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The Relativistic Charged Membrane and its Total Mass[1]

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Abstract. A general classical theory of a relativistic charged p-brane is formulated. Membrane's mass is calculated in various ways: from the radiation back reaction, from energy-momentum of the electromagnetic field around a moving membrane, and from the canonical momentum. This completes the initial Dirac's derivation of the charged membrane's mass.

1 Introduction

The general theory of relativistic p-branes has been thoroughly studied by many authors [1]. It is very interesting and instructive to put the electric charge distribution on a p-brane. A model was first proposed by Dirac [2], but his action does not contain a coupling term between the charge and the field potential A_μ. Dirac introduced the coupling by a suitable boundary conditions for A_μ, valid only in a particular gauge. A general form of the action was given in Ref. [3]:

$$I[X^\mu(\xi), A_\mu] = \int \mathrm{d}^d\xi(\kappa\sqrt{|f|} + e^a\,\partial_a X^\mu A_\mu)\delta^D(x - X(\xi))\mathrm{d}^D x + \frac{1}{16\pi}\int F_{\mu\nu}F^{\mu\nu}\sqrt{|g|}\,\mathrm{d}^D x \quad (1.1)$$

Here d is the worldsheet dimension and D the space-time dimension; $\xi^a, a = 0, 1, 2, ..., d-1$, are worldsheet coordinates (parameters) and $X^\mu(\xi), \mu = 0, 1, 2, ..., D-1$ the embedding functions, $f_{ab} = \partial_a X^\mu \partial_b X_\mu$ the induced metric, $f \equiv \det f_{ab}$, κ tension and e^a the electric charge current density on the worldsheet.

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2 Equations of motion for membrane's centre of mass

By varying (1.1) with respect to X^μ we obtain the membrane's equation of motion

$$\kappa\,\partial_a(\sqrt{|f|}\partial^a X^\mu) + e^a\partial_a X^\nu F_\nu{}^\mu = 0 \tag{2.1}$$

Integrating the latter equation over the worldsheet, using the Gauss law and assuming, as usual, that only the space-like hypersurfaces Σ_1 and Σ_2 do contribute to the first integral, and then taking Σ_1 and Σ_2 to be infinitesimally close to each other, we obtain

$$\frac{\mathrm{d}P^\mu_{\mathrm{m}}}{\mathrm{d}\tau} + \int \mathrm{d}\sigma\, e^a\, \partial_a X^\nu F_\nu{}^\mu(x) = 0 \tag{2.2}$$

where $\mathrm{d}\sigma = n^a\,\mathrm{d}\sigma_a$, n^a a normal vector to the hypersurface element $\mathrm{d}\sigma_a$, τ the time like parameter on the worldsheet, and $P^\mu_{\mathrm{m}} = \kappa\int \mathrm{d}\sigma_a\sqrt{|f|}\,\partial^a X^\mu$ the total kinetic momentum. This is the equation of motion for membrane's centre of mass. The electromagnetic field can be taken to consist of a fixed external field $F^{(\mathrm{ext})}_{\mu\nu}$ and the self-field generated by our membrane: $F_{\mu\nu} = F^{(\mathrm{ext})}_{\mu\nu} + F^{(\mathrm{self})}_{\mu\nu}$. Expanding the external field around the centroid worldline X^μ_{C} and writing $e^a\partial_a X^\nu = e\dot{X}^\nu = e\dot{X}^\nu_{\mathrm{C}} + e(\dot{X} - \dot{X}_{\mathrm{C}})^\nu$, where $e \equiv n^a e_a$ is charge density, the equation of motion (2.2) becomes

$$\frac{\mathrm{d}P^\mu_{\mathrm{m}}}{\mathrm{d}\tau} + q\dot{X}^\nu_{\mathrm{C}} F_\nu{}^{\mu(\mathrm{ext})} + \text{higher multipoles} + \int \mathrm{d}\sigma\, e\dot{X}^\nu F_\nu{}^{\mu(\mathrm{self})} = 0 \tag{2.3}$$

Going now to a specific case of a 2-dimensional spherical membrane, of radius r, without oscillations, with its centre of mass speed much smaller than the speed of light, we obtain the spatial components of the self force $F^r_{(\mathrm{self})} = -\frac{q^2}{2r}\ddot{X}^r + F^r_{(\mathrm{rad})} + (\text{higher derivatives})$, where $q = \int \mathrm{d}\sigma_a\, e^a$ is the total charge. For the kinetic momentum we obtain $P^\mu_{\mathrm{m}} = 4\pi\kappa r^2\,\dot{X}_\mu/\sqrt{\dot{X}^2}$. We now insert these last two expressions into Eq.(2.2) and identify the coefficient in front of acceleration as the renormalized or the observed mass:

$$M = 4\pi\kappa r^2 + \frac{q^2}{2r} \tag{2.4}$$

3 Energy-momentum of the electromagnetic field around a moving membrane

The second way to obtain the membrane's mass is to calculate the stress-energy tensor belonging to the action (1.1):

$$T^{\mu\nu} = 2\partial\mathcal{L}/\partial g^{\mu\nu} = T^{\mu\nu}_{\mathrm{m}} + T^{\mu\nu}_{\mathrm{EM}} \tag{3.1}$$

where

$$T^{\mu\nu}_{\mathrm{m}} = \kappa\int \mathrm{d}^d\xi\sqrt{|f|}\,\partial_a X^\mu\,\partial^a X^\nu\,\delta^D(x - X(\xi)) \tag{3.2}$$

$$T^{\mu\nu}_{\rm EM} = \frac{1}{16\pi} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} - \frac{1}{4\pi} F^{\rho\mu} F_\rho{}^\nu \tag{3.3}$$

