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Bäcklund Transformations for the $U(N)$ σ Models and the susy $U(N)$ σ Models

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Abstract. We construct Bäcklund transformations for the purely bosonic $U(N)$ σ models and for the supersymmetric $U(N)$ σ models. We follow, in our construction, a similar approach to that used for the AKNS systems. Riccati forms for these models are obtained via the linear systems. Conclusions and a few comments are given.

1 Introduction

It is known that all two-dimensional sigma models are solvable by the inverse scattering method [1]. A distinguishing feature of theories solvable by the inverse scattering method is that they possess many classical integrability properties, such as, the construction of Bäcklund transformations (BT), Kac-Moody algebra, infinite sets of conservation laws, etc. The BT are an important tool for constructing new solutions of the equations of motion of the model from a known solution of that equation [2, 3].

Moreover, it is well-known that many nonlinear partial differential equations of wide application, each with soliton solutions, may be solved by linear computations, using the linear scattering problem [2–5]. The Ablowitz-Kaup-Newell-Segur (AKNS) equations are nonlinear differential equations for two field functions, usually denoted by $q(x, t)$ and $r(x, t)$, in a two-dimensional $x - t$ coordinate space. They can be derived by cross differentiation from the following linear problem:

$$\partial\psi/\partial x = U\psi, \qquad U = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}, \tag{1.1}$$

$$\partial\psi/\partial t = V\psi, \qquad V = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{1.2}$$

where $\psi^T = (\psi_1 \quad \psi_2)$ and η is a parameter independent of x and t. The integrability condition

$$\partial U/\partial t - \partial V/\partial x + [U, V] = 0, \tag{1.3}$$

restricts the functions A, B, C to be functionals of q and r. By suitable choices of these functionals, a large class of the AKNS equations are obtained [4,5].

Konno and Wadati [6] introduced the function

$$\Gamma = \psi_1/\psi_2, \tag{1.4}$$

so that the linear equations (1.1) and (1.2) produce the Riccati forms

$$\begin{aligned} \partial\Gamma/\partial x &= 2\eta\Gamma + q - r\Gamma^2, \\ \partial\Gamma/\partial t &= 2A\Gamma + B - C\Gamma^2. \end{aligned} \tag{1.5}$$

Then, they derived some BT for the nonlinear equations of the AKNS class via the transformations Γ' and q' which leave the Riccati equations (1.5) invariant [6].

In this paper, we shall deal with the construction of BT for both the $U(N)$ σ models and the supersymmetric (susy) $U(N)$ σ models. We shall follow a procedure similar to that of Konno and Wadati [6] applied to the AKNS systems, however, here we shall work in the two-dimensional Euclidean space x and y. In the following section, we start with considering the linear system corresponding to equations (1.1) and (1.2), and define the matrices U and V which lead to the $U(N)$ σ model equations through the integrability condition of the linear system. Then we construct Riccati forms for the models and define a transformation which leave these Riccati forms invariant. Using this transformation and the Riccati forms we obtain the BT for the $U(N)$ σ models. In section 3, we shall consider another application, that is, the susy $U(N)$ σ models and apply the same procedure of section 2 and obtain BT for these models. In the last section, we give conclusions and some feature and comments.

2 BT for the $U(N)$ σ Models

Using the conventions and notations of ref. [7], the field equations of the $U(N)$ σ models, in two-dimensional Euclidian space, can be written as:

$$\partial_+ A_- + \partial_- A_+ = 0, \tag{2.1}$$

where $\partial_\pm = \partial/\partial x_\pm$ with $x_\pm = x \pm iy$ and the purely bosonic gauge potentials $A_\pm$ are given by

$$A_\pm = \tfrac{1}{2} g^\dagger \partial_\pm g, \tag{2.2}$$

with the $N \times N$ unitary matrix g which satisfies $g^\dagger g = gg^\dagger = 1$, where $\dagger$ denotes the hermitian conjugation. Moreover, the potentials $A_\pm$ satisfy the integrability condition

$$\partial_+ A_- - \partial_- A_+ + 2[A_+, A_-] = 0. \tag{2.3}$$

We consider the following linear system:

$$\psi_+ = U\psi, \qquad U = \begin{pmatrix} U^1 & U^2 \\ U^3 & U^4 \end{pmatrix}, \tag{2.4}$$

$$\psi_- = V\psi, \qquad V = \begin{pmatrix} V^1 & V^2 \\ V^3 & V^4 \end{pmatrix}, \tag{2.5}$$

