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**Autor:** Hoppe, Jens

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# On M-Algebras, the Quantisation of Nambu-Mechanics, and Volume Preserving Diffeomorphisms<sup>1</sup>

Jens Hoppe <sup>2</sup>  
Theoretische Physik  
ETH-Hönggerberg  
8093 Zürich  
Schweiz

*Dedicated to K. Hepp and W. Hunziker on the occasion of their 60th birthdays.*

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**Abstract:** M-branes are related to theories on function spaces  $\mathcal{A}$  involving  $M$ -linear non-commutative maps from  $\mathcal{A} \times \cdots \times \mathcal{A}$  to  $\mathcal{A}$ . While the Lie-symmetry-algebra of volume preserving diffeomorphisms of  $T^M$  cannot be deformed when  $M > 2$ , the arising  $M$ -algebras naturally relate to Nambu's generalisation of Hamiltonian mechanics, e.g. by providing a representation of the canonical  $M$ -commutation relations,  $[J_1, \cdots, J_M] = i\hbar$ . Concerning multidimensional integrability, an important generalisation of Lax-pairs is given.

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<sup>1</sup>ETH-TH/95-33

<sup>2</sup>Heisenberg Fellow. On leave of absence from Karlsruhe University.

## 1. Introduction

Generalizing fundamental concepts, such as Lie algebras or Hamiltonian dynamics, may have quite diverse merits; it can lead to new, interesting possibilities, – or reassure oneself of our present notions. While the result that volume preserving diffeomorphisms of toroidal  $M$ -branes, as a Lie-symmetry algebra, cannot be deformed (if  $M > 2$ ) is of the latter nature – the following ideas appear to be worthwhile pursuing:

– Using a  $*M$ -deformation of the algebra of functions on some  $M$ -dimensional manifold for representing the  $M$ -linear analogue to Heisenberg's commutation relations that Nambu [1] encountered in multi-Hamiltonian dynamics.

– Generalizing the Jacobi identity for Lie algebras to a (2-bracket) identity involving  $2M - 1$  elements of a vectorspace  $V$  for which an antisymmetric  $M$ -linear map ( $M$ -commutator) from  $V \times \cdots \times V$  to  $V$  is defined (in a dynamical context, an identity involving  $M$ , rather than 2, of the  $M$ -commutators, may be preferred).

– A potential relevance of  $M$ -algebras to the quantisation of space-time.

Perhaps most importantly (on a concrete, practical level), an explicit example is given (the multidimensional diffeomorphism-invariant integrable field theories found in [2]) for the usefulness (envisaged some time ago [3]) of generalizing Lax-pairs to -triples, ....

## 2. M-algebras from M-branes

A relativistic  $M$ -brane moving in  $D$ -dimensional space time may be described, in a light-cone gauge, by the  $\text{VDiff}\Sigma$ -invariant sector of ([4])

$$H = \frac{1}{2} \int_{\Sigma} \frac{d^M \varphi}{\rho(\varphi)} (\vec{p}^2 + g) \quad (1)$$

where  $g$  is the determinant of the  $M \times M$  matrix  $(g_{rs}) := (\nabla_r x^i \nabla_s x_i)_{r,s=1 \dots M}$ ,  $x^i$  and  $p_i$  ( $i = 1, \dots, D - 2 =: d$ ) are canonically conjugate fields, and  $\rho$  is a fixed non-dynamical density on the  $M$ -dimensional parameter-manifold  $\Sigma$  ( $M = 1$  for strings,  $M = 2$  for membranes, ...). Generators of  $\text{VDiff}\Sigma$ , the group of volume-preserving diffeomorphisms of  $\Sigma$  (resp. the component connected to the identity), are represented by

$$K := \int_{\Sigma} f^r p_i \partial_r x^i d^M \varphi \quad (2)$$

with  $\nabla_r f^r = 0$ .  $g$  may be written as

$$g = \sum_{i_1 < i_2 < \dots < i_M} \{x_{i_1}, \dots, x_{i_M}\} \{x^{i_1}, \dots, x^{i_M}\}, \quad (3)$$

where the 'Nambu-bracket'  $\{\dots\}$  is defined for functions  $f_1, \dots, f_M$  on  $\Sigma$  as

$$\{f_1, \dots, f_M\} := \epsilon^{r_1 \dots r_M} \partial_{r_1} f_1 \cdots \partial_{r_M} f_M. \quad (4)$$

This trivial, but important observation suggests to consider Hamiltonians

$$H_\lambda := \frac{1}{2} \text{Tr} \left( \vec{P}^2 \pm \sum_{i_1 < \dots < i_M} [X_{i_1}, \dots, X_{i_M}]_\lambda^2 \right), \quad (5)$$

resp.

$$H_\lambda = \frac{1}{2} \sum_{i=1}^d \beta(P_i, P_i) + \frac{1}{2} \sum_{i_1 < \dots < i_M} \beta([X_{i_1}, \dots, X_{i_M}]_\lambda, [X_{i_1}, \dots, X_{i_M}]_\lambda), \quad (6)$$

where  $X^i$  and  $P_i$  are elements of (possibly finite dimensional,  $\lambda$ -dependent) vectorspaces  $V$  on which antisymmetric  $M$ -linear maps  $[\dots]_\lambda : V \times \dots \times V \rightarrow V$  are defined, and  $\beta$  a positive definite hermitean form, preferably invariant with respect to some analogue of volume preserving diffeomorphisms (cp. (2)).

With

$$[T_{a_1}, \dots, T_{a_M}]_\lambda = f_{a_1 \dots a_M}^a(\lambda) T_a \quad (7)$$

and

$$\beta(T_a, T_b) = \delta_b^a \quad (8)$$

for some (possibly  $\lambda$ -dependent) basis  $\{T_a\}_{a=1}^{\dim V}$  of  $V$ , i.e.

$$f_{a_1 \dots a_M}^a(\lambda) = \beta(T_a, [T_{a_1}, \dots, T_{a_M}]_\lambda), \quad (9)$$

(6) reads

$$H_\lambda = \frac{1}{2} p_{ia}^* p_{ia} + \frac{1}{2} (f_{a_1 \dots a_M}^a(\lambda))^* f_{b_1 \dots b_M}^a(\lambda) \frac{1}{M!} x_{i_1 a_1}^* \dots x_{i_M a_M}^* x_{i_1 b_1} \dots x_{i_M b_M}, \quad (10)$$

while (1) may be written as

$$H = \frac{1}{2} p_{i\alpha}^* p_{i\alpha} + \frac{1}{2} (g_{\alpha_1 \dots \alpha_M}^\alpha)^* g_{\beta_1 \dots \beta_M}^\alpha \frac{1}{M!} x_{i_1 \alpha_1}^* \dots x_{i_M \beta_M}^* ; \quad (11)$$

$$g_{\alpha_1 \dots \alpha_M}^\alpha := \int_\Sigma Y_\alpha^* \{Y_{\alpha_1}, \dots, Y_{\alpha_M}\} \rho d^M \varphi \quad (12)$$

is defined with respect to some orthonormal basis of functions (on  $\Sigma$ ) satisfying

$$\int Y_\alpha^* Y_\beta \rho d^M \varphi = \delta_\beta^\alpha \quad \alpha, \beta = 1 \dots \infty \quad (13)$$

