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# Quasi-Einstein metrics and their renormalizability properties

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*Abstract.* We construct a family of kählerian quasi-Einstein metrics with an isometry group  $U(n)$  acting linearly on the holomorphic coordinates. Suitable restrictions on the parameters give rise to complete non-compact as well as compact metrics whose geometrical structure is studied in detail. And we discuss The two loop renormalizability properties of the bosonic  $\sigma$ -models.

## 1 Introduction

Quasi-Einstein metrics are defined by the constraints

$$Ric_{\mu\nu} = \lambda g_{\mu\nu} + \frac{1}{2}(D_{\mu}v_{\nu} + D_{\nu}v_{\mu}). \quad (1.1)$$

In section 2, using Kähler geometry and an isometry group  $U(n)$ , we reduce (1.1) to a non-linear differential equation. Since this equation cannot be integrated to get the Kähler potential we switch to non-holomorphic coordinates which are useful to display explicitly the distance.

In section 3 we give two classes of complete non-compact metrics and we describe the compact metrics of the family.

In section 4 we discuss the two-loop renormalizability of the bosonic  $\sigma$ -models built on kählerian quasi-Einstein metrics. Our main result is that in the chosen isometry class the requirement of two-loop renormalizability selects uniquely  $CP^n$ , which is not only Einstein but symmetric!

## 2 Quasi-Einstein Kähler metrics

### 2.1 Isometry group $U(n)$

The idea to obtain explicit quasi-Einstein metrics is to use holomorphic coordinates  $\{z_i, i = 1, \dots, n\}$  in  $\mathbb{C}^n \setminus \{0\}$  and to choose for isometry group a  $U(n)$  for which  $\{z_i\}$  transform according to its fundamental representation. This implies that the Kähler potential  $K$  depends solely on  $s = \bar{z} \cdot z = \sum_{i=1}^n \bar{z}_i z_i$ . It follows that the metric is

$$\frac{1}{2} g = g_{i\bar{j}} dz^i d\bar{z}^j = A d\bar{z} \cdot dz + A' |\bar{z} \cdot dz|^2, \quad A(s) = \frac{dK(s)}{ds}.$$

The Ricci tensor reads

$$Ric_{i\bar{j}} = -\partial_{i\bar{j}}^2 \ln D, \quad D = \det(g_{i\bar{j}}) = A^{n-1}(sA)'.$$

The restrictions put on the metric by the quasi-Einstein requirement (1.1) imply ([1])

$$v_i = c \partial_i(sA), \quad A^{n-1}(sA)' = e^{-\lambda K - (Re c)sA}. \tag{2.1}$$

where  $c$  is an a priori complex integration constant.

### 2.2 The coordinates choice

In view of the complexity of (2.1), there is little hope to get the explicit Kähler potential  $K(s)$  and the distance if we insist on using holomorphic coordinates. We define

$$z_i = \sqrt{s} \xi_i \quad i = 1, \dots, n$$

and

$$\xi = \frac{e^{i\tau}}{1 + \bar{u} \cdot u} (1, u), \quad \tau \in [0, 2\pi], \quad \{u_i, i = 1, \dots, n - 1\}.$$

This gives for the final form of our distance

$$\frac{1}{2} g = \frac{(dt)^2}{4\varrho} + \varrho (d\tau + \theta)^2 + t \frac{h}{2} (CP^{n-1})$$

with

$$\varrho = s \frac{dt}{ds}, \quad t = s K', \quad \theta = \frac{1}{1 + \bar{u} \cdot u} \frac{\bar{u} \cdot du - u \cdot d\bar{u}}{2i}.$$

The differential equation (2.1) becomes

$$\frac{d}{dt} (e^{ct} t^{n-1} \varrho(t)) = e^{ct} (n t^{n-1} - \lambda t^n). \tag{2.2}$$

The vector  $v$  is given by the holomorphic 1-form  $v = c dt$  for some real constant  $c$ .

### 3 Complete and compact metrics

Integrating (2.2) gives

$$t^{n-1} \varrho = \frac{e^{-ct} \int_0^{ct} e^u \left( n u^{n-1} - \frac{\lambda}{c} u^n \right) du}{c^n} - B e^{-ct}, \quad n = 1, 2, \dots \tag{3.1}$$

#### 3.1 Complete non-compact metrics

The first one appears for  $B = 0, c > 0, \lambda < 0$ .

