in I_2} \mathcal{L}(\mathcal{H}_0^{(\alpha_1, \alpha_2)})$ and $h(\mathcal{L}(\mathcal{H}_1^{(\alpha_1)}), \mathcal{L}(\mathcal{H}_2^{(\alpha_2)})) = \mathcal{L}(\mathcal{H}_0^{(\alpha_1 \alpha_2)})$ $\forall \alpha_1 \in I_1, \forall \alpha_2 \in I_2$. It is now clear that for all the maps $h^{\alpha_1 \alpha_2} \equiv h|_{\mathcal{L}(\mathcal{H}_1^{(\alpha_1)}) \times \mathcal{L}(\mathcal{H}_2^{(\alpha_2)})}$ one can apply the previous analysis.

**Remark 10:** It is obvious that the analysis contained in this Section can be easily extended to the case when the system $\mathcal{L}_0$ is composed of more that two subsystems

$\mathcal{L}_1, ..., \mathcal{L}_n$ $(n > 2)$. This kind of generalization seems to be more cumbersome to do if one adopts the line of argument in [3], [4].

## 3 Systems of Identical Particles

A. We try here to propose a definition for a system composed of two *identical* subsystems by modifying as little as possible Definition 1. It is clear from the begining that one must take $\mathcal{L}_1 = \mathcal{L}_2$ i.e. we have a map $h : \mathcal{L}_1 \times \mathcal{L}_1 \to \mathcal{L}_0$. Also, we expect that, because of the identity of the the subsystems, this map is symmetric. The physical interpetation of $h$ must be the following one: if $a, b \in \mathcal{L}_1$ are two properties then $h(a, b)$ correspond to the property: "one of the subsystem is in the state $a$ and the other is in the state $b$". Because of this (natural) interpretation it follows that we cannot expect that item (b) of Definition 1 holds in this case. If we could interpret the supremum operation $\vee$ as the logical "OR" we would be tempted to substitute (b) of Definition 1 by something of the type:

$$h(a, b) \wedge h(c, d) = h(a \wedge c, b \wedge d) \vee h(a \wedge d, b \wedge c).$$

However, it is known that $\vee$ can be interpreted as the logical "OR" only in the pure classical case. (Indeed, in the pure quantum case one can easily find two properties $a, b \in \mathcal{L}(\mathcal{H})$ such that the logical proposition $aORb$ corresponds to no element in $\mathcal{L}(\mathcal{H})$). So, the relation above cannot hold.

After this discussion we make an attempt for a convenient definition.

**Definition 2:** We say that the propositional system $\mathcal{L}_0$ is *composed of two identical subsystems* if there exists a propositional system $\mathcal{L}_1$ and a map $h : \mathcal{L}_1 \times \mathcal{L}_1 \to \mathcal{L}_0$ verifying:

$(a_1)$ if $p_1, p_2 \in \mathcal{L}_1$, $p_1 \neq p_2$ are atoms, then $h(p_1, p_2) \in \mathcal{L}_0$ is an atom

$(a_2)$ if $p \in \mathcal{L}_1$ is an atom then $h(p, p) \in \mathcal{L}_0$ is either 0 or an atom

$(a_3)$ if $a, b, c, d \in \mathcal{L}_1$ are atoms and

$$h(a, b) = h(c, d) \neq 0$$

then $a = c$, $b = d$ or $a = d$, $b = c$.

$(b_1)$ if $a < c, d$ and $b < c, d$, then $h(a, b) < h(c, d)$

$(b_2)$ the atoms $a_i \in \mathcal{L}_1$ $(i \in I)$ are linear independent *iff* all the distinct non-zero atoms of the form $h(a_i, a_j)$ are linear independent

$(c_1)$ $\vee_{p_1, p_2 = atoms} h(p_1, p_2) = I_0$

$(c_2)$ $h(I_1, I_1) = I_0$

(d) $h(a, b)' = h(a', b')$

(e) $h(a, b) = h(b, a)$.