(even for real  $x_i$ , it is often convenient to take a complex basis).  
Obvious questions are:

- 1) Does there exist a ‘natural’ sequence of finite dimensional vectorspaces  $V_n$  with basis  $\{T_a^{(n)}\}$  and antisymmetric maps  $F_n : V_n \times \cdots \times V_n \rightarrow V_n$  such that for each  $(M + 1)$ -tuple  $(a_1 \cdots a_M)$

$$\lim_{n \rightarrow \infty} f_{a_1 \cdots a_M}^a(\lambda_n) \stackrel{?}{=} g_{a_1 \cdots a_M}^a. \tag{14}$$

- 2) For which  $M$  do there exist finite dimensional analogues of (2),  $K(n)$ , leaving  $(10)_{\lambda_n}$  invariant, such that, as  $n \rightarrow \infty$ , the full invariance under volume-preserving diffeomorphisms is recovered?

- 3) What about  $\lambda$ -deformations with infinite dimensional  $V$ ’s ?

Let us look at the case of a  $M$ -torus,  $\Sigma = T^M$  :

Choosing

$$Y_{\vec{m}} = e^{i\vec{m}\vec{\varphi}}, \vec{m} = (m_1, \dots, m_M) \in \mathbb{Z}^M, \rho \equiv 1, \tag{15}$$

one gets

$$g_{\vec{m}_1 \cdots \vec{m}_M}^{\vec{m}} = i^M (\vec{m}_1, \dots, \vec{m}_M) \delta_{\vec{m}_1 + \cdots + \vec{m}_M}^{\vec{m}} \tag{16}$$

where  $(\vec{m}_1, \dots, \vec{m}_M) \in \mathbb{Z}$  denotes the determinant of the corresponding  $M \times M$  Matrix (an element of  $GL(M, \mathbb{Z})$ ).

Consider now the following ‘ $*M$ -product’ (a deformation of the ordinary commutative product of  $M$  functions  $f_1, \dots, f_M$  on  $\Sigma$ ):

$$(f_1 \cdots f_M)_* := f_1 \cdots f_M + \sum_{m=1}^{\infty} \frac{\left(\frac{(-i)^{M+1}\lambda}{M!}\right)^m}{m!} \frac{\epsilon^{r_1 r'_1 \cdots r_1^{(M)}}}{\epsilon^{r_m r'_m \cdots r_m^{(M)}}} \frac{\partial^m f_i}{\partial \varphi^{r_1} \cdots \partial \varphi^{r_m}} \cdots \frac{\partial^m f_M}{\partial \varphi^{r_1^{(M)}} \cdots \partial \varphi^{r_m^{(M)}}}. \tag{17}$$

One then finds that

$$(Y_{\vec{m}_1} \cdots Y_{\vec{m}_M})_* = \sqrt{\omega}^{- (\vec{m}_1, \dots, \vec{m}_M)} Y_{\vec{m}_1 + \cdots + \vec{m}_M} \\ \sqrt{\omega} = e^{i \frac{\lambda}{M!}}. \tag{18}$$

Defining

$$[f_1, \dots, f_M]_* := \sum_{\sigma \in S_M} (\text{sign } \sigma) (f_{\sigma_1} \cdots f_{\sigma_M})_* \tag{19}$$

to simply be the antisymmetrized  $*M$ -product, one gets

$$[T_{\vec{m}_1}, \dots, T_{\vec{m}_M}] = \frac{-i}{2\pi\Lambda} \sin(2\pi\Lambda(\vec{m}_1, \dots, \vec{m}_M)) T_{\vec{m}_1 + \cdots + \vec{m}_M} \tag{20}$$

$$\text{with } \Lambda := \frac{\lambda}{2\pi M!} \text{ and } T_{\vec{m}} := \lambda^{-\frac{1}{M-1}} Y_{\vec{m}}.$$

For  $M > 1$  arbitrary (but fixed), let  $V$  denote the vectorspace (over  $\mathbb{C}$ ) generated by  $\{T_{\vec{m}}\}_{\vec{m} \in \mathbb{Z}^M}$ ,  $\mathbb{M}^\Lambda$  denote  $(V, *)$  and  $\mathbb{A}^\Lambda$  denote  $(V, [\cdots]_*)$ .

The hermitean form  $\beta$  (cp. (8),(9)),

$$\beta(T_{\vec{m}}, T_{\vec{n}}) = \delta_{\vec{n}}^{\vec{m}}, \quad \beta(c_i X_i, d_j X_j) = c_i^* d_j \beta(X_i, X_j),$$

will have the important property ('invariance') that (for  $X_i = x_{i\vec{m}} T_{\vec{m}}$  with  $x_{i\vec{m}}^* = x_{i-\vec{m}}$ )

$$\beta(X, [X_{i_1}, \dots, X_{i_M}]) = -\beta(X_{i_r}, [X_{i_1}, \dots, X_{i_{r-1}}, X, X_{i_{r+1}}, \dots, X_{i_M}]),$$

as

$$\beta(T_{\vec{m}}, [T_{\vec{m}_1}, \dots, T_{\vec{m}_M}]) = \frac{-i}{2\pi\Lambda} \delta_{\vec{m}_1, \dots, \vec{m}_M}^{\vec{m}} \sin(2\pi\Lambda(\vec{m}_1, \dots, \vec{m}_M)).$$

For rational  $\Lambda = \frac{\tilde{N}}{N}$  (assuming  $N$  and  $\tilde{N} < N$  having no common divisor  $> 1$ ) both  $\mathbb{A}^\Lambda$  and  $\mathbb{M}^\Lambda$  may be divided by an ideal of finite codimension, namely (using the periodicity of the structure-constants) the vectorspace  $\mathbb{I}$  generated by all elements of the form  $T_{\vec{m}} - T_{\vec{m}+N}$  (anything). One thus arrives at considering (for arbitrary odd  $N$ )

$$V^{(N)} := \left\langle T_{\vec{m}} | m_r = -\frac{N-1}{2}, \dots, +\frac{N-1}{2} \right\rangle_{\mathbb{C}} \quad r = 1 \dots M \quad (21)$$

with a  $*_M$  product on  $V^{(N)}$  defined just as in (18):

$$\begin{aligned} (T_{\vec{m}_1} \dots T_{\vec{m}_M})_* &:= \frac{-iN}{2\pi\tilde{N}M!} \omega^{-\frac{1}{2}(\vec{m}_1, \dots, \vec{m}_M)} T_{\vec{m}_1 + \dots + \vec{m}_M \pmod{N}} \\ \omega &= e^{4\pi i \frac{\tilde{N}}{N}}, \end{aligned} \quad (22)$$

and a corresponding alternating product,

$$\begin{aligned} [T_{\vec{m}_1}, \dots, T_{\vec{m}_M}]_* &= \frac{-iN}{2\pi\tilde{N}} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1, \dots, \vec{m}_M)\right) T_{\vec{m}_1 + \dots + \vec{m}_M \pmod{N}} \\ \vec{m}_r &\in (\mathbb{Z}_N)^M. \end{aligned} \quad (23)$$

