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HAMILTONIAN AND BRST FORMULATIONS OF THE SCHWINGER MODEL

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Abstract
The Hamiltonian and BRST formulations of the Schwinger model are investigated.

1. Introduction

Electrodynamics in one–space one–time dimension with massless fermions, known as the Schwinger model [1] has been studied by several authors [1–7] in various contexts. In 1962, Schwinger was able to obtain the exact Green's functions for the theory [1] and since then many authors have obtained solutions of the theory in various gauges [2]. The path– integral solutions and operator solutions have also been studied [4,2]. The various solutions have been helpful in elucidating many properties (often nonperturbative) of the Schwinger model [1–7]. Further, the model is exactly solvable [3]. The solvability of the model is due to a remarkable property of one–dimensional fermion systems, namely, that they can be completely described in terms of canonical one–dimensional boson fields [3]. In fact, a study of the two–dimensional field theories in general, and of the Schwinger model, in particular, has led to the conclusion that any fermion (plus bosons if desired) field theory has its boson–equivalent field theory [3]. As a consequence of this, many surprising features of field theories (at least in two dimensions) have been revealed [1–7].

The Schwinger model is well known to describe a bonafide (pure) gauge–invariant theory possessing two first–class constraints. In the present work we study the Hamiltonian [8,9] and Becchi–Rouet–Stora–Tyutin (BRST) [10–13] formulations of the bosonized Schwinger model. The Hamiltonian formulation is considered in Sec. 2, and the BRST formulation in Sec. 3.

2. The Hamiltonian Formulation

The Schwinger model in one–space one–time dimension is described by the Lagrangian density [1]:

$$\mathcal{L} = \bar{\Psi}\gamma^{\mu}(i\partial_{\mu} + gA_{\mu})\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.1}$$

which is equivalent to its bosonized form [3]:

$$\mathcal{L} = \frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - g\epsilon^{\mu\nu}\partial_{\mu}\phi A_{\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.2a}$$

$$g^{\mu\nu} := \mathrm{diag}\,(+1,-1); \quad \epsilon^{\mu\nu} = -\epsilon^{\nu\mu}; \quad \epsilon^{01} = +1; \quad \mu,\nu = 0,1. \tag{2.2b}$$

In (2.1) ((2.2)) the first term corresponds to a massless fermion (boson). The second term represents the coupling of this fermion (boson) to the electromagnetic field A_{μ}. The last term is the kinetic energy term of the electromagnetic field. In component form (2.2) could be written:

$$\mathcal{L} = \frac{1}{2}(\dot{\phi}^2 - \phi'^2) + g(\phi' A_0 - \dot{\phi}A_1) + \frac{1}{2}(\dot{A}_1 - A_0')^2 \tag{2.3}$$

where overdots and primes denote time and space derivatives respectively. The Euler–Lagrange equations obtained from $\mathcal{L}$ (2.3) are:

$$(\ddot{\phi} - \phi'') = g(\dot{A}_1 - A_0') \tag{2.4a}$$

$$-gJ_1 := (\ddot{A}_1 - \dot{A}_0') = -g\dot{\phi} \tag{2.4b}$$

$$-gJ_0 := (\dot{A}_1' - A_0'') = -g\phi' \tag{2.4c}$$

It is easy to see from (2.4) that the vector current (J_μ) is conserved, i.e.,

$$\partial_\nu J^\nu = -\frac{1}{g}\,\partial_\nu(\partial_\mu F^{\mu\nu}) = (\dot{J}_0 - J_1') = (\dot{\phi}' - \dot{\phi}') = 0 \tag{2.5}$$

implying that the theory possesses (at the classical level) a vector–gauge symmetry.