**Remark 11:** All the axioms exept $(b_2)$ and $(c_1)$ are rather easy to interpret from the physical point of view. We note that if, in analogy to Definition 2, one would require that

$h(p_1, p_2)$ is an atom $\forall p_1, p_2 \in \mathcal{L}_1$, then one would obtain only Bose statistics as a result of the analysis below. On the other hand, if one requires that $h(p,p)$ is always 0, then one obtains the result in [5]. So, we accept a more general situation described by $(a_1)$ and $(a_2)$. This is the only possible generalization which preserves the general idea which had suggested Definition 2 (see [3]). In particular, $(a_2)$ takes into account the logical possibility that the two identical system cannot be in the same state.

Because $(b_2)$ and $(c_1)$ have a certain degree of naturalness we think it is interesting to analyse in detail the consequence of this definition.

We proceed now with this analysis on the lines of Subsection 2C in the case $\mathcal{L}_i = \mathcal{L}(\mathcal{H}_i)$ where $\mathcal{H}_i$ is a vector space of Hilbertian type over the division ring $D_i$ $(i = 0, 1)$ and $dim(\mathcal{H}_1) \geq 4$.

1. We define a map $B : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_0$ as follows:

- $B(0, x_2) = B(x_1, 0) = 0,\ \forall x_1, x_2 \in \mathcal{H}_1$

- if $x_1 \in \mathcal{H}_1 \backslash \{0\}$ and $h(D_1 \cdot x_1, D_1 \cdot x_1) = 0$ then $B(x_1, x_1) = 0$

- if $x_1, x_2 \in \mathcal{H}_1 \backslash \{0\}$ and $x_1 \neq x_2$ or if $x_1 = x_2$ but $h(D_1 \cdot x_1, D_1 \cdot x_1) \neq 0$, then $B(x_1, x_2)$ is an arbitrary non-zero element of the atom $h(D_1 \cdot x_1, D_1 \cdot x_2)$

It is clear that we have $D_0 \cdot B(x_1, x_2) = h(D_1 \cdot x_1, D_1 \cdot x_2)$.

2. From Definition 2 $(b_2)$ one can show that for any $x_2 \in \mathcal{H}_1 \backslash \{0\}$ there exists a function $\omega_{x_2} : (\mathcal{H}_1 \backslash \{0\}) \times (\mathcal{H}_1 \backslash \{0\}) \to D_0$ such that $\forall x_1, y_1, x_2$ linear independent we have:

$$B(x_1 + y_1, x_2) = \omega_{x_2}(x_1, x_1 + y_1)\ B(x_1, x_2) + \omega_{x_2}(y_1, x_1 + y_1)\ B(y_1, x_2). \tag{3.1}$$

3. Next, one takes $x_1, y_1, z_1, x_2 \in \mathcal{H}_1$ linear independent and shows that we have:

$$\omega_{x_2}(y_1, z_1)\ \omega_{x_2}(x_1, y_1) = \omega_{x_2}(x_1, z_1) \tag{3.2}$$

and

$$\omega_{x_2}(x_1, y_1)\ \omega_{x_2}(y_1, x_1) = 1 \tag{3.3}$$

4. Like at 2A, we now extend the function $\omega_{x_2}$ to other values of $x_1, y_1$. One must use definitions of the type (2.4) and prove their consistency. It is at this step that one needs the condition $dim(\mathcal{H}_1) \geq 4$.

