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An Estimate Regarding One-Dimensional Point Interactions

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Abstract. It is shown that the difference between the eigenfunction of the Hamiltonian of an attractive point interaction and the one of the Hamiltonian with the short-range potential converging to the point interaction goes to zero in the L^2-norm like a power of the scaling parameter with exponent less than 1/2.

1 Introduction

As is well known, point interactions are used as an approximation for short-range potentials. In addition, Schroedinger and Dirac Hamiltonians with point interactions have the advantage of being exactly solvable models, that is to say one can explicitly compute eigenvalues, resonances and other important physical quantities. Most of the results regarding Hamiltonians with point interactions can be found in [1] and the related literature.

In this paper we are concerned with the estimate of the L^2-norm of the difference between the eigenvector of the Schroedinger Hamiltonian with a point interaction $H_0 - \|V\|_1\delta$ with $V \geq 0, \|e^{2a|x|}V\|_1 < \infty$ for some $a > 0$, i.e. the exactly solvable model, and that of the Hamiltonian $H_\epsilon = -\frac{d^2}{dx^2} - V_\epsilon, V_\epsilon(x) = \epsilon^{-1}V(\frac{x}{\epsilon})$, that is to say the Hamiltonian with a short-range potential which converges to H_0 in the norm resolvent sense and represents the more realistic model.

The estimate will show that such a norm goes to zero like a power of the scaling parameter

ϵ with exponent less than 1/2.

The need for such an estimate has arisen in the context of the problem of the stabilisation of atoms in superintense laser fields for which we refer the reader to [4] and the literature cited therein.

2 The estimate

As we have anticipated in the introduction, we are going to find an estimate for $\|\psi_\epsilon - \psi_0\|_2$, the two functions inside the norm being defined by

$$H_\epsilon \psi_\epsilon = [-\frac{d^2}{dx^2} - V_\epsilon]\psi_\epsilon = E_\epsilon \psi_\epsilon \tag{2.1}$$

and

$$H_0 \psi_0 = [-\frac{d^2}{dx^2} - \|V\|_1 \delta]\psi_0 = E_0 \psi_0 \tag{2.2}$$

As is well known from [1, sect.I.3.2], H_ϵ converges to H_0 in the norm resolvent sense, which implies by means of Theor. 3.3.1 therein that the discrete spectrum of H_ϵ consists of only one simple negative eigenvalue E_ϵ given by an analytic function of the scaling parameter ϵ in a neighbourhood of the origin.

Then we can state our result as follows:

Theorem 2.1 *Let V, ψ_0, ψ_ϵ be the functions defined above. Then,*

$$\frac{\|\psi_\epsilon - \psi_0\|_2}{\epsilon^{\frac{\gamma}{2}}} < \infty \tag{2.3}$$

for some $\gamma < 1$.

The main tool to derive the estimate will be the well-known Birman-Schwinger technique. Actually, as we are going to work in p-space, we shall use the integral operators

$$B(E_\epsilon) = (-\frac{d^2}{dx^2} - E_\epsilon)^{-\frac{1}{2}} V_\epsilon (-\frac{d^2}{dx^2} - E_\epsilon)^{-\frac{1}{2}} \tag{2.4}$$

and

$$B(E_0) = (-\frac{d^2}{dx^2} - E_0)^{-\frac{1}{2}} \delta (-\frac{d^2}{dx^2} - E_0)^{-\frac{1}{2}} \tag{2.5}$$

whose integral kernels are given by

$$B(p, p'; E_\epsilon) = (p^2 - E_\epsilon)^{-\frac{1}{2}} \frac{\hat{V}(\epsilon(p - p'))}{(2\pi)^{\frac{1}{2}}} (p'^2 - E_\epsilon)^{-\frac{1}{2}} \tag{2.6}$$

and

$$B(p,p';E_0) = (p^2 - E_0)^{-\frac{1}{2}} \frac{\hat{V}(0)}{(2\pi)^{\frac{1}{2}}} (p'^2 - E_0)^{-\frac{1}{2}} \tag{2.7}$$

It is immediate to realise that $B(E_0)$ is a rank one operator.