The 'structure constants' of the alternating finite dimensional  $M$ -algebras

$$\begin{aligned} \mathbb{A}_N &:= (V^{(N)}, [\dots]_*), \\ f_{\vec{m}_1 \dots \vec{m}_M}^{(N)\vec{m}} &:= \frac{-iN}{2\pi\tilde{N}} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1, \dots, \vec{m}_M)\right) \cdot \delta_{\vec{m}_1 + \dots + \vec{m}_M \pmod{N}}^{\vec{m}} \end{aligned} \quad (24)$$

satisfy (14) (up to an  $N$  and  $\mathbb{Z}_N^M$ -independent rescaling of the generators, resp. factors of  $i$ , which anyway drop out in (10) and (11);  $n = N^M$ ,  $f^{(N)} \stackrel{\Delta}{=} f(\lambda_n)$ ,  $\vec{m} \in \mathbb{Z}_N^M$ ,  $V^{(N)} = V_{n=N^3}$ , and  $\lim_{N \rightarrow \infty} V^{(N)} = V$ ).

$$\begin{aligned} H_N &= \frac{1}{2} p_{i-\vec{m}} p_{i\vec{m}} \\ &+ \frac{1}{2} \frac{N^2}{4\pi^2 \tilde{N}^2} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1 \dots \vec{m}_M)\right) \cdot \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{n}_1, \dots, \vec{n}_M)\right) \\ &\frac{1}{M!} \cdot x_{i_1-\vec{m}_1} \dots x_{i_M-\vec{m}_M} x_{i_1\vec{n}_1} \dots x_{i_M\vec{n}_M} \delta_{\vec{n}_1 + \dots + \vec{n}_M \pmod{N}}^{\vec{m}_1 + \dots + \vec{m}_M} \end{aligned} \quad (25)$$

could therefore be considered as a finite-dimensional analogue of (1).

### 3. Multidimensional Commutation Relations

Before turning to questions of symmetry, let me discuss in a little more detail the  $*M$ -algebras  $\mathbb{M}^\Lambda$ , with defining relations (cp. (18); note the slight change of notation/normalisation)

$$(T_{\vec{m}_1} \cdots T_{\vec{m}_M})_* = \omega^{-\frac{1}{2}(\vec{m}_1, \dots, \vec{m}_M)} T_{\vec{m}_1 + \dots + \vec{m}_M} (*),$$

and as vectorspaces generated by basis-elements  $T_{\vec{m}}$ ,  $\vec{m} \in S$  (where  $S = \mathbb{Z}^M$ ,  $S = (\mathbb{Z}_N)^M$ , or any combination thereof – in the  $M$ -brane context, depending on whether  $\Sigma = T^M$ , resp. a fully, or partially, discretized  $M$ -torus).

First of all note, that for any  $M$  elements,  $A_1, \dots, A_M \in V$ , any even permutation  $\sigma \in S_M$  (the symmetric group in  $M$  objects), and any choice of  $S$  (even  $\mathbb{R}^M$ ),

$$(A_1 \cdots A_M)_* = (A_{\sigma(1)} \cdots A_{\sigma(M)}) \quad (\text{sign } \sigma = +), \quad (26)$$

and that  $E := T_{\vec{0}}$  acts as a ‘unity’ in the sense that if one of the  $A_r$  is equal to  $T_{\vec{0}}$ , the  $*M$ -product becomes commutative (i.e. independent of the order of its  $M$  entries).

Using  $E$ , one may identify  $T_{(m=\pm|m|, 0, \dots, 0)}$  with the  $|m|$ -th power of  $E_{\pm 1} := T_{(\pm 1, 0, \dots, 0)}$ ,

$$T_{(m, 0, \dots, 0)} = \underbrace{\left( \left( \left( \left( E \cdots E E_{\pm 1} \right)_* \cdots E E_{\pm 1} \right)_* \cdots \right)_* \cdots E E_{\pm 1} \right)_*}_{\substack{\uparrow \\ |m| \text{ brackets}}}, \quad (27)$$

so that one may wonder whether  $\mathbb{M}^\Lambda$  can be thought of as being generated by

$$E = T_{\vec{0}}, E_{\pm 1} = T_{(\pm 1, 0, \dots, 0)}, \dots, E_{\pm M} = T_{(0, \dots, 0, \pm 1)}.$$

This is indeed the case: Let  $\mathbb{F}^M$  be the free (non associative)  $M$ -algebra generated by  $2M + 1$  elements  $E, E_{\pm 1}, \dots, E_{\pm M}$ ; define arbitrary powers  $(E_r)^m$  of the generating elements as in (27) (from now on  $E_{-r}^{|m|} := E_r^{-|m|}$ , a notation that will be justified via (29)), and let

$$E_{\vec{m}} := E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}. \quad (28)$$

Divide  $\mathbb{F}^M$  by the ideal generated by elements

$$E_{\vec{m}'} E_{\vec{m}''} \cdots E_{\vec{m}^{(M)}} - \omega^{\gamma(\vec{m}', \vec{m}'', \dots, \vec{m}^{(M)})} \cdot E_{\vec{m}' + \dots + \vec{m}^{(M)}} \quad (29)$$

where  $\omega = e^{4\pi i \Lambda}$  and

$$\begin{aligned} 2\gamma(\vec{m}', \dots, \vec{m}^{(M)}) &:= (m_1 \cdot m_2 \cdot \cdots \cdot m_M) - (\vec{m}', \vec{m}'', \dots, \vec{m}^{(M)}) \\ &\quad - \sum_{r=1}^M \left( \prod_{s=1}^M m_s^{(r)} \right) \\ (\vec{m} &:= \vec{m}' + \vec{m}'' + \cdots + \vec{m}^{(M)}). \end{aligned} \quad (30)$$

This quotient then is isomorphic to  $\mathbb{M}^\Lambda$ , as can be seen by defining

$$T_{\vec{m}} := \omega^{\frac{1}{2} m_1 m_2 \cdots m_M} E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}, \quad (31)$$

which (due to (29) being zero in  $\mathbb{F}^\Lambda/I$ ) satisfies (18) (with  $Y$  standing for  $T$ ).

Note that

$$E_2^{m_2} E_1^{m_1} E_3^{m_3} \cdots E_M^{m_M} = \omega^{m_1 m_2 \cdots m_M} \cdot E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}, \tag{32}$$

in particular:

$$E_2 E_1 E_3 \cdots E_M = \omega E_1 E_2 \cdots E_M \tag{33}$$

(while any even permutation does not alter the product, cp. (26)).