5. Using the definition of $\omega$ from above, one extends (3.1)-(3.3) to all $x_1, y_1, z_1, x_2 \in \mathcal{H}_1 \backslash \{0\}$. So, applying the trick 2A.10 one succeds to make the map $B$ additive in the first argument:

$$B(x_1 + y_1, x_2) = B(x_1, x_2) + B(y_1, x_2). \tag{3.4}$$

Next, we apply the trick 2C.5-6 and obtain additivity in the second argument also:

$$B(x_1, x_2 + y_2) = B(x_1, x_2) + B(x_1, y_2). \tag{3.5}$$

Moreover, we still have:

$$D_0 \cdot B(x_1, x_2) = h(D_1 \cdot x_1, D_1 \cdot x_2). \tag{3.6}$$

Of course, in obtaining (3.4) and (3.5) there are more cases to study (by comparison with Subsection 2C) because there are more possibilities of linear dependence which has to be studied case by case. Exept for tediousness, the proof is not very difficult.

**Remark 12:** One can extend the result above for a system of $n$ identical subsystems ($n > 2$) if one takes $dim(\mathcal{H}_1)$ sufficiently large.

6. From Definition 2 ($c_1$) one gets immediately:

$$Im(B) = \mathcal{H}_0. \tag{3.7}$$

7. Now we come to the most interesting part, namely we use Definition 2 (e) which expresses the identity of the subsystems. This will impose some additional restrictions on the map $B$.

Indeed, Fron Definition 2 (e) it follows the existence of a map $\varepsilon : \mathcal{H}_1 \times \mathcal{H}_1 \to D_0$ such that:

$$B(x_1, x_2) = \varepsilon(x_1, x_2) B(x_2, x_1). \tag{3.8}$$

It is an easy matter now to use the biadditivity of $B$ and prove that in fact $\varepsilon$ is a constant element of $D_0$:

$$B(x_1, x_2) = \varepsilon B(x_2, x_1). \tag{3.9}$$

It is clear that $\varepsilon$ is constrained by:

$$\varepsilon^2 = 1 \quad \Leftrightarrow (\varepsilon + 1)(\varepsilon - 1) = 0.$$

We have exactly two possibilities: $\varepsilon = 1$ and $\varepsilon = -1$. We call these possibilities *statistics*. When we have:

$$B(x_2, x_1) = B(x_1, x_2) \tag{3.10}$$

we say that we have *Bose-Einstein statistics* and when we have:

$$B(x_2, x_1) = -B(x_1, x_2) \tag{3.11}$$

we say that we have *Fermi-Dirac statistics.*

**Remark 13:** The result above can be extended to the case of $n$ identical subsystems (for $n > 2$). Indeed, if we have already the additivity of the map $B : \mathcal{H}_1^{\times n} \to \mathcal{H}_0$ in all arguments, then instead of (3.9) we will find:

$$B(x_{\sigma(1)}, ..., x_{\sigma(n)}) = \varepsilon(\sigma) B(x_1, ..., x_n) \tag{3.12}$$

for any permutation $\sigma \in \mathcal{P}_n$ of the numbers $1, ..., n$. From (3.12) it follows that $\varepsilon : \mathcal{P}_n \to D_0$ is an one-dimensional representation of the permutation group $\mathcal{P}_n$. If one denotes the transposition of $i$ with $i+1$ by $\sigma_i$, then one knows that for any $i$ one has: $\sigma_i^2 = id$ and $\sigma_i\ \sigma_{i+1}\ \sigma_i = \sigma_{i+1}\ \sigma_i\ \sigma_{i+1}$. From here one immediately has for any $i$: $\varepsilon(\sigma_i)^2 = 1$ and $\varepsilon(\sigma_i)\ \varepsilon(\sigma_{i+1})\ \varepsilon(\sigma_i) = \varepsilon(\sigma_{i+1})\ \varepsilon(\sigma_i)\varepsilon(\sigma_{i+1})$. The first relation gives $\varepsilon(\sigma_i) = \pm 1$; then the second one gives $\varepsilon(\sigma_i) = \varepsilon(\sigma_{i+1})$. If $\varepsilon(\sigma_i) = 1$ $(\forall i)$, then one has $\varepsilon = id$, and if $\varepsilon(\sigma_i) = -1$ $(\forall i)$, then one has $\varepsilon(\sigma) = sign(\sigma)$.