We now study the Hamiltonian formulation [8,9] of the bosonized model described by $\mathcal{L}$ (2.2). The canonical momenta obtained from $\mathcal{L}$ are:

$$\pi_0 := \frac{\partial\mathcal{L}}{\partial\dot{A}_0} = 0; \quad E := \frac{\partial\mathcal{L}}{\partial\dot{A}_1} = (\dot{A}_1 - A_0'); \; \pi := \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = (\dot{\phi} - gA_1) \tag{2.6}$$

Here π_0, $E\,(=\pi^1)$ and π, are the momenta canonically conjugate respectively to A_0, A_1 and ϕ. The first equation in (2.6) implies that the theory possesses a primary constraint

$$\Omega_1 := \pi_0 \approx 0 \tag{2.7}$$

The canonical Hamiltonian density corresponding to $\mathcal{L}$ is

$$\begin{aligned}\mathcal{H}_c &= \pi_0\dot{A}_0 + E\dot{A}_1 + \pi\dot{\phi} - \mathcal{L}\\ &= \tfrac{1}{2}(E^2 + \pi^2 + \phi'^2) + EA_0' + g\pi A_1 + \tfrac{1}{2}g^2A_1^2 - g\phi' A_0\end{aligned} \tag{2.8}$$

After including the primary constraint Ω_1, in the canonical Hamiltonian density $\mathcal{H}_c$ with the help of Lagrange multiplier u, one can write the total Hamiltonian density $\mathcal{H}_T$ as [8,9]:

$$\mathcal{H}_T = \tfrac{1}{2}(E^2+\pi^2+\phi'^2) + EA_0' + g\pi A_1 + \tfrac{1}{2}g^2A_1^2 - g\phi' A_0 + \pi_0 u \tag{2.9}$$

The Hamilton's equations obtained from the total Hamiltonian $H_T^I = \int \mathcal{H}_T^I\, dx$ are:

$$\dot{\phi} = \frac{\partial H_T}{\partial \pi} = (\pi + gA_1); \quad -\dot{\pi} = \frac{\partial H_T}{\partial \phi} = (gA_0' - \phi'') \tag{2.10a}$$

$$\dot{A}_0 = \frac{\partial H_T}{\partial \pi_0} = u; \quad -\dot{\pi}_0 = \frac{\partial H_T}{\partial A_0} = -(E' + g\phi') \tag{2.10b}$$

$$\dot{A}_1 = \frac{\partial H_T}{\partial E} = (E + A_0'); \quad -\dot{E} = \frac{\partial H_T}{\partial A_1} = g(\pi + gA_1) \tag{2.10c}$$

$$\dot{u} = \frac{\partial H_T}{\partial p_u} = 0; \quad -\dot{p}_u = \frac{\partial H_T}{\partial u} = \pi_0 \tag{2.10d}$$

These are the equations of motion of the theory that preserve the constraints of the theory in the course of time. For the Poisson bracket $\{,\}_p$ of two functions A and B, we choose the convention:

$$\{A(x),B(y)\}_p := \int dz \sum_\alpha \left[\frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)} \right] \tag{2.11}$$

Demanding that the primary constraint Ω_1 be preserved in the course of time, we obtain the secondary constraint

$$\Omega_2 := \{\Omega_1, \mathcal{H}_T\}_p = (E' + g\phi') \approx 0 \tag{2.12}$$

The preservation of Ω_2 for all time does not give rise to any further constraints. The theory is thus seen to possess only two constraints Ω_1 and Ω_2. The matrix of the Poisson brackets of the constraints Ω_i is a 2×2 null matrix and therefore singular implying that the set of constraints Ω_i is first–class and that the theory described by $\mathcal{L}$ is a bonafide (pure) gauge–invariant theory. In fact, the Lagrangian density $\mathcal{L}$ is seen to be invariant under the time–dependent gauge transformations:

$$\delta A_0 = +\dot{\beta}(x,t), \quad \delta A_1 = +\beta'(x,t), \quad \delta\phi = 0, \quad \delta u = 0 \tag{2.13a}$$

$$\delta\pi_0 = 0, \quad \delta E = 0, \quad \delta\pi = -\beta'(x,t), \quad \delta p_u = 0 \tag{2.13b}$$

up to a total divergence:

$$\delta\mathcal{L} = g\epsilon^{\mu\nu}\partial_{\mu}(\beta\partial_{\nu}\phi) \tag{2.14}$$

where $\beta(x,t)$ is an arbitrary function of the coordinates. The action $S = \int\mathcal{L}\, dx\, dt$ is therefore gauge–invariant. The reduced Hamiltonian density of the theory ($\mathcal{H}_R$), obtained from $\mathcal{H}_T$ after the implementation of constraints Ω_i, is finally given by [8,9]

$$\mathcal{H}_R = \frac{1}{2}(E^2 + \pi^2 + \phi'^2 + g^2 A_1^2 + 2g\pi A_1) = \frac{1}{4}(\mathcal{J}_+^2 + \mathcal{J}_-^2 + 2E^2) \tag{2.15}$$

where

$$\mathcal{J}_{\pm} = \mathcal{J}_0 \pm \mathcal{J}_1 = \phi' \pm (\pi + gA_1) \tag{2.16}$$

are the U(1) kac–Moody currents. $\mathcal{H}_R$ is thus seen to be positive semi–definite.

