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On M-Algebras, the Quantisation of Nambu-Mechanics, and Volume Preserving Diffeomorphisms¹

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Dedicated to K. Hepp and W. Hunziker on the occasion of their 60th birthdays.

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Abstract: M-branes are related to theories on function spaces \mathcal{A} involving M -linear non-commutative maps from $\mathcal{A} \times \cdots \times \mathcal{A}$ to \mathcal{A} . While the Lie-symmetry-algebra of volume preserving diffeomorphisms of T^M cannot be deformed when $M > 2$, the arising M -algebras naturally relate to Nambu's generalisation of Hamiltonian mechanics, e.g. by providing a representation of the canonical M -commutation relations, $[J_1, \cdots, J_M] = i\hbar$. Concerning multidimensional integrability, an important generalisation of Lax-pairs is given.

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1. Introduction

Generalizing fundamental concepts, such as Lie algebras or Hamiltonian dynamics, may have quite diverse merits; it can lead to new, interesting possibilities, – or reassure oneself of our present notions. While the result that volume preserving diffeomorphisms of toroidal M -branes, as a Lie-symmetry algebra, cannot be deformed (if $M > 2$) is of the latter nature – the following ideas appear to be worthwhile pursuing:

– Using a $*M$ -deformation of the algebra of functions on some M -dimensional manifold for representing the M -linear analogue to Heisenberg's commutation relations that Nambu [1] encountered in multi-Hamiltonian dynamics.

– Generalizing the Jacobi identity for Lie algebras to a (2-bracket) identity involving $2M - 1$ elements of a vectorspace V for which an antisymmetric M -linear map (M -commutator) from $V \times \cdots \times V$ to V is defined (in a dynamical context, an identity involving M , rather than 2, of the M -commutators, may be preferred).

– A potential relevance of M -algebras to the quantisation of space-time.

Perhaps most importantly (on a concrete, practical level), an explicit example is given (the multidimensional diffeomorphism-invariant integrable field theories found in [2]) for the usefulness (envisaged some time ago [3]) of generalizing Lax-pairs to -triples,

2. M-algebras from M-branes

A relativistic M -brane moving in D -dimensional space time may be described, in a light-cone gauge, by the $\text{VDiff}\Sigma$ -invariant sector of ([4])

$$H = \frac{1}{2} \int_{\Sigma} \frac{d^M \varphi}{\rho(\varphi)} (\vec{p}^2 + g) \quad (1)$$

where g is the determinant of the $M \times M$ matrix $(g_{rs}) := (\nabla_r x^i \nabla_s x_i)_{r,s=1 \dots M}$, x^i and p_i ($i = 1, \dots, D - 2 =: d$) are canonically conjugate fields, and ρ is a fixed non-dynamical density on the M -dimensional parameter-manifold Σ ($M = 1$ for strings, $M = 2$ for membranes, ...). Generators of $\text{VDiff}\Sigma$, the group of volume-preserving diffeomorphisms of Σ (resp. the component connected to the identity), are represented by

$$K := \int_{\Sigma} f^r p_i \partial_r x^i d^M \varphi \quad (2)$$

with $\nabla_r f^r = 0$. g may be written as

$$g = \sum_{i_1 < i_2 < \dots < i_M} \{x_{i_1}, \dots, x_{i_M}\} \{x^{i_1}, \dots, x^{i_M}\}, \quad (3)$$

where the 'Nambu-bracket' $\{\dots\}$ is defined for functions f_1, \dots, f_M on Σ as

$$\{f_1, \dots, f_M\} := \epsilon^{r_1 \dots r_M} \partial_{r_1} f_1 \cdots \partial_{r_M} f_M. \quad (4)$$

This trivial, but important observation suggests to consider Hamiltonians

$$H_\lambda := \frac{1}{2} \text{Tr} \left(\vec{P}^2 \pm \sum_{i_1 < \dots < i_M} [X_{i_1}, \dots, X_{i_M}]_\lambda^2 \right), \quad (5)$$

resp.