8. Like at 2C.8 one finds out that there exists a map $\varphi : D_1 \times D_1 \to D_0$ such that:

$$B(\lambda_1 x_1, \lambda_2 x_2) = \varphi(\lambda_1, \lambda_2)\ B(x_1, x_2). \tag{3.13}$$

This map verifies:

$$\varphi(\lambda_1 + \mu_1, \lambda_2 + \mu_2) = \varphi(\lambda_1, \lambda_2) + \varphi(\lambda_1, \mu_2) + \varphi(\mu_1, \lambda_2) + \varphi(\mu_1, \mu_2) \tag{3.14}$$

$$\varphi(\lambda_1 \mu_1, \lambda_2 \mu_2) = \varphi(\lambda_1, \lambda_2)\ \varphi(\mu_1, \mu_2). \tag{3.15}$$

$$\varphi(\lambda_2, \lambda_1) = \varphi(\lambda_1, \lambda_2). \tag{3.16}$$

If $D_0$ is a commutative division ring $\varphi$ also satisfies:

$$\bar{\varphi}(\lambda_1, \lambda_2) = \varphi(\bar{\lambda}_1, \bar{\lambda}_2) \tag{3.17}$$

For $D_0 = R$ we have $\varphi(\lambda_1, \lambda_2) = \lambda_1 \lambda_2$ and for $D_0 = C$ we have another possibility, namely $\varphi(\lambda_1, \lambda_2) = \bar{\lambda}_1 \bar{\lambda}_2$.

9. Like in the precceding Section it is interesting to consider the particular case when $D_0 = D_1$ is a commutative division ring and $\varphi : D_0 \times D_0 \to D_0$ is:

$$\varphi(\lambda_1, \lambda_2) = \lambda_1 \lambda_2. \tag{3.18}$$

We have in analogy to Theorem 1:

**Theorem 2:** In the conditions above one can take $\mathcal{H}_0 = \vee^2 \mathcal{H}_1$ if $B$ is symmetric and $\mathcal{H}_0 = \wedge^2 \mathcal{H}_1$ if $B$ is antisymmetric.

**Remark 14:** We see that the structure of symmetric and antisymmetric tensor product emerges naturally in our framework. In particular, this is the case when $D_0 = R, C$.

**Remark 15:** One can also prove a formula of the type (2.38):

$$< B(x_1, x_2), B(y_1, y_2) >_{\mathcal{H}_0} =$$

$$\frac{1}{2}\delta[\varphi(< x_1, y_1 >_{\mathcal{H}_1}, < x_2, y_2 >_{\mathcal{H}_1}) \pm \varphi(< x_1, y_2 >_{\mathcal{H}_1}, < x_2, y_1 >_{\mathcal{H}_1})] \tag{3.19}$$

with $+(-)$ if $B$ is symmetric (antisymmetric).

For the case of $n$ identical subsystems, the coresponding formulae are:

$$< B(x_1, ..., x_n), B(y_1, ..., y_n) >_{\mathcal{H}_0} =$$

$$\frac{1}{n!}\delta \sum_{\sigma \in \mathcal{P}_n} \varphi(< x_1, y_{\sigma(1)} >_{\mathcal{H}_1}, ..., < x_n, y_{\sigma(n)} >_{\mathcal{H}_1}) \tag{3.20}$$

if $B$ is symmetric, and:

$$< B(x_1, ..., x_n), B(y_1, ..., y_n) >_{\mathcal{H}_0} =$$

$$\frac{1}{n!}\delta \sum_{\sigma \in \mathcal{P}_n} (-1)^{|\sigma|} \varphi(< x_1, y_{\sigma(1)} >_{\mathcal{H}_1}, ..., < x_n, y_{\sigma(n)} >_{\mathcal{H}_1}) \tag{3.21}$$

if $B$ is antisymmetric. In (3.19)-(3.20), $\delta \in D_0 \backslash \{0\}$.