In order to get a feeling for (29)/(30) it may be instructive to consider  $M = 3$ : due to (29),

$$\begin{aligned} & (E_1^{n_1} E_2^{n_2} E_3^{n_3})(E_1^{l_1} E_2^{l_2} E_3^{l_3})(E_1^{k_1} E_2^{k_2} E_3^{k_3}) \\ = & E_1^{n_1+l_1+k_1} E_2^{n_2+l_2+k_2} E_3^{n_3+l_3+k_3} \\ & \cdot \omega^{n_1 l_3 k_2 + n_2 l_1 k_3 + n_3 l_2 k_1} \\ & \cdot \sqrt{\omega}^{n_1(l_2 l_3 + k_2 k_3) + n_2(l_1 l_3 + k_1 k_3) + n_3(l_1 l_2 + k_1 k_2)} \\ & \cdot \sqrt{\omega}^{n_1 n_2(l_3 + k_3) + n_1 n_3(l_2 + k_2) + n_2 n_3(l_1 + k_1)} \end{aligned} \tag{34}$$

The general rule (30) can hence be stated as follows:

Consider all possible triples (resp.  $M$ -tuples) containing powers of each of the  $E_r$  ( $r = 1 \cdots M$ ) exactly once. If the ‘contraction’ picks out exactly one factor from each of the 3 (resp.  $M$ ) factors in (34) it does not contribute if they are already in the correct order, modulo even permutations (cp. 26), (like  $E_1^{n_1} E_2^{l_2} E_3^{k_3}$ , or  $E_2^{n_2} E_3^{l_3} E_1^{k_1}$ ), while they contribute a factor  $\omega^{(\text{product of the } E\text{-powers})}$ , when they are not in the correct (modulo even permutation) order (like  $E_2^{n_2} E_1^{l_1} E_3^{k_3}$ ). Contractions entirely within one of the factors don’t contribute, while mixed contractions (involving at least 2, but not all, of the factors in (34)), all contribute a factor  $\sqrt{\omega^{(\text{product of the } E\text{-powers})}}$ , irrespective of their order.

Due to (32), all ‘monomials’ are proportional to one of the elements  $E_{\vec{m}}$  (cp. (28)) – which therefore form a basis (with the convention  $E_{\vec{0}} \equiv E$ ). Note that  $2\pi M! \Lambda = \lambda \rightarrow 0$  is a ‘classical limit’ (resp.  $\lambda \neq 0$  a ‘quantisation’ of the classical Nambu-structure) as, formally,

$$[\ln E_1, \ln E_2, \cdots, \ln E_M] = i \lambda E. \tag{35}$$

Having obtained this relation, one could of course start with objects  $\ln E_r =: J_r$ ,  $[J_1, J_2, \cdots, J_M] = i \lambda E$ , and derive generalized ‘Hausdorff-formulae’ for products involving the  $e^{i m_r J_r}$ .

Of course, (35) cannot be true in any  $M$ -algebra containing only finite linear combinations of the basis-elements  $E_{\vec{m}}$ , as  $T_{\vec{0}} = E$  never appears on the r.h.s. of (20); this is similar to the fact that the canonical commutation relations of ordinary quantum mechanics,  $[q, p] = i \hbar \mathbf{1}$ , cannot hold for trace-class operators. (35) may be justified by formally

expanding  $\ln E_r = - \sum_{n_r=1}^{\infty} \sum_{k_r=0}^{n_r} \binom{n_r}{k_r} \frac{(-)^{k_r}}{n_r} E_r^k$ , using

$$[E_1^{k_1}, E_2^{k_2}, \cdots, E_M^{k_M}] = \frac{M!}{2} (1 - \omega^{k_1 \cdots k_M}) E_1^{k_1} \cdots E_M^{k_M}$$



and then resumming recursively, after the first step obtaining

$$\frac{M!}{2} \ln E_1 \cdots \ln E_M - \frac{M!}{2} \sum_{\substack{n_r, k_r \\ r > 1}} \ln(E_1 \omega^{k_2 \cdots k_M}) E_2^{k_2} \cdots E_M^{k_M} = \frac{M!}{2} (\ln \omega) \cdot E, \quad (36)$$

as formally,

$$\sum_{n_r=1}^{\infty} \sum_{k_r=1}^{n_r} \binom{n_r}{k_r} \frac{(-)^{k_r}}{n_r} k_r E_r^k = E_r \cdot \sum_{n'=0}^{\infty} (E - E_r)^{n'} = E.$$

## 4. Breakdown of Conventional Symmetries

Let us now discuss the question, whether theories like (5) or (6) can have symmetries reminiscent of volume preserving diffeomorphisms; in particular whether the generators (2) may be ‘translated’ to finite dimensional analogues. \* For simplicity, consider again  $\Sigma = T^M$ .

As  $f^r = \partial_s \omega^{rs} = \epsilon^{rsr_1 \cdots r_{M-2}} \partial_s \omega_{r_1 \cdots r_{M-2}}$  for non-constant (divergence-free) vector-fields on  $T^M$ , (2) may be written in the form

$$K_{r_1 \cdots r_{M-2}} = \int d^M \varphi \omega_{r_1 \cdots r_{M-2}} \{p_i, x^i, \varphi^{r_1}, \dots, \varphi^{r_{M-2}}\}, \quad (37)$$

resp., in Fourier-components,

$$K_{r_1 \cdots r_{M-2}}^{\vec{l}} = \sum_{\substack{\vec{m}, \vec{n} \\ \in \mathbb{Z}^M}} \delta_{\vec{m}+\vec{n}}^{\vec{l}} p_{i\vec{m}} x_{i\vec{n}} (\vec{m}, \vec{n}, \vec{e}_{r_1}, \dots, \vec{e}_{r_{M-2}}) \quad (38)$$

(where  $\vec{e}_r$  denotes the unit vector in the  $r$ -direction).

Suppose the deformed theory was invariant under transformations that are still generated in a conventional way by phase-space functions of the form

$$K^{\vec{l}} = \sum_{\vec{m}, \vec{n} \in S} p_{i\vec{m}} x_{i\vec{n}} \delta_{\vec{m}+\vec{n}}^{\vec{l}} c_{\vec{m}\vec{n}}. \quad (39)$$

Using  $[x_{i\vec{m}}, p_{j\vec{n}}] = \delta_{ij} \delta_{\vec{m}\vec{n}}^{-}$ , while leaving open whether  $S = \mathbb{Z}^M$  or  $S = (\mathbb{Z}_N)^M$  as well as (independently) whether  $\delta$  is defined mod  $N$ , or not, one has

$$[K^{\vec{l}}, \tilde{K}^{\vec{l}'}] = \sum_{\substack{\vec{m}_1, \vec{n}_1 \\ \in S}} p_{i\vec{m}_1} x_{i\vec{n}_1} \delta_{\vec{m}_1+\vec{n}_1}^{\vec{l}+\vec{l}'} \tilde{c}_{\vec{m}_1\vec{n}_1} \quad (40)$$

with

$$\tilde{c}_{\vec{m}\vec{n}} = \sum_{\vec{k} \in S} \left( \delta_{\vec{k}}^{\vec{l}-\vec{m}} \delta_{-\vec{k}}^{\vec{l}'-\vec{n}} c_{\vec{m}\vec{k}} \tilde{c}_{-\vec{k}\vec{n}} - \left( \vec{l} \leftrightarrow \vec{l}' \right) \right),$$