At  $t = 0$ , setting  $t = r^2$  the distance becomes

$$\frac{1}{2} g \approx dr^2 + r^2 \left[ (d\tau + \theta)^2 + \frac{h}{2} (CP^{n-1}) \right] = dr^2 + r^2 d\bar{\xi} \cdot d\xi$$

which shows that  $r = 0$  is a “nut”, [4].

For  $t \rightarrow \infty$ , setting  $t = \frac{|\lambda|}{c} r^2$  the distance becomes

$$\frac{1}{2} g \approx dr^2 + r^2 \left\{ \left( \frac{|\lambda|}{c} \right)^2 |\bar{u} \cdot du|^2 + \frac{|\lambda|}{c} (d\bar{u} \cdot du - |\bar{u} \cdot du|^2) \right\}$$

showing that infinity is asymptotically flat (albeit not euclidean).

For  $\lambda = 0$  the infinity is taubian, [2, p. 252].

The second one appears for  $B > 0, c > 0, \lambda \leq 0$ . But now  $t_n$  define by  $\varrho(t_n) = 0$  is a “bolt” of twist  $k = n + 1, n + 2, \dots$ , [4].

#### 3.2 Compact metrics

Let us suppose  $c < 0, \lambda > |c|$  and  $n \geq 2$ . From (3.1) we can deduce there does exist a finite interval  $[t_n^{(1)}, t_n^{(2)}]$  on which  $\varrho(t)$  is positive. The constraints for  $t = t_n^{(1)}$  and  $t = t_n^{(2)}$  to be bolts of twist  $k$  are [5]

$$\begin{cases} \varrho(t_n^{(1)}) = 0, & \varrho'(t_n^{(1)}) = k, \\ \varrho(t_n^{(2)}) = 0, & \varrho'(t_n^{(2)}) = -k. \end{cases} \quad k = 1, 2, 3, \dots$$

which reduce, using the differential equation,  $\varrho' + \left( c + \frac{n-1}{t} \right) \varrho = n - \lambda t$ , to the transcendental equation

$$\int_{-k}^{+k} e^{-\xi u} u (n + u)^{n-1} du = 0, \quad \xi = \frac{|c|}{t}, \quad k = 1, 2, \dots, n - 1.$$

And this equation has a unique solution  $\xi \in ]0, 1[$ .

## 4 Bosonic $\sigma$ -models at two loops

For a given bosonic  $\sigma$ -model with classical action

$$\frac{1}{g^2} \int d^2x \frac{1}{2} g_{ij}(\varphi) \partial_\mu \varphi^i \partial_\mu \varphi^j$$

the one and two-loop on-shell divergences follow from Friedan [3] and are given by

$$\left(\frac{\hbar}{4\pi}\right) \frac{1}{\epsilon} \int d^2x Ric_{ij} \partial_\mu \varphi^i \partial_\mu \varphi^j + \left(\frac{\hbar}{4\pi}\right)^2 \frac{g^2}{\epsilon} \int d^2x \frac{1}{2} R_{is,tu} R_j^{s,tu} \partial_\mu \varphi^i \partial_\mu \varphi^j, \quad \epsilon = d - 2.$$

Two-loop renormalizability requires

$$X_{ij} = R_{is,tu} R_j^{s,tu} = 0 = D_i w_j + D_j w_i \quad \Rightarrow \quad w_i = c_2 \partial_i (sA)$$

and

$$X_{i\bar{j}} = R_{i\bar{k},l\bar{n}} R_{\bar{j}}^{\bar{k},l\bar{n}} = \lambda_2 g_{i\bar{j}} + D_i w_{\bar{j}} + D_{\bar{j}} w_i = \partial_{i\bar{j}}^2 (\lambda_2 K + (c_2 + \bar{c}_2) sA).$$

For the divergence  $X_{i\bar{j}}$  to be absorbable it is therefore necessary that  $X_{i\bar{j}}$  be Kähler. In the class of metrics considered in this article, the large isometry group enables one to compute the curvature tensor and

$$X_{i\bar{j}} = \mu(s) \delta_{ij} + \nu(s) \bar{z}_i z_j.$$

The Kähler condition requires therefore  $\nu = \mu'$  which is integrated to  $K(s) = a \ln(s + b)$ . Among all kählerian metrics of dimension  $n \geq 2$ , with an isometry group  $U(n)$  acting linearly on the holomorphic coordinates, only  $CP^n$  is one-loop and two-loop renormalizable.

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