**Remark 16:** Let us supplement Definition 2 with:

(f) if $a, b \in \mathcal{L}_1$ are in the center of $\mathcal{L}_1$, then $h(a, b) \in \mathcal{L}_0$ is in the center of $\mathcal{L}_0$

Then one can extend the results above to the case when $\mathcal{L}_0$ and $\mathcal{L}_1$ are propositional systems with superselection rules (see Remark 9).

B. It is interesting to see what gives our analysis in the case when the one-particle system is an anyon i.e. a projective unitary irreducible representation of the Poincaré group in $1 + 2$ dimensions. It is tempting to see if one can recover the multi-anyonic wave function.

For the identification of the one-particle system we rely on the results of [13] where the complete list of projective unitary irreducible representations of the Poincaré group in $1 + 2$ dimensions is given.

We provide here the necessary information. One should consider the unitary irreducible representations of the universal covering group of the Poincaré group in $1 + 2$ dimensions.

First, one identifies the universal covering group of the Lorentz group in $1 + 2$ dimensions with:

$$G \equiv R \times \{u \in C| \ |u| < 1\} \tag{3.22}$$

with the composition law:

$$(x, u) \cdot (y, v) \equiv \left( x + y + \frac{1}{2i} ln \frac{1 + e^{-2iy} u\bar{v}}{1 + e^{2iy} v\bar{u}}, \frac{u + e^{2iy} v}{e^{2iy} + u\bar{v}} \right). \tag{3.23}$$

(one writes $\frac{1+z}{1+\bar{z}} \in C_1 - \{-1\}$ uniquely as $e^{2it}$ with $t \in (-\pi/2, \pi/2)$).

Next, one provides the covering homomorphism $\delta : G \rightarrow \mathcal{L}^{\uparrow}_{+}$ by composing $\delta_1 : G \rightarrow SL(2,R)$ given by:

$$\delta_1(x,u) \equiv \frac{1}{2\sqrt{1-|u|^2}} \begin{pmatrix} e^{ix}(1+u)+e^{-ix}(1+\bar{u}) & ie^{ix}(1-u)-ie^{-ix}(1-\bar{u}) \\ -ie^{ix}(1+u)+ie^{-ix}(1+\bar{u}) & ie^{ix}(1-u)+ie^{-ix}(1-\bar{u}) \end{pmatrix} \tag{3.24}$$

with $\delta_2 : SL(2,R) \rightarrow \mathcal{L}^{\uparrow}_{+}$ constructed in analogy to the covering map $SL(2,C) \rightarrow \mathcal{L}^{\uparrow}_{+}$ (in $1+3$ dimensions).

Finally, one identifies the universal covering group of the Poincaré group in $1+2$ dimensions with the inhomogeneous group associated to $G$:

$$in(G) = G \times R^3 \tag{3.25}$$

with the composition law:

$$((x,u),a) \circ ((y,v),b) = ((x,u)\circ(y,v), a + \delta(x,u)b). \tag{3.26}$$

The analysis [13] provides a complete list of all unitary irreducible representations of $in(G)$. We give only the formulae for the systems of non-zero mass and of zero mass, leaving aside the tachions and the representations of null momentum. We denote by $X^{\pm}_m$ the upper (lower) hyperboloid of mass $m \in R_+ \cup \{0\}$ and by $\alpha^{\pm}_m$ the corresponding Lorentz invariant measures. Then, the particle of non-zero mass are identified to the representation $W^{m,\eta,s}$ $(m \in R_+, \eta = \pm, s \in R)$ acting in $L^2(X^{\eta}_m, d\alpha^{\eta}_m)$ as follows:

$$\left(W^{m,\eta,s}_{x,u,a} f\right)(p) = e^{i\{a,p\}} e^{isx} \left[\frac{p^0 + \eta m - \bar{u}e^{2ix} < p >}{p^0 + \eta m - ue^{-2ix} < \bar{p} >}\right]^{s/2} f(\delta(x,u)^{-1}p). \tag{3.27}$$