\*For  $M = 2$ , this question was already considered in [4] and answered positively.

while  $K^{\vec{l}} = 0$  would require  $c_{\vec{m}\vec{n}} = \dots = c_{\vec{n}\vec{m}}$  and

$$\begin{aligned} & \sin(2\pi\Lambda(\vec{a}_1, \dots, \vec{a}_M)) \sin(2\pi\Lambda(\vec{a}_1 + \dots + \vec{a}_M, \vec{a}_2', \dots, \vec{a}_M')) \\ & \cdot c_{\vec{a}_1 + \dots + \vec{a}_1' + \dots + \vec{a}_M', \vec{a}_1'} \cdot x_{i_1 \vec{a}_1} x_{i_1 \vec{a}_1'} \cdots x_{i_M \vec{a}_M} x_{i_M \vec{a}_M'} = 0 \end{aligned} \tag{41}$$

(where for (41) consistency of the  $\delta$ -functions used in (39) and  $(25)_\Lambda$  with the index set  $S$  was assumed).

The effect of the  $x_{i\vec{m}}$ -factors in (41) is to make the product  $\sin \cdot \sin \cdot c$ , symmetric under any interchange  $\vec{a}_r \leftrightarrow \vec{a}_r'$ , as well as any simultaneous interchange  $\vec{a}_r \leftrightarrow \vec{a}_s$ ,  $\vec{a}_r' \leftrightarrow \vec{a}_s'$ . Choosing, e.g.,  $\vec{a}_r' = \vec{a}_r$  ( $r = 1 \cdots M$ ), with  $\sin(2\pi\Lambda(\vec{a}_1 \cdots \vec{a}_M)) \neq 0$ , (41) requires that

$$\sum_{\sigma \in S_M} c_{\vec{a}_{\sigma 1} + 2(\vec{a}_{\sigma 2} + \dots + \vec{a}_{\sigma M}), \vec{a}_{\sigma 1}} = 0. \tag{42}$$

This condition is insensitive to any alteration of the deformation: replacing the sine-function in (41) (resp.  $(25)_\Lambda, \dots$ ) by any other function of the determinant will still result in (42) as a necessary condition. Apart from  $M = 2$  ( $c_{\vec{a}_1 + 2\vec{a}_2, \vec{a}_1} + c_{\vec{a}_2 + 2\vec{a}_1, \vec{a}_2} = 0$  is trivially satisfied by any odd function) (42) is not satisfied by

$$c_{\vec{m}\vec{n}} = \sin(2\pi\Lambda(\vec{m}, \vec{n}, \vec{k}_1, \dots, \vec{k}_{M-2})), \tag{43}$$

--- nor would (40) be a linear combination of the generators (39), for such a  $c_{\vec{m}\vec{n}}$ ; for  $M = 3$ , e.g., one would obtain

$$\begin{aligned} & \widetilde{c}_{\vec{m}\vec{n}}(\vec{l}, \vec{l}'; \vec{k}, \vec{k}') \\ & = \sin\left(2\pi\Lambda\left(\vec{l}, \vec{l}', \frac{\vec{k} + \vec{k}'}{2}\right)\right) \\ & \quad \cdot \sin\left(2\pi\Lambda\left(\left(\vec{m}, \vec{n}, \frac{\vec{k} + \vec{k}'}{2}\right) + \left(\vec{m} - \vec{n}, \frac{\vec{l} - \vec{l}'}{2}, \frac{\vec{k} - \vec{k}'}{2}\right)\right)\right) \\ & - \sin\left(2\pi\Lambda\left(\vec{l}, \vec{l}', \frac{\vec{k} - \vec{k}'}{2}\right)\right) \\ & \quad \cdot \sin\left(2\pi\Lambda\left(\vec{m}, \vec{n}, \frac{\vec{k} - \vec{k}'}{2}\right) + \left(\vec{m} - \vec{n}, \frac{\vec{l} - \vec{l}'}{2}, \frac{\vec{k} + \vec{k}'}{2}\right)\right) \end{aligned} \tag{44}$$

--- which means that the algebra closes only for  $\vec{k}' = \vec{k}$  (for  $\Lambda = \frac{1}{N}$  this would give  $N^3$  closed Lie algebras, each of dimension  $N^3$ ; in fact, each consisting of  $N$  copies of  $gl(N)$ ).

– In any case, if  $c_{\vec{m}\vec{n}}$  was a function of  $(\vec{m}_1 \vec{n}_1 \vec{k}_1, \dots, \vec{k}_{M-2})$ , one could let  $\vec{a}_2, \vec{a}_3, \dots, \vec{a}_M$  differ only in the ('irrelevant')  $\vec{k}_1, \dots, \vec{k}_{M-2}$  directions and obtain

$$f(((2M - 2)\vec{a}_2, \vec{a}_1, \dots)) + (M - 1)f((2\vec{a}_1, \vec{a}_2, \dots)) = 0, \tag{45}$$

which eliminates all  $c_{\vec{m}\vec{n}}$  that are non-linear functions of the determinant.

Interestingly,  $c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{something})_{if M > 2}$  is suggested by yet another consideration: replacing  $\{p_i, x_i, \varphi^3, \dots, \varphi^M\}$  (cp. (37); for notational simplicity taking  $r_1 = 3, \dots, r_{M-2} = M$ ) by

$$[P_i, X_i, \ln E_3, \dots, \ln E_M], \tag{46}$$

(with  $P_i = p_{i\vec{m}} T_{\vec{m}}, X_i = x_{i\vec{m}} T_{\vec{m}}$ ) formally expanding the logarithms in a power series, using (20), and then (formally) summing again, one obtains something proportional to

$$p_{i\vec{m}} x_{i\vec{n}} T_{\vec{m}+\vec{n}} \cdot (m_1 n_2 - m_2 n_1) . \tag{47}$$