$$H_\lambda = \frac{1}{2} \sum_{i=1}^d \beta(P_i, P_i) + \frac{1}{2} \sum_{i_1 < \dots < i_M} \beta([X_{i_1}, \dots, X_{i_M}]_\lambda, [X_{i_1}, \dots, X_{i_M}]_\lambda), \quad (6)$$

where X^i and P_i are elements of (possibly finite dimensional, λ -dependent) vectorspaces V on which antisymmetric M -linear maps $[\dots]_\lambda : V \times \dots \times V \rightarrow V$ are defined, and β a positive definite hermitean form, preferably invariant with respect to some analogue of volume preserving diffeomorphisms (cp. (2)).

With

$$[T_{a_1}, \dots, T_{a_M}]_\lambda = f_{a_1 \dots a_M}^a(\lambda) T_a \quad (7)$$

and

$$\beta(T_a, T_b) = \delta_b^a \quad (8)$$

for some (possibly λ -dependent) basis $\{T_a\}_{a=1}^{\dim V}$ of V , i.e.

$$f_{a_1 \dots a_M}^a(\lambda) = \beta(T_a, [T_{a_1}, \dots, T_{a_M}]_\lambda), \quad (9)$$

(6) reads

$$H_\lambda = \frac{1}{2} p_{ia}^* p_{ia} + \frac{1}{2} (f_{a_1 \dots a_M}^a(\lambda))^* f_{b_1 \dots b_M}^a(\lambda) \frac{1}{M!} x_{i_1 a_1}^* \dots x_{i_M a_M}^* x_{i_1 b_1} \dots x_{i_M b_M}, \quad (10)$$

while (1) may be written as

$$H = \frac{1}{2} p_{i\alpha}^* p_{i\alpha} + \frac{1}{2} (g_{\alpha_1 \dots \alpha_M}^\alpha)^* g_{\beta_1 \dots \beta_M}^\alpha \frac{1}{M!} x_{i_1 \alpha_1}^* \dots x_{i_M \beta_M}^* ; \quad (11)$$

$$g_{\alpha_1 \dots \alpha_M}^\alpha := \int_\Sigma Y_\alpha^* \{Y_{\alpha_1}, \dots, Y_{\alpha_M}\} \rho d^M \varphi \quad (12)$$

is defined with respect to some orthonormal basis of functions (on Σ) satisfying

$$\int Y_\alpha^* Y_\beta \rho d^M \varphi = \delta_\beta^\alpha \quad \alpha, \beta = 1 \dots \infty \quad (13)$$

(even for real x_i , it is often convenient to take a complex basis).
Obvious questions are:

- 1) Does there exist a ‘natural’ sequence of finite dimensional vectorspaces V_n with basis $\{T_a^{(n)}\}$ and antisymmetric maps $F_n : V_n \times \cdots \times V_n \rightarrow V_n$ such that for each $(M + 1)$ -tuple $(a_1 \cdots a_M)$

$$\lim_{n \rightarrow \infty} f_{a_1 \cdots a_M}^a(\lambda_n) \stackrel{?}{=} g_{a_1 \cdots a_M}^a. \tag{14}$$

- 2) For which M do there exist finite dimensional analogues of (2), $K(n)$, leaving $(10)_{\lambda_n}$ invariant, such that, as $n \rightarrow \infty$, the full invariance under volume-preserving diffeomorphisms is recovered?

- 3) What about λ -deformations with infinite dimensional V ’s ?

Let us look at the case of a M -torus, $\Sigma = T^M$:

Choosing

$$Y_{\vec{m}} = e^{i\vec{m}\vec{\varphi}}, \vec{m} = (m_1, \dots, m_M) \in \mathbb{Z}^M, \rho \equiv 1, \tag{15}$$

one gets

$$g_{\vec{m}_1 \cdots \vec{m}_M}^{\vec{m}} = i^M(\vec{m}_1, \dots, \vec{m}_M) \delta_{\vec{m}_1 + \cdots + \vec{m}_M}^{\vec{m}} \tag{16}$$

where $(\vec{m}_1, \dots, \vec{m}_M) \in \mathbb{Z}$ denotes the determinant of the corresponding $M \times M$ Matrix (an element of $GL(M, \mathbb{Z})$).