where $\psi_\pm = \partial\psi/\partial x_\pm$, the matrix $\psi = \begin{pmatrix} \psi^1 & \psi^2 \\ \psi^3 & \psi^4 \end{pmatrix}$ and U^i, V^i and ψ^i are $N \times N$ matrices. This linear system has the integrability condition

$$\partial_- U - \partial_+ V + [U, V] = 0. \tag{2.6}$$

Let us define a $N \times N$ matrix Γ by

$$\Gamma = (\psi^1\gamma_0 + \psi^2)(\psi^3\gamma_0 + \psi^4)^{-1}, \tag{2.7}$$

where γ_0 is the asymptotic form of Γ. Then by using the linear system (2.4) and (2.5), one can show that Γ satisfies the following matrix Riccati forms:

$$\partial_+\Gamma = -\Gamma U^3\Gamma - \Gamma U^4 + U^1\Gamma + U^2, \tag{2.8}$$

$$\partial_-\Gamma = -\Gamma V^3\Gamma - \Gamma V^4 + V^1\Gamma + V^2. \tag{2.9}$$

If we choose now the matrices U and V in the following form:

$$U = \frac{1}{(1-\eta)}\begin{pmatrix} -(1-2\eta)A_+ & -A_+ \\ A_+ & -A_+ \end{pmatrix}, \qquad V = \frac{1}{(1+\eta)}\begin{pmatrix} -(1+2\eta)A_- & A_- \\ -A_- & -A \end{pmatrix}, \tag{2.10}$$

where η is a constant parameter, then the integrability condition (2.6) implies the equations of motion and the integrability condition, (2.1) and (2.3), for the potentials $A_\pm$ of the $U(N)$ σ models. For this choice of U and V given by (2.10), equations (2.8) and (2.9) imply the following Riccati forms for the $U(N)$ σ models

$$(1-\eta)\partial_+\Gamma = -\Gamma A_+\Gamma + \Gamma A_+ - (1-2\eta)A_+\Gamma - A_+, \tag{2.11}$$

$$(1+\eta)\partial_-\Gamma = \Gamma A_-\Gamma + \Gamma A_- - (1+2\eta)A_-\Gamma + A_-. \tag{2.12}$$

It is easy now to see that equations (2.11) and (2.12) are invariant under the transformations

$$\Gamma' = \Gamma, \tag{2.13}$$

$$A'_\pm = \Gamma^\dagger A_\pm\Gamma + \tfrac{1}{2}\Gamma^\dagger\partial_\pm\Gamma, \tag{2.14}$$

with the constraints

$$\Gamma + \Gamma^\dagger = 2\eta I, \qquad \Gamma\Gamma^\dagger = \Gamma^\dagger\Gamma = 1, \tag{2.15}$$

where $A'_\pm = \frac{1}{2}g'^\dagger\partial_\pm g'$ with $g'g'^\dagger = g'^\dagger g' = 1$.

From the transformations (2.14) and the Riccati forms (2.11) and (2.12) we obtain the BT of the $U(N)$ σ models in the form

$$A'_+ - A_+ = \tfrac{1}{2}\partial_+\Gamma, \qquad (2.16)$$

$$A'_- - A_- = -\tfrac{1}{2}\partial_-\Gamma, \qquad (2.17)$$

with the constraints (2.15).

3 BT for the susy $U(N)$ σ Models

Introducing the conventions of ref. [8], the field equations of the susy $U(N)$ σ models, in the superspace x, y, ϑ_1 and ϑ_2, can be written as:

$$D_+B_- - D_-B_+ = 0, \qquad (3.1)$$

where $D_\pm = \partial_{\vartheta_\pm} + i\vartheta_\pm\partial_\pm$ with $\partial_{\vartheta_\pm} = \partial/\partial\vartheta_\pm$, $\vartheta_\pm = \vartheta_1 \pm i\vartheta_2$ and the susy gauge potentials $B_\pm$ are defined by

$$B_\pm = \tfrac{1}{2}\varphi^\dagger D_\pm\varphi, \qquad (3.2)$$

with the $N \times N$ unitary matrix superfield φ which satisfies $\varphi\varphi^\dagger = \varphi^\dagger\varphi = 1$. Moreover, $B_\pm$ satisfty the integrability condition

$$D_+B_- + D_-B_+ + 2\{B_+, B_-\} = 0. \qquad (3.3)$$

We consider the following susy linear system

$$D_+\psi = U\psi, \qquad (3.4)$$

$$D_-\psi = V\psi, \qquad (3.5)$$

where U, V and ψ are given as in (2.4) and (2.5) but here U^i, V^i and ψ^i are $N \times N$ susy matrices. The integrability condition for (3.4) and (3.5) is