In order to quantize the theory using Dirac's procedure [9], we ought to convert the set of first–class constraints of the theory Ω_i into a set of second–class constraints, by imposing, arbitrarily, some additional constraints on the system called gauge–fixing conditions or the gauge–constraints. For the theory under condsideration, we could choose, for example, the gauge–fixing conditions: $A_0 = 0$ and $A_1' = 0$. Corresponding to this choice of the gauge– fixing condition, we have the following set of constraints under which the quantization of the theory could be studied [8,9]:

$$\psi_1 = \Omega_1 = \pi_0 \approx 0 \tag{2.17a}$$
$$\psi_2 = \Omega_2 = (E' + g\phi') \approx 0 \tag{2.17b}$$
$$\psi_3 = A_0 \approx 0 \tag{2.17c}$$
$$\psi_4 = A_1' \approx 0 \tag{2.17d}$$

We now calculate the Poisson brackets among the constraints ψ_i and obtain the matrix:

$$A_{\alpha\beta}(z,z') := \{\psi_\alpha(z), \psi_\beta(z')\}_P = \begin{bmatrix} 0 & 0 & -\delta(z-z') & 0 \\ 0 & 0 & 0 & \delta''(z-z') \\ \delta(z-z') & 0 & 0 & 0 \\ 0 & -\delta''(z-z') & 0 & 0 \end{bmatrix} \tag{2.18}$$

with the inverse

$$A^{-1}_{\alpha\beta}(z,z') = \begin{bmatrix} 0 & 0 & \delta(z-z') & 0 \\ 0 & 0 & 0 & -\frac{1}{2}|z-z'| \\ -\delta(z-z') & 0 & 0 & 0 \\ 0 & \frac{1}{2}|z-z'| & 0 & 0 \end{bmatrix} \tag{2.19}$$

and

$$\int dz\; A(x,z)\; A^{-1}(z,y) = 1_{4\times 4}\; \delta(x-y) \tag{2.20}$$

The Dirac bracket $\{,\}_D$ of the two functions A and B is defined as [8]:

$$\{A,B\}_D := \{A,B\}_p - \iint dz\, dz' \sum_{\alpha,\beta} \left[\{A,\Gamma_\alpha(z)\}_p \left[\Delta^{-1}_{\alpha\beta}(z,z')\right] \{\Gamma_\beta(z'),B\}_p\right] \tag{2.21}$$

where Γ_i are the constraints of the theory and $\Delta_{\alpha\beta}(z,z')$ $[:= \{\Gamma_\alpha(z),\ \Gamma_\beta(z')\}_p]$ is the matrix of the Poisson brackets of the constraints Γ_i. The transition to quantum mechanics is made by the replacement of the Dirac brackets by the operator commutation relations according to

$$\{A,B\}_D \longrightarrow (-i)[A,B]; \quad i = \sqrt{-1}. \tag{2.22}$$

Finally, the nonvanishing equal time commutators of the theory in the gauge $A_0 = 0$ and $A_1' = 0$, are obtained as:

$$2[\phi(x),\pi(y)] = \frac{2}{g}[E(x),\pi(y)] = [A_1(x),E(y)] = 2i\delta(x-y) \tag{2.23}$$

For the later use (in the next section), for considering the BRST formulation [10,11] of the theory described by $\mathcal{L}$, we convert the total Hamiltonian density $\mathcal{H}_T$ into the first-order Lagrangian density [11,12]:

$$\begin{aligned} \mathcal{L}_{I0} &= \pi\dot{\phi} + E\dot{A}_1 + \pi_0\dot{A}_0 + p_u\dot{u} - \mathcal{H}_T \\ &= \pi\dot{\phi} + E\dot{A}_1 + p_u\dot{u} - \frac{1}{2}(\pi^2 + \phi'^2 + E^2 + g^2A_1^2) - EA_0' - g\pi A_1 + g\phi' A_0 \end{aligned} \tag{2.24}$$

In (2.24) the term $\pi_0(\dot{A}_0-u)$ drops out in view of the Hamilton's equation (2.10b).