$$\begin{aligned} & [P_i, X_i, \ln E_3, \dots, \ln E_M] \\ = & p_{i\vec{m}} x_{i\vec{n}} (-)^{M-2} \sum_{n_3=1}^{\infty} \sum_{k_3=0}^{n_3} \dots \sum_{n_M=1}^{\infty} \sum_{k_M=0}^{n_M} \binom{n_3}{k_3} \dots \binom{n_M}{k_M} \frac{(-)^{k_3+\dots+k_M}}{n_3 \dots n_M} \\ & \cdot [T_{\vec{m}}, T_{\vec{n}}, E_3^{k_3}, \dots, E_M^{k_M}] \\ \sim & \sum \dots \sin(2\pi \Lambda(\vec{m}, \vec{n}, k_3 \vec{e}_3, \dots, k_M \vec{e}_M)) \cdot T_{\vec{m}+\vec{n}+\vec{k}} \\ \sim & \sum \dots \left( \sqrt{\omega}^{k_3 \dots k_M} z - \sqrt{\omega}^{-k_3 \dots k_M} z \right) (\sqrt{\omega})^{\prod_{r=1}^M (m_r+n_r+k_r)} \cdot \\ & \cdot E_1^{m_1+n_1} E_2^{m_2+n_2} E_3^{m_3+n_3+k_3} \dots E_M^{m_M+n_M+k_M} \\ \sim & \sum \dots \left( \ln \left( \sqrt{\omega}^{k_4 \dots k_M} z + \prod_{r \neq 3} (m_r+n_r+k_r) E_3 \right) \right. \\ & \left. - \ln \left( \sqrt{\omega}^{-k_4 \dots k_M} z + \prod_{r \neq 3} (\dots) E_3 \right) \right) \cdot \sqrt{\omega}^{(m_3+n_3) \cdot \prod_{r \neq 3} (\dots)} \\ & \cdot E_1^{m_1+n_1} E_2^{m_2+n_2} E_3^{m_3+n_3} E_4^{m_4+n_4+k_4} \dots E_M^{m_M+n_M+k_M} \\ & \left( z := (\vec{m}, \vec{n}, \vec{e}_3, \dots, \vec{e}_M) = m_1 n_2 - m_2 n_1 \right) \\ & \left( \vec{k} = (0, 0, k_3, \dots, k_M) \right) \\ = & (\ln \omega) p_{i\vec{m}} x_{i\vec{n}} z (\vec{m}, \vec{n}) \sqrt{\omega}^{\prod_{r=1}^M (m_r+n_r)} E_1^{m_1+n_1} \dots E_M^{m_M+n_M} \\ = & (m_1 n_2 - m_2 n_1) p_{i\vec{m}} x_{i\vec{n}} (\ln \omega) \cdot T_{\vec{m}+\vec{n}} \\ & \text{where (for } r > 3) - \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} \frac{(-)^k}{n} k \cdot E_r^k \cdot (\omega^{\dots})^k = E \text{ was used.} \end{aligned}$$

However,

$$c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{anything}) \tag{48}$$

does not satisfy (41). Moreover, even if one considers more general deformations of the Hamiltonian, i.e. replacing the sine-function in (41) by an arbitrary odd (power-series) function  $f$  of the determinant, the corresponding condition,

$$\begin{aligned} & f(\vec{a}_1, \dots, \vec{a}_M) f(\vec{a}_1 + \dots + \vec{a}_M, \vec{a}'_2, \dots, \vec{a}'_M) \cdot (\vec{e}, \vec{a}'_1, \dots) = 0 \\ & + (M \cdot 2^M - 1) \text{ permutations,} \end{aligned} \tag{49}$$

$\vec{e} = \sum_{r=1}^M (\vec{a}_r + \vec{a}'_r)$ , can never be satisfied by any non-linear function  $f$  – as one can see, e.g., by choosing  $\vec{a}'_r = \mu_r \vec{a}_r$ . Supposing  $f(x) = \alpha x + \beta x^{2n+1} = \dots$ , and denoting  $(\vec{a}_1, \dots, \vec{a}_M)$  by  $z$ ,  $\prod_{r=1}^M \mu_r$  by  $\mu$ , the terms  $\mu_1, \alpha z \beta (\mu z)^{2n+1}$ , e.g., (occurring only twice, with the same sign) could never cancel.

The preceding arguments possibly suffice to prove that, independent of the above dynamical context, the Lie algebra of volume-preserving diffeomorphisms of  $T^{M>2}$  does not possess any non-trivial deformations.\*

## 5. Rigidity of Canonical Nambu-Poisson Manifolds

For the multilinear antisymmetric map (4), and  $2M - 1$  arbitrary functions  $f_1, \dots, f_{2M-1}$ , one has (cp. [5]):

$$\begin{aligned} & \{ \{ f_M, f_1, \dots, f_{M-1} \}, f_{M+1}, \dots, f_{2M-1} \} \\ & + \{ f_M, \{ f_{M+1}, f_1, \dots, f_{M-1} \}, f_{M+2}, \dots, f_{2M-1} \} \\ & + \dots + \{ f_M, \dots, f_{2M-2}, \{ f_{2M-1}, f_1, \dots, f_{M-1} \} \} \\ & = \{ \{ f_M, \dots, f_{2M-1} \}, f_1, \dots, f_{M-1} \} . \end{aligned} \quad (50)$$

Takhtajan [5], stressing its relevance for time-evolution in Nambu-mechanics [1], named (50) ‘Fundamental Identity (FI)’, and defined a ‘Nambu-Poisson-manifold of order  $M$ ’ to be a manifold  $X$  together with a multilinear antisymmetric map  $\{\dots\}$  satisfying (50) and the Leibniz-rule

$$\{ f_1 \tilde{f}_1, f_2, \dots, f_M \} = f_1 \{ \tilde{f}_1, f_2, \dots, f_M \} + \{ f_1, \dots, f_M \} \tilde{f}_1 \quad (51)$$

for functions  $f_r : X \rightarrow \mathbb{R}$  (or  $\mathbb{C}$ ).

Without (51), i.e. just demanding (50) for an antisymmetric  $M$  linear map:  $V \times \dots \times V \rightarrow V$ ,  $V$  some vectorspace, Takhtajan defines a ‘Nambu-Lie-gebra’ [5], – also called ‘Fillipov [6] Lie algebra’ [7]). I would like to point out various other identities satisfied by canonical Nambu-Poisson brackets (4), and show that all of them – including (50)! – do not allow deformations (of certain natural type), if  $M > 2$ .

At least from a non-dynamical point of view, all identities involving Nambu-brackets obtained from antisymmetrizing the product of two determinants formed from  $2M - M$  vectors,

$$(\vec{a}_1 \dots \vec{a}_M)(\vec{a}_{M+1} \dots \vec{a}_{2M}) \quad (52)$$

with respect to  $M + 1$  of the  $\vec{a}_\alpha$  ( $\alpha = 1 \dots 2M$ ) should be treated on an equal footing. For  $M = 3$ , e.g., one has – apart from

$$\begin{aligned} & (\vec{a} \vec{b} \vec{c}_1)(\vec{c}_2 \vec{c}_3 \vec{c}_4) - (\vec{a} \vec{b} \vec{c}_2)(\vec{c}_3 \vec{c}_4 \vec{c}_1) \\ & + (\vec{a} \vec{b} \vec{c}_3)(\vec{c}_4 \vec{c}_1 \vec{c}_2) - (\vec{a} \vec{b} \vec{c}_4)(\vec{c}_1 \vec{c}_2 \vec{c}_3) = 0 , \end{aligned} \quad (53)$$

which gives rise to  $(50)_{M=3}$  for functions  $f \in T^3$  – also

$$(a \vec{c}_1 \vec{c}_2)(\vec{c}_3 \vec{c}_4 \vec{b}) = 0 , \quad (54)$$

---

\*M. Bordemann has informed me that apparently an even more general statement of this nature has recently been proven in [19].