The integral operator $B(E_\epsilon)$ is shown to be trace class by using a well-known result (see [2]), since the operator is positive and its kernel is continuous due to the fact that $\hat{V}$ is a continuous function vanishing at infinity because of the L^1-decay of V. We can easily compute its trace by using the lemma invoked:

$$\|B(E_\epsilon)\|_1 = \frac{\hat{V}(0)}{(2\pi)^{\frac{1}{2}}} \int \frac{dp}{p^2 + |E_\epsilon|} = \frac{\pi \hat{V}(0)}{(2\pi|E_\epsilon|)^{\frac{1}{2}}} = \frac{\|V\|_1}{2|E_\epsilon|^{\frac{1}{2}}} \tag{2.8}$$

As a consequence of the Birman-Schwinger principle, $B(E_\epsilon)$ has an eigenvalue equal to one. Let $\hat{\chi}_\epsilon^{(1)}$ be the associated normalised eigenfunction. We recall that 1 is also the only eigenvalue of B_0 and therefore we are allowed to apply the Kato-Rellich theorem (Theor. XII.8 in [3]) at $\lambda_1(0) = 1 = \lambda_1(\epsilon)$ with $\epsilon \in [0,1]$ to obtain that the analytic eigenfunction can be written in terms of $\hat{\chi}_0$, the normalised eigenfunction of B_0 explicitly given by $(\frac{|E_0|^{\frac{1}{2}}}{\pi})^{\frac{1}{2}}(p^2 + |E_0|)^{-\frac{1}{2}}$, as follows:

$$P(\epsilon)\hat{\chi}_0 = |\hat{\chi}_\epsilon^{(1)} >< \hat{\chi}_\epsilon^{(1)}|\hat{\chi}_0 \tag{2.9}$$

Let us therefore estimate the norm of the difference of the two eigenvectors:

$$\begin{aligned} \|[1 - P(\epsilon)]\hat{\chi}_0\|_2^2 &= (\hat{\chi}_0, [1 - P(\epsilon)]\hat{\chi}_0) = \\ &= (\hat{\chi}_0, [B(0) - |\hat{\chi}_\epsilon^{(1)} >< \hat{\chi}_\epsilon^{(1)}| - \sum_{l>1} \lambda^{(l)}(\epsilon)|\hat{\chi}_\epsilon^{(l)} >< \hat{\chi}_\epsilon^{(l)}|]\hat{\chi}_0) + \\ &+ (\hat{\chi}_0, [\sum_{l>1} \lambda^{(l)}(\epsilon)|\hat{\chi}_\epsilon^{(l)} >< \hat{\chi}_\epsilon^{(l)}|]\hat{\chi}_0) = (\hat{\chi}_0, [B(0) - B(E_\epsilon)]\hat{\chi}_0) + \\ &+ (\hat{\chi}_0, [B(E_\epsilon)(1 - P(\epsilon))]\hat{\chi}_0) \end{aligned} \tag{2.10}$$

Let us first consider the second summand. It can easily be seen that it is bounded by

$$\|B(E_\epsilon)\| - 1 = \frac{\|V\|_1}{2|E_\epsilon|^{\frac{1}{2}}} - 1 = \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}} \tag{2.11}$$

exploiting the fact that $\frac{\|V\|_1}{2|E_0|^{\frac{1}{2}}} = 1$.

Going back to the first summand, its positivity can be shown as follows:

$$(\hat{\chi}_0, B(E_\epsilon)\hat{\chi}_0) = \alpha_0^2(\epsilon)(\hat{\chi}_\epsilon^{(1)}, B(E_\epsilon)\hat{\chi}_\epsilon^{(1)}) + (\tilde{\chi}_\epsilon, B(E_\epsilon)\tilde{\chi}_\epsilon) \tag{2.12}$$

having set $\hat{\chi}_0 = \alpha_0(\epsilon)\hat{\chi}_\epsilon^{(1)} + \tilde{\chi}_\epsilon$ with $\tilde{\chi}_\epsilon$ belonging to the orthogonal set of $\hat{\chi}_\epsilon^{(1)}$ and exploited the definition of $\hat{\chi}_\epsilon^{(1)}$ as well as the self-adjointness of $B(E_\epsilon)$. The quantity written above is bounded by $\alpha_0^2(\epsilon) + \|\tilde{\chi}_\epsilon\|_2^2 = \|\hat{\chi}_0\|_2^2$ provided $B(E_\epsilon)$ has no eigenvalues greater than 1 on the orthogonal set of $\hat{\chi}_\epsilon^{(1)}$. This is actually the case since, as we have already seen, the trace of $B(E_\epsilon)$ restricted to the orthogonal set of $\hat{\chi}_\epsilon^{(1)}$ is equal to $\frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}}$, which is certainly less than 1 due to the fact that $|E_0| < 4|E_\epsilon|$ for any ϵ sufficiently small.