3. The BRST Formulation

3A. The Schwinger Model and BRST Invariance

Following Ref. [11], we rewrite the gauge–invariant theory of Schwinger model [1] as a quantum system which possesses the generalized gauge invariance called BRST symmetry. For this, we first enlarge the Hilbert space of our gauge–invariant model [1] and replace the notion of gauge transformation which shifts operators by c–number functions by a BRST transformation which mixes operators having different statistics.

We then introduce new anti–commuting variables c and $\bar{c}$ called Faddeev–Popov ghost and anti–ghost fields respectively (Grassmann numbers on the classical level, operators in the quantized theory) and a commuting variable b called the Nakanishi–Lautrup field such that [11,12]:

$$\hat{\delta}\phi = 0, \quad \hat{\delta}A_0 = \dot{c}, \quad \hat{\delta}A_1 = c', \quad \hat{\delta}\pi = -c', \quad \hat{\delta}E = 0, \quad \hat{\delta}\pi_0 = 0; \tag{3.1a}$$

$$\hat{\delta}c = 0, \quad \hat{\delta}\bar{c} = b, \quad \hat{\delta}b = 0, \quad \hat{\delta}u = 0, \quad \hat{\delta}p_u = 0 \tag{3.1b}$$

with the property $\hat{\delta}^2 = 0$. We now define a BRST–invariant function of the dynamical variables to be a function $f(\pi,\pi_0,E,p_b,\pi_c,\pi_{\bar{c}},\phi,A_0,A_1,b,c,\bar{c})$ such that $\hat{\delta}f = 0$.

3B. Gauge–Fixing in the BRST Formalism

Performing gauge–fixing in the BRST formalism implies adding to the first–order Lagrangian density (2.24) a trivial BRST–invariant function [11,12]. We could thus write the quantum Lagrangian density (taking e.g., a trivial BRST–invariant function as follows [11,12]:

$$\begin{aligned}\mathcal{L}_{BRST} &= \mathcal{L}_{IO} + \hat{\delta}[\,\bar{c}(\dot{A}_0 + \tfrac{1}{2}b + A_1 + \pi)] \\ &= \pi\dot{\phi} + E\dot{A}_1 + p_u\dot{u} - \tfrac{1}{2}(\pi^2 + \phi'^2 + E^2 + g^2A_1^2) - EA_0' - g\pi A_1 + g\phi' A_0 \\ &\quad + \hat{\delta}[\,\bar{c}(\dot{A}_0 + \tfrac{1}{2}b + A_1 + \pi)]\end{aligned} \tag{3.2}$$

The last term in the above equation (Eq. (3.2)) is the extra BRST–invariant gauge–fixing term. Using the definition of $\hat{\delta}$ we can rewrite $\mathcal{L}_{\mathrm{BRST}}$ (with one integration by parts):

$$\mathcal{L}_{\mathrm{BRST}} = \pi\dot{\phi} + E\dot{A}_1 + p_u\dot{u} - \tfrac{1}{2}(\pi^2+\phi'^2+E^2+g^2A_1^2) - EA_0' - g\pi A_1 + g\phi' A_0 + \tfrac{1}{2}b^2 + b(\dot{A}_0 + A_1 + \pi) + \dot{\bar{c}}\dot{c} \tag{3.3}$$

Proceeding classically, the Euler–Lagrange equation for b reads:

$$-b = (\dot{A}_0 + A_1 + \pi) \tag{3.4}$$

Also, the requirement $\hat{\delta}b = 0$ (cf. Eq. (3.1b)) implies:

$$-\hat{\delta}b = (\hat{\delta}\dot{A}_0 + \hat{\delta}A_1 + \hat{\delta}\pi) = 0 \tag{3.5}$$

which in turn implies

$$\ddot{c} = 0 \tag{3.6}$$

The above equation is also an Euler–Lagrange equation obtained by the variation of $\mathcal{L}_{\mathrm{BRST}}$ with respect to $\bar{c}$.