yielding the following 6-term identity (FI)<sub>6</sub> (which can of course also be proven by using just the definition (4),  $\{f, g, h\} = \epsilon_{\alpha\beta\gamma} \partial_\alpha f \partial_\beta g \partial_\gamma h$ , rather than (54); i.e. not necessarily specifying the manifold  $X$ ):

$$\{\{f, f_{[1}, f_2\} f_3, f_4\} = 0 \tag{55}$$

as well as the 4-term identity ( $\tilde{\text{FI}}$ ),

$$\begin{aligned} & \{\{f, f_1, f_2\}, g, f_3\} \\ & + \{\{f, f_2, f_3\}, g, f_1\} \\ & + \{\{f, f_3, f_1\}, g, f_2\} = - \{f, g, \{f_1, f_2, f_3\}\} \end{aligned} \tag{56}$$

- - - each of which is independent of (50)<sub>M=3</sub> (while any 2 of the 3 identities yield the 3<sup>rd</sup>).

Naively, one would think that (56) should follow from (50)<sub>3</sub> alone, as (54) follows from (53) (perhaps one should note that for general  $M$ , a theorem concerning vector invariants [8] states, that any (!) vector-bracket identity is an algebraic consequence of

$$(\vec{a}_{[1} \vec{a}_2 \cdots \vec{a}_M) (\vec{a}_{M+1]} \cdots \vec{a}_{2M}) = 0 ;$$

however, in the proof of (56) via vector-bracket identities, one in particular needs (54) for the special case  $\vec{a} = \vec{b}$  – which cannot be stated as an identity between functions on  $X$ .) Curiously (with respect to a statistical approach to  $M$ -branes), vector-bracket identities (‘Basis Exchange Properties’ [9]) also play an important role in combinatorial geometry.

From an aesthetic point of view, the most natural quadratic identity for (4) is

$$\sum_{\sigma \in S_{2M-1}} (\text{sign } \sigma) \{\{f_{\sigma_1}, \dots, f_{\sigma_M}\} f_{\sigma_{M+1}}, \dots, f_{\sigma_{2M-1}}\} = 0 . \tag{57}$$

For  $M = 3$ , e.g., one could see this to be a consequence of (50)<sub>3</sub> and (56) by grouping the 10 distinct terms in (57) according to whether  $\{f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}\}$  contains both  $f_4$  and  $f_5$  (3 terms, ‘type A’), only one of them (3 ‘B-terms’ and 3 ‘C-terms’) or none of them (1 term, ‘type D’); for the B (resp. C)-terms one can use (56) while (50) for the A-terms, to get  $\pm \{f_4, f_5, \{f_1 f_2 f_3\}\}$  for each of the 4 types, and for the B and C-terms with a sign opposite to the one obtained from the D (and A) term(s). (57) (taken without the derivation-requirement) is a beautiful generalisation of Lie-algebras ( $M = 2$ ), and has recently started to attract the attention of mathematicians – mostly under the name of  $(M - 1)$ -ary Lie algebras [10 - 13]. \*

Unfortunately, all identities (50), (55)–(57), can be shown to be rigid, in the following sense: assuming that

$$[T_{\vec{m}_1}, \dots, T_{\vec{m}_M}]_\lambda = g_\lambda ((\vec{m}_1, \dots, \vec{m}_M)) T_{\vec{m}_1 + \dots + \vec{m}_M} \tag{58}$$

with  $g_\lambda(x)$  a smooth odd function proportional to  $x + \lambda^n c x^n$  as  $\lambda \rightarrow 0$  ( $n > 1$ ) any of the above identities will require the constant  $c$  to be equal to zero (I have proved this

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\*I would like to thank W. Soergel for mentioning refs. [10]/[11] to me and J.L. Loday for sending me a copy of [10] and [12]; also, I would like to express my gratitude to R. Chatterjee and L. Takhtajan for sending me their papers on Nambu Mechanics (cp. [5]).

only for  $M = 3$ , and in the case of (57) – the a priori most promising case – for general  $M > 2$ ).

Concerning

$$\begin{aligned}
 & g_\lambda \left( (\vec{a}, \vec{b}, \vec{c}_1) \right) g_\lambda \left( (\vec{a} + \vec{b} + \vec{c}_1, \vec{c}_2, \vec{c}_3) \right) \\
 + & g_\lambda \left( (\vec{a}, \vec{b}, \vec{c}_2) \right) g_\lambda \left( (\vec{a} + \vec{b} + \vec{c}_2, \vec{c}_3, \vec{c}_1) \right) \\
 + & g_\lambda \left( (\vec{a}, \vec{b}, \vec{c}_3) \right) g_\lambda \left( (\vec{a} + \vec{b} + \vec{c}_3, \vec{c}_1, \vec{c}_2) \right) \\
 \stackrel{!}{=} & g_\lambda \left( (\vec{c}_1, \vec{c}_2, \vec{c}_3) \right) g_\lambda \left( (\vec{c}_1 + \vec{c}_2 + \vec{c}_3, \vec{a}, \vec{b}) \right) , \tag{59}
 \end{aligned}$$

i.e. the deformation of  $(50)_{M=3}$ , one could assume  $z := (\vec{c}_1, \vec{c}_2, \vec{c}_3) \neq 0$ ,  $\vec{a} = \sum_1^3 \alpha_r \vec{c}_r$ ,  $\vec{b} = \sum_1^3 \beta_r \vec{c}_r$ , such that  $g(y) := \bar{g}_\lambda(y) := g_\lambda(zy)$  must satisfy

$$\begin{aligned}
 & g(\alpha_2 \beta_3 - \alpha_3 \beta_2) \cdot g(1 + \alpha_1 + \beta_1) \\
 & + \text{cyclic permutations} \tag{60} \\
 & = g(1) \cdot g(\alpha_2 \beta_3 - \alpha_3 \beta_2 + \text{cycl.})
 \end{aligned}$$

for all  $\alpha_r, \beta_r$ ; which is clearly impossible for any nonlinear  $g$  of the required form. (e.g., as in next to lowest order in  $\lambda$  the terms  $\alpha_1(\alpha_2 \beta_3)^{n>1}$  appear only once).

