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Relativistic Maclaurin Discs and Bifurcations

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Abstract. Sequences of rotating relativistic discs with internal two-dimensional pressure have been constructed. It is shown that in weaker relativistic configurations the sequences undergo a bifurcation from a disc to a ring structure, while in stronger relativistic cases the sequences terminate at the mass-shed limit where gravitational forces are exactly balanced by centrifugal forces.

1 Introduction

Relativistic discs may play an important role during the process of galaxy formation in the early universe. In addition, the final state of a close binary system, consisting of two neutron stars just after coalescence, can be described as a highly flattened disc-like structure. Relativistic discs with zero internal pressure have been studied previously [1] and recently an analytical solution has been presented [2]. In the case of non-vanishing internal pressure the bifurcation process has been studied recently in the Newtonian limit [3]. This work extends the analysis of relativistic discs with zero pressure and the Newtonian calculations including pressure.

We have constructed simple models of self-gravitating discs which are infinitesimally thin, having zero vertical extent. A constant angular velocity Ω has been assumed. We use a cylindrical coordinate system (ρ, z, φ, t) where the disc is located in the equatorial plane $(z = 0)$. The line element is written in the form used in [1]

$$ds^2 = e^{2\mu}\left(d\rho^2 + dz^2\right) + \rho^2 B^2 e^{-2\nu}(d\varphi - \omega dt)^2 - e^{2\nu} dt^2, \tag{1.1}$$

which describes axisymmetric, rotating and stationary configurations in general relativity. Four metric functions (μ, ν, B, and ω) which depend only on the coordinates ρ and z have to be determined. For the energy-momentum tensor T_{ab} we use the expression for an ideal gas where the pressure acts only in the equatorial plane. The vacuum Einstein equations $G_{ab} = 0$ yield four second order partial differential equations for the four metric potentials, see [1]. Flat relativistic discs can indeed be described by the vacuum Einstein equations, where the surface mass and pressure distributions create just the jump conditions for the derivatives of the metric potentials in the equatorial plane. These jump conditions are obtained by vertical integration of the Einstein equations [1]. Two different equations of state relating surface density with pressure were used.

$$p_p = K\sigma_p^\gamma, \tag{1.2}$$

where p_p and σ_p denote the proper values of the surface pressure and total mass energy density and, in analogy to the three-dimensional case, we use the isentropic relations:

$$p_p = K\sigma_0^\gamma = (\gamma - 1)\sigma_i. \tag{1.3}$$

Here σ_0 denotes the proper rest mass density, σ_i the proper internal energy density. K is constant and the adiabatic exponent γ is set to 3, in both cases. In these cases there exists an analytic disc solution in the Newtonian limit, where Ω and K are related by

$$K/K_{max} = 1 - (\Omega/\Omega_c)^2, \tag{1.4}$$

where K_{max} is the maximum possible value of K which is taken for zero rotation, and Ω_c is the maximum rotation rate in the case of a pressure-less disc of dust.

The partial differential equations are solved using numerical methods. To properly treat the boundary conditions at large distances from the disc, new coordinates $u = \rho/(\rho + a), v = z/(z + a)$ were used, where infinity has been transformed onto a finite distance, where a denotes the disc radius (normalised to 1). To obtain a numerical solution, the set of partial differential equations are discretised on a grid of typically 128×128 grid cells. The resulting matrix equations are solved by the successive over relaxation method. The jump conditions of the potentials in the disc region and the radial hydrostatic equation are iterated simultaneously with the matrix iterations. The numerically generated solutions where tested on the known dust disc solution [2]. In the case with internal pressure the bifurcation diagramm of the limiting Newtonian case [3] was reproduced accurately.

2 Results

The non-rotating discs with $\Omega = 0$, which are supported purely by the pressure have a maximum mass, similar to stars (Fig. 1a). The dashed line refers to the first equation of state (1.2) and the solid line to (1.3). For higher relativistic cases the surface density is concentrated more and more towards the centre, terminating finally into a black hole configuration.