$$D_-U + D_+V - \{U, V\} = 0. \qquad (3.6)$$

If we define the susy matrix Γ as in (2.7), then one can show that Γ satisfies the following susy matrix Riccati forms:

$$D_+\Gamma = -\Gamma U^3\Gamma - \Gamma U^4 + U^1\Gamma + U^2, \qquad (3.7)$$

$$D_-\Gamma = -\Gamma V^3\Gamma - \Gamma V^4 + V^4\Gamma + V^2. \qquad (3.8)$$

Let us choose here the susy matrices U and V as:

$$U = \frac{1}{(1-\eta)}\begin{pmatrix} -(1-2\eta)B_+ & -B_+ \\ B_+ & -B_+ \end{pmatrix}, \qquad V = \frac{1}{(1+\eta)}\begin{pmatrix} -(1+2\eta)B_- & B_- \\ -B_- & -B_- \end{pmatrix}, \qquad (3.9)$$

where η is a susy constant parameter. In this case, the integrability condition (3.6) implies the field equations and the integrability condition for the susy $U(N)$ σ models given by (3.1) and (3.3). Moreover, the choice of U and V given by (3.9) implies the following Riccati forms for the susy $U(N)$ σ models:

$$(1-\eta)D_+\Gamma = -\Gamma B_+\Gamma + \Gamma B_+ - (1-2\eta)B_+\Gamma - B_+, \tag{3.10}$$

$$(1+\eta)D_-\Gamma = \Gamma B_-\Gamma + \Gamma B_- - (1+2\eta)B_-\Gamma + B_-. \tag{3.11}$$

Consider the transformations

$$\Gamma' = \Gamma, \tag{3.12}$$

$$B'_\pm = \Gamma^\dagger B_\pm \Gamma + \tfrac{1}{2}\Gamma^\dagger D_\pm \Gamma. \tag{3.13}$$

Then, one can show that equations (3.10) and (3.11) are invariant under these transformations with imposing the same constraints as (2.15). Also, here $B'_\pm = \frac{1}{2}\varphi'^\dagger D_\pm \varphi'$ with $\varphi'\varphi'^\dagger = \varphi'^\dagger\varphi' = 1$. Using the transformations (3.13) into Riccati equations (3.10) and (3.11) we obtain the BT for the susy $U(N)$ σ models as:

$$B'_+ - B_+ = \tfrac{1}{2}D_+\Gamma, \tag{3.14}$$

$$B'_- - B_- = -\tfrac{1}{2}D_-\Gamma, \tag{3.15}$$

with the constraints (2.15).

4 Conclusion and Comments

In conclusion, we constructed BT for both the $U(N)$ σ models in two-dimensional Euclidean space (x, y) and the susy $U(N)$ σ models defined on the superspace $(x, y, \vartheta_1, \vartheta_2)$. In our construction, we followed a procedure similar to that of Konno and Wadati [6] applied for the AKNS systems. For each model, we defined the suitable choices of the interies of the matrices U and V in the considered linear systems so that they lead to the field equations of the models. Then we obtained the Riccati forms for our models. We defined the transformations which leave these Riccati forms invariant and we used them with the Riccati forms and obtained the BT for each model.

To see that equations (2.16) and (2.17) are the BT for the $U(N)$ σ models, we differentiate both (2.16) with respect to x_- and (2.17) with respect to x_+, then adding together to obtain

$$\partial_+ A'_- + \partial_- A'_+ - \partial_+ A_- - \partial_- A_+ = 0. \tag{4.1}$$

Thus $A'_\pm$ satisfy the field equations (2.1) if $A_\pm$ do. Similarly, to show that equations (3.14) and (3.15) are the BT for the susy $U(N)$ σ models, one can find the procedure: $D_+(3.15) - D_-(3.14)$ and obtain

$$D_+B'_- - D_-B'_+ - D_+B_- + D_-B_+ = 0, \tag{4.2}$$

where we have to notice that $\{D_+, D_-\} = 0$. So $B'_\pm$ satisfy the field equations (3.1) if $B_\pm$ do.

Solving the equation (2.14) for Γ, one obtains

$$\Gamma = g^\dagger g'. \tag{4.3}$$

In this case, the BT (2.16) and (2.17) coincide with the BT obtained in ref. [9]. Also, (3.13) implies that $\Gamma = \varphi^\dagger \varphi'$, then the BT (3.14) and (3.15) coincide with the BT of ref. [10]. However, in [9] and [10], they used a method involving the so-called Darboux transformations. One other comment here is that our approach can be applied in the same way exactly and construct BT for both the $U(N)$ σ models and the susy $U(N)$ σ models when we add the WZW-term to each of them.

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