Similarly, the deformation of (56) is impossible due to the analogous requirement

$$\begin{aligned}
 & g(\alpha_3) g(\beta_2 - \beta_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1)) + \text{cycl.} \\
 \stackrel{!}{=} & -g(1) g((\alpha_1 \beta_2 - \alpha_2 \beta_1) + \text{cycl.}) . \tag{61}
 \end{aligned}$$

Finally, concerning possible deformations of (57), let  $(\vec{a}_1, \dots, \vec{a}_M) \neq 0$ , and

$$\vec{a}_{M+\bar{r}} = \sum_{s=1}^M \alpha_s^{(\bar{r})} \vec{a}_s \quad (\bar{r} = 1, \dots, M-1);$$

$$\text{then } g(1 + \alpha_1^{(1)} + \dots + \alpha_1^{(M-1)}) \cdot g \left( \underbrace{\begin{pmatrix} 1 \\ 0 \ \vec{\alpha}^{(1)} \dots \vec{\alpha}^{(M-1)} \\ \vdots \\ 0 \end{pmatrix}}_{=: [1]} \right) ,$$

e.g., contains (in next to lowest order in  $\lambda$ ) a term  $\alpha_1^{(1)} \cdot \alpha_1^{(2)} \cdot [1]$  (of total degree  $(M + 1)$  in the  $\alpha_s^{(\bar{r})}$ ), which cannot appear anywhere else (in the same order in  $\lambda$ ), – in contradiction to the assumption that (57) should hold for  $[\dots]_\lambda$  (cp. (58)) replacing the curly bracket (4).

## 6. A Remark on Generalized Schild Actions

Consider

$$S := - \int d\varphi^0 d^M \varphi f(G), \quad (62)$$

where  $G := (-)^M \det (G_{\alpha\beta})$ ,  $G_{\alpha\beta} := \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\mu\nu}$ ,  $\eta_{\mu\nu} = \text{diag} (1, -1, \dots, -1)$ ,  $\alpha, \beta = 0, \dots, M$  and  $f$  some smooth monotonic function like  $G^\gamma$  ( $\gamma = 1$  resp.  $\frac{1}{2}$  corresponding to a generalized Schild-, resp. Nambu-Goto, action for  $M$ -branes). Apart from a few subtleties (like  $\gamma = 1$  allowing for vanishing  $G$ , while  $\gamma = \frac{1}{2}$  does not) actions with different  $f$  are equivalent, in the sense that the equations of motion,

$$\partial_\alpha (f'(G) G^{\alpha\beta} \partial_\beta x^\mu) = 0 \quad \mu = 0 \dots D-1 \quad (63)$$

are easily seen to imply

$$\partial_\alpha G = 0 \quad \alpha = 0, \dots, M \quad (64)$$

(just multiply (63) by  $\partial_\epsilon x_\mu$  and sum) – unless  $f(G) = \text{const.} \sqrt{G}$ , in which case (62) is fully reparametrisation invariant and a parametrisation may be assumed in which  $G = \text{const.}$  (such that (63) becomes proportional to  $\partial_\alpha (G^{\alpha\beta} \partial_\beta x^\mu)$  also in this case). Due to

$$G = \sum_{\mu_1 < \dots < \mu_{M+1}} \{x^{\mu_1}, \dots, x^{\mu_{M+1}}\} \{x_{\mu_1}, \dots, x_{\mu_{M+1}}\} \quad (65)$$

(63) may be written as (cp. [14] for strings, and [15] for membranes, in the case of  $\gamma = 1$  resp.  $\frac{1}{2}$ )

$$\{f'(G)\{x^{\mu_1}, \dots, x^{\mu_{M+1}}\}, x_{\mu_2}, \dots, x_{\mu_{M+1}}\} = 0, \quad (66)$$

whose deformed analogue (note the similarity between  $G = \text{const.}$  and condition (3.9) of [16])

$$[[x^{\mu_1}, \dots, x^{\mu_{M+1}}], x_{\mu_2}, \dots, x_{\mu_{M+1}}] = 0 \quad (67)$$

looks very suggestive when thinking about space-time quantization in  $M$ -brane theories.

## 7. Multidimensional Integrable Systems from M-algebras

Several ideas used in the context of integrable systems are based on bilinear operations. Our problems to extend results about low (especially 1+1) dimensional integrable field theories to higher dimensions may well rest on precisely this fact. Already some time ago, attempts were made to overcome this difficulty by generalizing Lax-pairs to -triples ([3], p. 72) and Hirota's bilinear equations for ' $\tau$ -functions' [17] to multilinear equations ([3], p. 107-111).

At that time, good examples were lacking, and – not being an exception to the rule that generalisations involving the number of dimensions (of one sort or an other) are usually hindered by implicitly low dimensional point(s) of view – the proposed generalisation of

Hirota-operators may have still been too naive; while hoping to come back to the question of multidimensional  $\tau$ -functions in the near future, I would now like to give an example ( $M > 3$  will then be obvious) for an equation of the form

$$\dot{\mathcal{L}} = \frac{1}{\rho} \{\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2\} \quad (68)$$

being equivalent to the equations of motion of a compact 3 dimensional manifold  $\widehat{\Sigma} \subset \mathbb{R}^4$  (described by a time-dependent 4-vector  $x^i(\varphi^1, \varphi^2, \varphi^3, t)$ ), moving in such a way that its normal velocity is always equal to the induced volume density  $\sqrt{g}$  (on  $\widehat{\Sigma}$ ) divided by a fixed non-dynamical density  $\rho(\varphi)$  ('the' volume density of the parameter manifold):

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\rho} \{x_2, x_3, x_4\} \\ \dot{x}_2 &= -\frac{1}{\rho} \{x_3, x_4, x_1\} \\ \dot{x}_3 &= \frac{1}{\rho} \{x_4, x_1, x_2\} \\ \dot{x}_4 &= -\frac{1}{\rho} \{x_1, x_2, x_3\} . \end{aligned} \quad (69)$$

With the curly bracket defined as before (cp. (4)), it will be an immediate consequence of (68) that

$$Q_n := \int_{\Sigma} d^3\varphi \rho(\varphi) \mathcal{L}^n \quad (70)$$

is time-independent (for any  $n$ ).

In [2] evolution-equations of the form (69) (in any number of dimensions) were shown to correspond to the diffeomorphism invariant part of an integrable Hamiltonian field theory (as well as to a gradient flow); one way to solve such equations is to note ([18], [2]) that the time at which the hypersurface will pass a point  $\vec{x}$  in space will simply be a harmonic function.

In any case, the (a) form of  $(\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2)$  that will yield the equivalence of (69) with (68) is:

$$\begin{aligned} \mathcal{L} &= (x_1 + ix_2) \frac{1}{\lambda} + (x_3 + ix_4) \frac{1}{\mu} + \mu(x_3 - ix_4) - \lambda(x_1 - ix_2) \\ \mathcal{M}_1 &= \frac{\mu}{2}(x_3 - ix_4) - \frac{1}{2\mu}(x_3 + ix_4) \\ \mathcal{M}_2 &= \frac{\lambda}{2}(x_1 - ix_2) + \frac{1}{2\lambda}(x_1 + ix_2) \end{aligned} \quad (71)$$

(involving two spectral parameters,  $\lambda$  and  $\mu$ ). Surely, this observation will have much more elegant formulations, and conclusions.



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