Since $|E_\epsilon| < |E_0|$ we have:

$$(\hat{\chi}_0, [B(0) - B(E_\epsilon)]\hat{\chi}_0) \leq \frac{|E_0|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\pi} \times \int\int (p^2 + |E_0|)^{-\frac{1}{2}} \times$$

$$\times (p^2 + |E_\epsilon|)^{-\frac{1}{2}} [\hat{V}(0) - \hat{V}(\epsilon(p - p'))] (p'^2 + |E_\epsilon|)^{-\frac{1}{2}} (p'^2 + |E_0|)^{-\frac{1}{2}} dp dp' \tag{2.13}$$

which goes to 0 as $\epsilon \to 0$ by dominated convergence. Hence,

$$\|[1 - P(\epsilon)]\hat{\chi}_0\|_2^2 \leq F(\epsilon) + \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}}, \tag{2.14}$$

$F(\epsilon)$ denoting the right hand side of (13).

Since $P(\epsilon)\hat{\chi}_0$ is eigenvector of $B(E_\epsilon)$ with eigenvalue 1, it follows that

$$\hat{\psi}_\epsilon = (p^2 + |E_\epsilon|)^{-\frac{1}{2}} P(\epsilon)\hat{\chi}_0 \tag{2.15}$$

is the solution of the Schroedinger equation with the V_ϵ potential in p-space. Similarly,

$$\hat{\psi}_0 = (p^2 + |E_0|)^{-\frac{1}{2}} \hat{\chi}_0 \tag{2.16}$$

is the eigenfunction of H_0 in p-space.

Therefore,

$$\|\hat{\psi}_\epsilon - \hat{\psi}_0\|_2 \leq \|(p^2 + |E_\epsilon|)^{-\frac{1}{2}}[1 - P(\epsilon)]\hat{\chi}_0\|_2 + \|[(p^2 + |E_\epsilon|)^{-\frac{1}{2}} - (p^2 + |E_0|)^{-\frac{1}{2}}]\hat{\chi}_0\|_2 \leq$$

$$\leq \|(p^2 + |E_\epsilon|)^{-\frac{1}{2}}\|_\infty \|[1 - P(\epsilon)]\chi_0\|_2 + \|(p^2 + |E_\epsilon|)^{-\frac{1}{2}} - (p^2 + |E_0|)^{-\frac{1}{2}}\|_\infty \leq$$

$$\leq \frac{1}{|E_\epsilon|^{\frac{1}{2}}}[F(\epsilon) + \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_\epsilon|^{\frac{1}{2}}}]^{\frac{1}{2}} + \frac{|E_0|^{\frac{1}{2}} - |E_\epsilon|^{\frac{1}{2}}}{|E_0|^{\frac{1}{2}}|E_\epsilon|^{\frac{1}{2}}} \tag{2.17}$$

By using the assumption on the potential V we can now estimate the quantity $F(\epsilon)$.

As a consequence of Lemma 1.XIII.11 in [3], we have:

$$\hat{V}(0) - \hat{V}(\epsilon(p - p')) \leq 2^{1-\gamma}(2\pi)^{-\frac{1}{2}}\epsilon^\gamma |p - p'|^\gamma \|(1 + |x|)^\gamma V\|_1 \tag{2.18}$$

for some $\gamma < 1$.

Furthermore, by noting that $1 + |p - p'| \leq 1 + |p| + |p'| \leq (1 + |p|)(1 + |p'|) \leq 2(1 + p^2)^{\frac{1}{2}}(1 + p'^2)^{\frac{1}{2}}$ we get:

$$F(\epsilon) \leq 2\epsilon^\gamma (2\pi)^{-\frac{1}{2}} \|(1 + |x|)^\gamma V\|_1 (\int \frac{(1 + p^2)^{\frac{\gamma}{2}}}{p^2 + |E_\epsilon|} dp)^2, \tag{2.19}$$

which implies (3) after being inserted into (17).

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