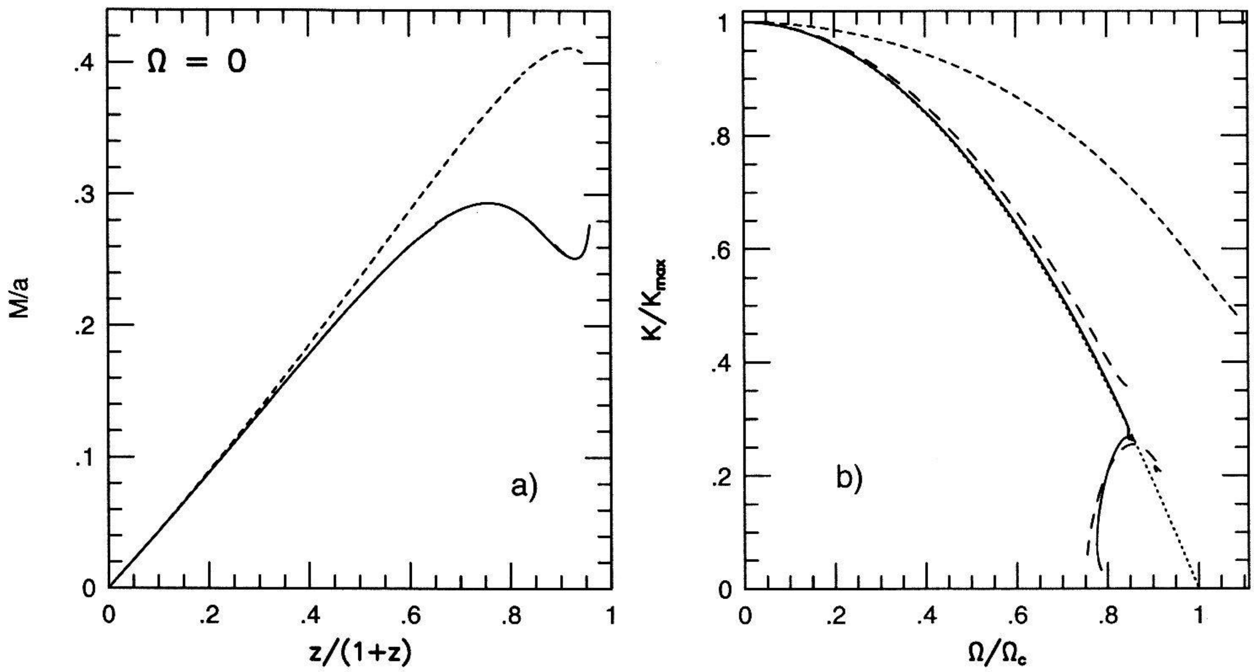

Figure 1: a) The gravitational mass of the disc as a function of the central redshift z in the case of zero rotation for two different equations of state. b) The pressure constant K versus angular velocity Ω given in units of the angular velocity of the dust disc Ω_c. Sequences for different values of the central redshift (solid $z = 0.025$, long-dashed $z = 0.05$, dashed $z = 1.89$) are shown. The dotted line refers to the Newtonian Maclaurin disc. $\Omega = 0$ refers to purely pressure supported discs having $K = K_{max}$. $\Omega = \Omega_c$ implies K=0 and refers to the dust disc.

Rotating discs were studied for the second equation of state only (1.3), and the results are displayed in Fig. 1b. Weaker relativistic discs (solid line) follow the Newtonian curve closely and bifurcate continuously into a ring-like structure at $\Omega/\Omega_c \approx 0.84$. For intermediate central redshifts (long-dashed line), the discs and rings coexist with no apparent connection between them. Stronger relativistic (short-dashed line) discs terminate in a mass shed limit, where gravity is balanced exactly by centrifugal forces. It should be noted that the angular velocity reached in these stronger relativistic cases exceeds the value of the pressure-free case $\Omega = \Omega_c$ for the same central redshift. This is caused by the non-linearity of the Einstein-equations.