In introducing momenta we have to be careful in defining those for fermionic variables. Thus we define the bosonic momenta in the usual way so that [11,12]

$$\pi_0 = \frac{\partial}{\partial \dot{A}_0}\mathcal{L}_{\mathrm{BRST}} = +b \tag{3.7}$$

but for the fermionic momenta with directional derivatives we set [11,12]

$$\pi_c := \mathcal{L}_{\mathrm{BRST}}\frac{\overleftarrow{\partial}}{\partial \dot{c}} = \dot{\bar{c}}\,; \quad \pi_{\bar{c}} := \frac{\overrightarrow{\partial}}{\partial \dot{\bar{c}}}\mathcal{L}_{\mathrm{BRST}} = \dot{c} \tag{3.8}$$

implying that the variable canonically conjugate to c is $\dot{\bar{c}}$ and the variable conjugate to $\bar{c}$

is $\dot{c}$. In forming the Hamiltonian density $\mathcal{H}_{BRST}$ from the Lagrangian density in the usual way we remember that the former has to be Hermitian. Then [11,12]

$$\mathcal{H}_{BRST} = \pi\dot{\phi} + E\dot{A}_1 + \pi_0\dot{A}_0 + p_u\dot{u} + \pi_c\dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - \mathcal{L}_{BRST}$$

$$= \frac{1}{2}(\pi^2+\phi'^2+E^2+g^2A_1^2) + EA_0' + g\pi A_1 - g\phi' A_0 - \frac{1}{2}\pi_0^2 - \pi_0(A_1+\pi) + \pi_c\pi_{\bar{c}} \quad (3.9)$$

We can check the consistency of (3.8) with (3.9) by looking at Hamilton's equations for the fermionic variables, i.e. (cf. Ref. [12]).

$$\dot{c} = \frac{\overrightarrow{\partial}}{\partial \pi_c}\mathcal{H}_{BRST}, \qquad \dot{\bar{c}} = \mathcal{H}_{BRST}\frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} \tag{3.10}$$

Thus

$$\dot{c} = \frac{\overrightarrow{\partial}}{\partial \pi_c}\mathcal{H}_{BRST} = \pi_{\bar{c}}\,; \qquad \dot{\bar{c}} = \mathcal{H}_{BRST}\frac{\overleftarrow{\partial}}{\partial \pi_{\bar{c}}} = \pi_c \tag{3.11}$$

in agreement with (3.8).

For the operators $c, \bar{c}, \dot{c}$ and $\dot{\bar{c}}$, one needs to specify the anti–commutation relations of $\dot{c}$ with $\bar{c}$ or of $\dot{\bar{c}}$ with c, but not of c with $\bar{c}$. c and $\bar{c}$ are, in general, independent canonical variables and one assumes that [11,12]:

$$\{\pi_c, \pi_{\bar{c}}\} = \{\bar{c}, c\} = 0\,; \qquad \frac{d}{dt}\{\bar{c}, c\} = 0 \tag{3.12a}$$

$$\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} \tag{3.12b}$$

where {,} means anticommutator.

We thus see that the anti–commutators in (3.12b) are non–trivial and need to be fixed. In order to fix these we demand that c satisfy the Heisenberg equation [11,12]:

$$[c, \mathscr{H}_{BRST}] = i\dot{c} \tag{3.13}$$

and using the property $c^2 = \bar{c}^2 = 0$, one obtains

$$[c, \mathscr{H}_{BRST}] = \{\dot{\bar{c}}, c\}\ \dot{c}. \tag{3.14}$$

Eqs. (3.12) – (3.14) then imply:

$$\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} = i\ . \tag{3.15}$$

Here the minus sign in the above equation is non–trivial and implies the existence of states with negative norm in the space of state vectors of the theory [11,12].

3C. The BRST Charge Operator

The BRST charge operator Q is the generator of the BRST transformation (3.1). It is nilpotent and therefore satisfies $Q^2 = 0$. It mixes operators which satisfy Bose and Fermi statistics. According to its conventional definition, its commutators with Bose operators and its anti–commutators with Fermi operators in the present case satisfy:

$$[\phi, Q] = -c\ ; \quad [A_0, Q] = \dot{c}\ ; \quad [E, Q] = c \tag{3.16a}$$

$$\{\bar{c}, Q\} = -\pi_0\ ; \quad \{\dot{\bar{c}}, Q\} = -(E' + g\phi') \tag{3.16b}$$

All other commutators and anti–commutators involving Q vanish. In view of (3.16), the BRST charge operator of the present gauge–invariant theory can be written as

$$Q = \int dx\,[ic(E' + g\phi') - i\dot{c}\pi_0] \tag{3.17}$$

This equation implies that the set of states satisfying the condition $\pi_0|\psi\rangle = 0$ and $(E'+g\phi')|\psi\rangle = 0$ belongs to the dynamically stable subspace of states $|\psi\rangle$ satisfying $Q|\psi\rangle = 0$, i.e., it belongs to the set of BRST–invariant states.