The particles of zero mass are identified to the representations $W^{\eta,s,t}$ $(\eta = \pm, s \in R(mod\ 2), t \in R)$ acting in $L^2(X^{\eta}_0, d\alpha^{\eta}_0)$ as follows:

$$\left(W^{\eta,s,t}_{x,u,a} f\right)(p) = e^{i\{a,p\}} e^{isx} \left[\frac{p^0 - \bar{u}e^{2ix} < p >}{p^0 - ue^{-2ix} < \bar{p} >}\right]^{s/2} \times$$

$$exp\left\{i\eta t \frac{Im(ue^{-2ix} < \bar{p} >)}{p^0[(1+|u|^2)p^0 - 2Re(ue^{-2ix} < \bar{p} >)}\right\} f(\delta(x,u)^{-1}p). \tag{3.28}$$

Here $< p > \equiv p^1 + ip^2$ and $\{a,p\} \equiv a^0p^0 - a^1p^1 - a^2p^2$

We will take in the scheme of A as the one-particle space $\mathcal{H}_1$ one of the two possibilities above. Next, we should decide about the statistics: Bose or Fermi? To discriminate between these two possibilities we procceed as follows. We apply the standard construction of the field operator for both statistics and check the causality. In the Bose case the commutator $[\phi(x), \phi(y)]$ should vanish for $x - y$ a space-like vector and in the Fermi case the same should happen for the anti-commutator $\{[\phi(x), \phi(y)\}$. Simple computations show (see e.g. [14], ch. 3) that in the first case the commutator is proportional to the antisymmetric Pauli-Jordan distribution and in the second case the anti-commutator is proportional to the symmetric Pauli-Jordan distribution. Only the first case verifies the causality condition required above. So, we are entitled to conclude that we should choose Bose statistics. This statement is a very primitive spin-statistics type theorem.

Now we restrict (3.27) or (3.28) to the universal covering group of the Euclidean group in $1+2$ dimensions, i.e. we put $u=0$ and $a=(0,\mathbf{a})$. We get in both cases a representation which can be realized in $\mathcal{H}_1 = L^2(R^2, d\mathbf{p})$ and acts as follows:

$$\left(U_{x,\mathbf{a}} f\right)(\mathbf{p}) = e^{-i\mathbf{a}\cdot\mathbf{p}} e^{isx} f\left(R(x)^{-1}\mathbf{p}\right). \tag{3.29}$$

Here:

$$R(x) = \begin{pmatrix} cos(x) & -sin(x) \\ sin(x) & cos(x) \end{pmatrix}. \tag{3.30}$$

For the composite system of $n$ such subsystems we use Bose statistics (as justified above) and obtain the Hilbert space:

$$\vee^n \mathcal{H}_1 \cong \{f : R^{\times n} \to C | \int |f|^2 d\mathbf{p}_1 ... d\mathbf{p}_n < \infty,$$

$$f(\mathbf{p}_{\sigma(1)}, ..., \mathbf{p}_{\sigma(n)}) = f(\mathbf{p}_1, ..., \mathbf{p}_n), \ \forall \sigma \in \mathcal{P}_n\}$$

and the representation

$$\left(U_{x,\mathbf{a}}^{\otimes n} f\right)(\mathbf{p}_1, ..., \mathbf{p}_n) = e^{-i\mathbf{a}\cdot(\mathbf{p}_1 + ... + \mathbf{p}_n)} e^{isnx} f(R(x)^{-1}\mathbf{p}_1, ..., R(x)^{-1}\mathbf{p}_n). \tag{3.31}$$