The momentum is

$$P^\mu = \int \mathrm{d}\Sigma_\nu \, T^{\mu\nu} = P^\mu_{\rm m} + P^\mu_{\rm EM} \tag{3.4}$$

For a specific 2-dimensional membrane (as described above), and taking $\mathrm{d}\Sigma_\nu$ oriented along membrane's 4-velocity $\dot{X}_\nu$, we obtain (at $v << c$):

$$P^0 \; = \; \left(4\pi\kappa r^2 + \frac{q^2}{2r}\right) = M \tag{3.5}$$

$$P^r \; = \; \left(4\pi\kappa r^2 + \frac{q^2}{2r}\right) v^r = M\, v^r \quad , \qquad r = 1,2,3 \tag{3.6}$$

where v^r is membrane's centre of mass velocity. In Eqs.(3.5), (3.6) we have the same result for the membrane's mass as calculated from Eq.(2.3), where the radiation back reaction has been taken into account.

The old problem of 3/4 does not arise in our calculation of $P^\mu_{\rm EM}$. As already stated by Rohlrich [4] and Barut [5] (see also [3]) one obtains consistent electromagnetic mass, provided that the hypersurface element $\mathrm{d}\Sigma_\nu$ is chosen properly.

4 The canonical momentum and the Hamiltonian

From the action (1.1) we obtain the following expression for canonical momentum

$$p^a{}_\mu = \frac{\partial \mathcal{L}}{\partial \partial_a X^\mu} = \kappa \sqrt{|f|}\, \partial^a X_\mu + e^a A_\mu \tag{4.1}$$

which consists of the kinetic and the minimal coupling term. The Hamiltonian density $\mathcal{H}^a{}_b = p^a{}_\mu \partial_b X^\mu - \mathcal{L}\, \delta^a{}_b$ is identically zero and represents d independent worldsheet constraints which are a consequence of the reparametrization invariance of the action (1.1). According to Dirac [7], the Hamiltonian [6, 3] is a superposition of constraints:

$$H = \int \mathrm{d}\sigma\, \mathcal{H}^{ab} n_a n_b = \frac{1}{2} \int \mathrm{d}\sigma \frac{\sqrt{|\bar{f}|}\sqrt{n^2}}{\kappa} \left(\frac{\pi^\mu \pi_\mu}{|\bar{f}|} - \kappa^2 \right) \approx 0 \tag{4.2}$$

and is weakly zero. Here $\pi^a{}_\mu \equiv p^a{}_\mu - e^a A_\mu$, $\pi_\mu \equiv \pi^a{}_\mu n_a$, where n_a is the normal vector to the hypersurface element $\mathrm{d}\sigma_a$. The Hamiltonian equations of motion $\dot{p}_\mu = -\,\delta H / \delta X^\mu(\sigma)$, $\dot{X}^\mu = \delta H / \delta p_\mu(\sigma)$ give the correct Lorentz-force equation (2.1).

The canonical momentum of the whole membrane is given by

$$P^{(c)}_\mu = \int p^a{}_\mu \, \mathrm{d}\sigma_a = \int \pi_\mu \, \mathrm{d}\sigma + \frac{1}{2} \int e A_\mu \, \mathrm{d}\sigma \tag{4.3}$$

where $\frac{1}{2}$ in the electromagnetic term is neded in order to avoid double counting in the integration over the membrane. By using the constraint [8] $\pi^\mu \pi_\mu - |\bar{f}|\kappa^2 = 0$ we find for the time component $\pi_0 = (|\bar{f}|\kappa^2 + \vec{\pi}^2)^{1/2}$, $\vec{\pi}^2 = -\pi^r \pi_r$, $r = 1, 2, ..., D-1$. This can be inserted into the expression $P_0^{(c)}$ of Eq.(4.3) and we obtain

$$P_0^{(c)} = \int \mathrm{d}\sigma \sqrt{|\bar{f}|} \left(\kappa^2 + \frac{\vec{\pi}^2}{|\bar{f}|} \right)^{1/2} + \frac{1}{2} \int \mathrm{d}\sigma \, A_0 \tag{4.4}$$

For a 2-dimensional, spherically symmetric membrane Eq.(4.4) gives

$$P_0^{(c)} = \left((4\pi\kappa r^2)^2 + p_{(\mathrm{r})}^2 \right)^{1/2} + \frac{q^2}{2r} \tag{4.5}$$

where $p_{(\mathrm{r})} = -\, 4\pi\kappa r^2 \dot{r}/(1-\dot{r}^2)^{1/2}$. The time-like component of the total canonical momentum $P_0^{(c)}$ has the role of (non covariant) Hamiltonian and gives the equations of motion which are equivalent to the equations (2.1). When $p_{(\mathrm{r})} = 0$ the Hamiltonian $P_0^{(c)}$ coincides with membrane's mass (2.4) and (3.5).

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