In order to understand the condition needed for recovering the physical states of the theory we rewrite the operators c and $\bar{c}$ in terms of fermionic annihilation and creation operators. For this purpose we consider Eq. (3.6) (namely, $\ddot{c} = 0$). The solution of this equation gives the Heisenberg operator c(t) (and correspondingly $\bar{c}(t)$) as:

$$c(t) = Gt + F\ ; \quad \bar{c}(t) = G^{\dagger}t + F^{\dagger}\ ; \tag{3.18}$$

which at time t = 0 imply

$$c \equiv c(0) = F, \quad \bar{c} \equiv \bar{c}(0) = F^{\dagger} \tag{3.19a}$$

$$\dot{c} \equiv \dot{c}(0) = G, \quad \dot{\bar{c}} \equiv \dot{\bar{c}}(0) = G^{\dagger} \tag{3.19b}$$

By imposing the conditions

$$c^2 = \bar{c}^2 = \{\bar{c},c\} = \{\dot{\bar{c}},\dot{c}\} = 0; \quad \{\dot{\bar{c}},c\} = i = -\{\dot{c},\bar{c}\} \tag{3.20}$$

one then obtains

$$F^2 = F^{\dagger 2} = \{F^{\dagger},F\} = \{G^{\dagger},G\} = 0 \tag{3.21}$$

$$\{G^{\dagger},F\} = -\{G,F^{\dagger}\} = i \tag{3.22}$$

We now let $|0>$ denote the fermionic vacuum for which

$$G|0> = F|0> = 0; \tag{3.23}$$

Defining $|0>$ to have norm one, (3.22) implies

$$\left\langle 0|FG^{\dagger}|0\right\rangle = i; \quad \left\langle 0|GF^{\dagger}|0\right\rangle = -i \tag{3.24}$$

so that

$$G^{\dagger}|0> \neq 0; \qquad F^{\dagger}|0> \neq 0 \tag{3.25}$$

The theory is thus seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathscr{H}_{BRST}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

In terms of annihilation and creation operators the Hamiltonian density is

$$\mathscr{H}_{BRST} = \frac{1}{2}(\pi^2+\phi'^2+E^2+g^2A_1^2) + EA_0' + g\pi A_1 - g\phi' A_0 - \frac{1}{2}\pi_0^2 - \pi_0(A_1+\pi) + G^\dagger G \tag{3.26}$$

and the BRST charge operator Q is

$$Q = \int dx\{+i[F(E'+g\phi') - G\pi_0]\} \tag{3.27}$$

Now, because $Q|\psi\rangle = 0$, the set of states annihilated by Q contains not only the set of states for which $\pi_0|\psi\rangle = 0$ and $(E'+g\phi')|\psi\rangle = 0$ but also additional states for which $G|\psi\rangle = F|\psi\rangle = 0$ with $\pi_0|\psi\rangle \neq 0$ and $(E'+g\phi')|\psi\rangle \neq 0$. However, the Hamiltonian is also invariant under the anti–BRST transformation (in which the role of c and $-\bar{c}$ is interchanged) given by

$$\bar{\hat{\delta}}\phi = 0, \quad \bar{\hat{\delta}}A_0 = -\dot{\bar{c}}, \quad \bar{\hat{\delta}}A_1 = -\bar{c}', \quad \bar{\hat{\delta}}\pi = \bar{c}', \quad \bar{\hat{\delta}}E = 0, \quad \bar{\hat{\delta}}\pi_0 = 0; \tag{3.28a}$$

$$\bar{\hat{\delta}}\bar{c} = 0, \quad \bar{\hat{\delta}}c = -b, \quad \bar{\hat{\delta}}b = 0, \quad \bar{\hat{\delta}}u = 0, \quad \bar{\hat{\delta}}p_u = 0 \tag{3.28b}$$

with generator or anti–BRST charge

$$\begin{aligned}\bar{Q} &= \int dx[-i\bar{c}(E' + g\phi') + i\dot{\bar{c}}\pi_0] \\ &= \int dx\{-i[F^\dagger(E'+g\phi') - G^\dagger\pi_0]\}\end{aligned} \tag{3.29}$$