Next, we perform a Fourier transform and end up with a representation acting in the Hilbert space:

$$\mathcal{H}_0 = \{f : (R^2)^{\times n} \to C | \int |f|^2 d\mathbf{x}_1 ... d\mathbf{x}_n < \infty, f(\mathbf{x}_{\sigma(1)}, ..., \mathbf{x}_{\sigma(n)}) = f(\mathbf{x}_1, ..., \mathbf{x}_n), \ \forall \sigma \in \mathcal{P}_n\}$$

according to the formula:

$$\left(U_{x,\mathbf{a}}^{\otimes n} f\right)(\mathbf{x}_1, ..., \mathbf{x}_n) = e^{isnx} f(R(x)^{-1}(\mathbf{x}_1 - \mathbf{a}), ..., R(x)^{-1}(\mathbf{x}_n - \mathbf{a})). \tag{3.32}$$

**Remark 17:** We note that this formula indicates that the total spin of the system is $ns$.

**Remark 18:** Let us define the configuration space:

$$Q_n \equiv ((R^2)^{\times n} \backslash C)/\mathcal{P}_n. \tag{3.33}$$

Here $C$ is the so-called *collision set*:

$$C \equiv \{(\mathbf{x}_1, ..., \mathbf{x}_n) \in (R^2)^{\times n} | \mathbf{x}_i = \mathbf{x}_j \ for some \ \ i \neq j\} \tag{3.34}$$

and one factorizes to the natural action of $\mathcal{P}_n$.

Then, it is clear from (3.32) that the composite system we have described is localizable, in the sense of Newton-Wigner-Wightman [2] on the configuration space $Q_n$ i.e. one can work in $\mathcal{H}_0 = L^2(Q_n)$.

It is more convenient to identify $R^2$ with the complex plane $C$ as follows: $z_j = x_j^1 + ix_j^2$ $(j = 1, ..., n)$, $\alpha = a^1 + ia^2$. In these new variables the Hilbert space is:

$$\mathcal{H}_0 = \{f : C^{\times n} \to C | \int |f|^2 dz_1 d\bar{z}_1 ... dz_n d\bar{z}_n < \infty, f(z_{\sigma(1)}, ..., z_{\sigma(n)}) = f(z_1, ..., z_n)\}. \tag{3.35}$$

The formula for the representation (3.32) takes the form:

$$\left(U^{\otimes n}_{x,\mathbf{a}} f\right)(z_1, ..., z_n) = e^{isnx} f(e^{-ix}(z_1 - \alpha), ..., e^{-ix}(z_n - \alpha)). \tag{3.36}$$

Now we proceed as follows [15]. To every element $f \in \mathcal{H}_0$ we associate the multiform function $F$ defined on the universal covering space of $Q_n$:

$$F(z_1, ..., z_n) = \prod_{j<k} (z_j - z_k)^{2\theta} f(z_1, ..., z_n) \tag{3.37}$$

(see [15], eq. (1.15)).

In this new representation (3.36) becomes:

$$\left(U^{\otimes n}_{x,\mathbf{a}} F\right)(z_1, ..., z_n) = e^{i(sn+\theta n(n-1))x} F(e^{-ix}(z_1 - \alpha), ..., e^{-ix}(z_n - \alpha)). \tag{3.38}$$

Now we concentrate on the expression of the total Hamiltonian. It is clear that in the representation (3.35)+(3.36) the total Hamiltonian is the sum of the free one-particle Hamiltonians:

$$H = \sum_{j=1}^{n} H_0(\nabla_j). \tag{3.39}$$

Let us try however like in [15], eq.(1.17) to consider that the total Hamiltonian is the sum of free one-particle Hamiltonians in the new representation (3.37). In this case, reverting to the old representation, one obtains a topological interaction, i.e. the total Hamiltonian is:

$$H = \sum_{j=1}^{n} H_0(\nabla_j - iA(z_j)) \tag{3.40}$$

where:

$$A_\mu(z_j) = -\theta \sum_{k\neq j} \epsilon_{\mu\nu} \frac{x_j^\nu - x_k^\nu}{|z_j - z_k|^2}. \tag{3.41}$$

We have obtained the so-called system of $n$ "free" anyons, as presented in [16]-[21]. So, we can conclude, like in [15] that a system of "free" anyons is equivalent to a system of bosons carrying a charge $e$ and a magnetic vorticity $\Phi$ with: $e\Phi = -\theta$.