In Fig. 2 an enlargement of the bifurcation region is displayed. The ring bifurcates smoothly from the disc solution through a sequence of dumb-bell shaped density distributions (line 2 in Fig. 2b). At point 3 the density in the centre has reached zero, and the ring structures begin (line 3 and 4). The solid line in Fig. 2b is given approximately by the Newtonian relation $\sigma(r) = \sqrt{1-(r/a)^2}$. With increasing inner ring radius the maximum surface density increases (Fig. 2b), and the ring sequence terminates eventually into a ring with zero radial extent.

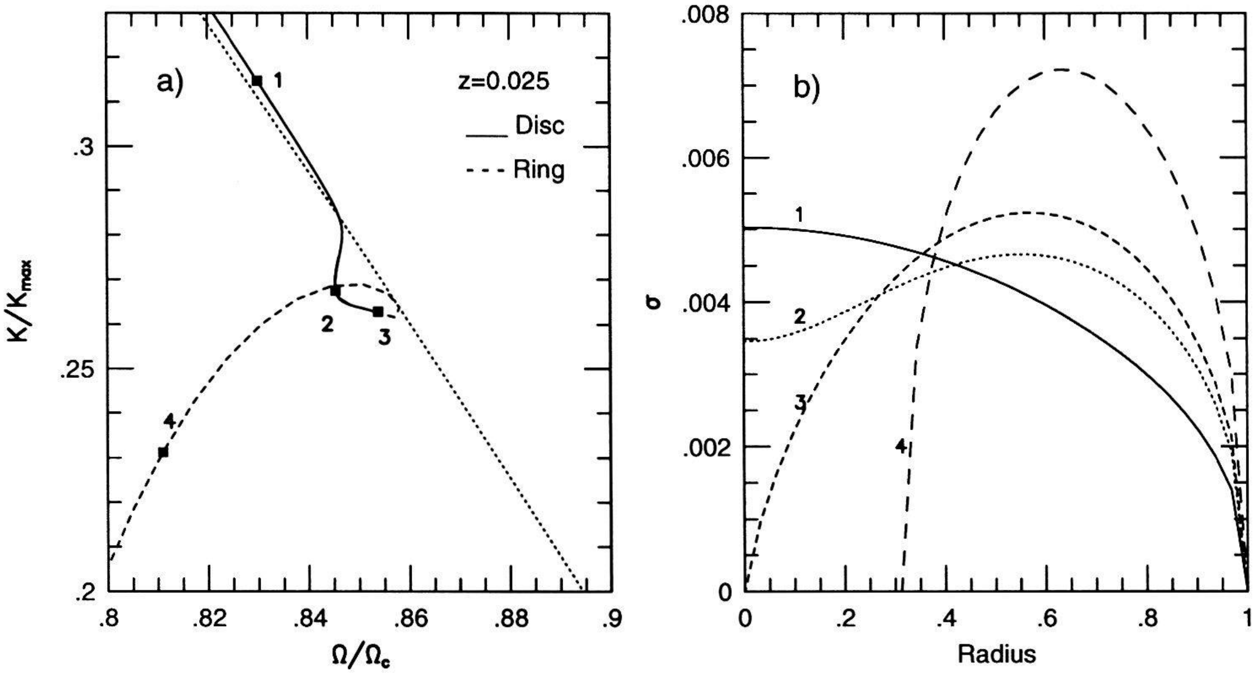

Figure 2: a) Enlargement of Fig. 1b displaying in detail the bifurcation from the disc to the ring configuration. The dotted line refers to the Newtonian disc solution. The solid line refers to the disc solution which turns into a ring struture (dashed line) at point (3). b) Plot of the surface density distribution at the locations marked in a).

Numerically, it was not possible to find the connection to the pressure-free dust case which is known to exist in the Newtonian case (dotted line). Even in the purely Newtonian calculations [3] this could not be achieved. This may be related to the fact, that the dust disc is (at least in the Newtonian case) dynamically unstable. The developed method can be extended relatively easily to the three-dimensional case and the study of bifurcations for extended objects may be possible.

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