Like we had $[Q,H] = 0$ before, we now have $[\bar{Q},H] = 0$, with all other commutators and anti–commutators also vanishing. We can thus impose the dual condition that both Q and $\bar{Q}$ annihilate physical states implying that

$$Q|\psi\rangle = 0 \text{ and } \overline{Q}|\psi\rangle = 0 \tag{3.30}$$

The states for which $\pi_0|\psi\rangle = 0$ and $(E'+g\phi')|\psi\rangle = 0$ satisfy both of these conditions and, in fact, are the only states satisfying both conditions since, although with (3.21) and (3.22)

$$G^\dagger G = -GG^\dagger \tag{3.31}$$

there are no states of this operator with $G^\dagger|0\rangle = 0$ and $F^\dagger|0\rangle = 0$ (cf. (3.25)), and hence no free eigenstates of the fermionic part of $\mathscr{H}_{BRST}$ which are annihilated by each of $G, G^\dagger, F, F^\dagger$. Thus the only states satisfying (3.30) are those satisfying the constraints $\pi_0 = 0$ and $(E'+g\phi') = 0$.

Further, the states for which $\pi_0|\psi\rangle = 0$ and $(E'+g\phi')|\psi\rangle = 0$ satisfy both of these conditions (3.30) and, in fact, are the only states satisfying both of these conditions (3.30) because in view of (3.20), one cannot have simultaneously $c, \dot{c}$ and $\bar{c}, \dot{\bar{c}}$, applied to $|\psi\rangle$ to give zero. Thus the only states satisfying (3.30) are those that satisfy the constraints of the theory (2.7) and (2.12), and they belong to the set of BRST–invariant and anti–BRST– invariant states.

One can understand the above point in terms of fermionic annihilation and creation operators as follows. The condition $Q|\psi\rangle = 0$ implies that the set of states annihilated by Q contains not only the states for which $\pi_0|\psi\rangle = 0$ and $(E'+g\phi')|\psi\rangle = 0$, but also additional states for which $G|\psi\rangle = F|\psi\rangle = 0$, but $\pi_0|\psi\rangle \neq 0$ and $(E'+g\phi')|\psi\rangle \neq 0$. However, $\overline{Q}|\psi\rangle = 0$ guarantees that the set of states annihilated by $\overline{Q}$ contains only the states for which $\pi_0|\psi\rangle = 0$ and $(E'+g\phi')|\psi\rangle = 0$, simply because $G^\dagger|\psi\rangle \neq 0$ and $F^\dagger|\psi\rangle \neq 0$. Thus in this alternative way also we see that the states satisfying $Q|\psi\rangle = \overline{Q}|\psi\rangle = 0$ (i.e. satisfying (3.30)) are only those that satisfy the constraints of the theory (2.7) and (2.12) and also that these states belong to the set of BRST–invariant and anti–BRST–invariant states.

In the usual Hamiltonian formulation of a gauge–invariant theory (like the present one) under some gauge–fixing conditions, one necessarily destroys the gauge–invariance of the

($\mathcal{L}$) into a BRST–invariant system, the new (BRST) symmetry is maintained even under gauge–fixing and hence projecting any state onto the sector of BRST and anti–BRST–invariant states yields a theory which is isomorphic to $\mathcal{L}$. The unitarity and consistency of the BRST–invariant theory described by $\mathcal{L}_{\text{BRST}}$ is guaranteed by the conservation and nilpotency of the BRST charge Q.

Towards the end, we like to make an important observation that some interesting work exists in the literature [6] where a bosonic version of the Schwinger model which couples a Dirac fermion to a U(1) gauge field has been proposed and Dirac quantized in a reduced phase space formalism [6]. This model has further been shown to be equivalent to the fermionic massless Schwinger model considered in our present work. For the details of this work we refere to the work of Ref. [6].

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