**Remark 19:** One notes from (3.37) that by exchanging two variables , say: $z_j \leftrightarrow z_k$, one gets a sign $(-1)^{2\theta}$. So, it is tempting to call $\theta$ the "statistics" of the system and say that one has interpolating statistics between Bose and Fermi statistics. However, we think that it is more natural to stick to the interpretation above, namely to conclude that a system of anyons is nothing else but a system of bosons with a special topological interaction.

**Remark 20:** One may wonder if there exists a connection between $\theta$ and $s$. An analysis based on the algebraic framework of quantum theory gives: $s = \theta \ (mod\ 1)$ [6], [9]. Probably the same conclusion can be derived in a more simpler way, using, as above, the argument of causality with the modified Hamiltonian (3.40).

C. Finally, we try to explain why one does not obtain in our analysis exotic statistics as parastatistics and infinite statistics [6] which appear naturally in the algebraic formulation of quantum theory. For this, one has to make a comparison between the algebraic and the lattice-theoretical formulation of quantum theory.

The first question to settle is: what is the algebra of observables for a pure quantum system $\mathcal{L} = \mathcal{L}(\mathcal{H})$? According to [1] (see also Subsection 2A), every orthogonal projector in $\mathcal{H}$ is an (elementary) observable of the sytem. This implies that the algebra of observables of $\mathcal{L}(\mathcal{H})$ is $\mathcal{B}(\mathcal{H})$ i.e. the set of all bounded self-adjoint operators in $\mathcal{H}$. That's it, the algebra of observables is the "largest" possible one. But comparing to the analysis of [6] we note that in this case we are left only with Bose and Fermi statistics. In fact, exotic statistics can appear only if the algebra of observables is "smaller". Even in this case one can prove that any system with parastatistics can be converted into a system with normal statistics (Bose or Fermi) by enlarging the algebra of observables: namely, one can prove the existence of a gauge group of symmetry $G$ and then a system with parastatistics can be transformed into a system with normal statistics but living in a non-trivial representation of the gauge group $G$ [22].

So, there seems to be a physical agreement between our result and the corresponding algebraic analysis. We might note however that our result is quite independent of the space-time localization properties of the physicsl system. On the contrary, such properties play a major rôle in the algebraic framework.

## 4 Conclusions

We have succeeded to prove that the lattice-theoretical (or propositional) point of view on quantum physics is compatible, under very general assumptions, only with two kind of statistics: Bose-Einstein and Fermi-Dirac. We have also succeeded to show that there exists some "philosophical" agreement between our result and the similar analysis appearing in the litterature, in the algebraic framework.

As we have explained in Subsection 3D, the game seeems to be more simpler than in the algebraic formalism because the algebra of observables is "too large". In fact, for the same reason, all automorphisms of the algebra of observables are unitary (or antiunitary) implementable in the lattice-theoretical framework, so we cannot describe the phenomenon of spontaneous breakdown of symmetry. (This phenomenon appears when the algebra of observables admits automorphisms which are not unitary implementable).

So, an interesting direction is suggested. Namely, one should try to generalize somehow the lattice-thoretical framework such that the corresponding algebra of observables is strictly smaller than $\mathcal{B}(\mathcal{H})$. If this can be accomplished, then it is plausible that more phenomenae as the spontaneous breakdown of symmetry could be accomodated in this framework.

We finally note that one can approach the problem of describing systems composed of subsystems using the more refined concepts of entities and separated entities of Aerts [23]. However, this approach is not compatible with the Hilbert space formalism of quantum mechanics, so there is no hope of obtaining tensor-like structures as in our approach.

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