Consider now the following ‘ $*M$ -product’ (a deformation of the ordinary commutative product of M functions f_1, \dots, f_M on Σ):

$$(f_1 \cdots f_M)_* := f_1 \cdots f_M + \sum_{m=1}^{\infty} \frac{\left(\frac{(-i)^{M+1}\lambda}{M!}\right)^m}{m!} \frac{\epsilon^{r_1 r'_1 \cdots r_1^{(M)}}}{\epsilon^{r_m r'_m \cdots r_m^{(M)}}} \frac{\partial^m f_i}{\partial \varphi^{r_1} \cdots \partial \varphi^{r_m}} \cdots \frac{\partial^m f_M}{\partial \varphi^{r_1^{(M)}} \cdots \partial \varphi^{r_m^{(M)}}}. \tag{17}$$

One then finds that

$$(Y_{\vec{m}_1} \cdots Y_{\vec{m}_M})_* = \sqrt{\omega}^{- (\vec{m}_1, \dots, \vec{m}_M)} Y_{\vec{m}_1 + \cdots + \vec{m}_M} \\ \sqrt{\omega} = e^{i \frac{\lambda}{M!}}. \tag{18}$$

Defining

$$[f_1, \dots, f_M]_* := \sum_{\sigma \in S_M} (\text{sign } \sigma) (f_{\sigma_1} \cdots f_{\sigma_M})_* \tag{19}$$

to simply be the antisymmetrized $*M$ -product, one gets

$$[T_{\vec{m}_1}, \dots, T_{\vec{m}_M}] = \frac{-i}{2\pi\Lambda} \sin(2\pi\Lambda(\vec{m}_1, \dots, \vec{m}_M)) T_{\vec{m}_1 + \cdots + \vec{m}_M} \tag{20}$$

$$\text{with } \Lambda := \frac{\lambda}{2\pi M!} \text{ and } T_{\vec{m}} := \lambda^{-\frac{1}{M-1}} Y_{\vec{m}}.$$

For $M > 1$ arbitrary (but fixed), let V denote the vectorspace (over \mathbb{C}) generated by $\{T_{\vec{m}}\}_{\vec{m} \in \mathbb{Z}^M}$, \mathbb{M}^Λ denote $(V, *)$ and \mathbb{A}^Λ denote $(V, [\cdots]_*)$.

The hermitean form β (cp. (8),(9)),

$$\beta(T_{\vec{m}}, T_{\vec{n}}) = \delta_{\vec{n}}^{\vec{m}}, \quad \beta(c_i X_i, d_j X_j) = c_i^* d_j \beta(X_i, X_j),$$

will have the important property ('invariance') that (for $X_i = x_{i\vec{m}} T_{\vec{m}}$ with $x_{i\vec{m}}^* = x_{i-\vec{m}}$)

$$\beta(X, [X_{i_1}, \dots, X_{i_M}]) = -\beta(X_{i_r}, [X_{i_1}, \dots, X_{i_{r-1}}, X, X_{i_{r+1}}, \dots, X_{i_M}]),$$

as

$$\beta(T_{\vec{m}}, [T_{\vec{m}_1}, \dots, T_{\vec{m}_M}]) = \frac{-i}{2\pi\Lambda} \delta_{\vec{m}_1, \dots, \vec{m}_M}^{\vec{m}} \sin(2\pi\Lambda(\vec{m}_1, \dots, \vec{m}_M)).$$

For rational $\Lambda = \frac{\tilde{N}}{N}$ (assuming N and $\tilde{N} < N$ having no common divisor > 1) both \mathbb{A}^Λ and \mathbb{M}^Λ may be divided by an ideal of finite codimension, namely (using the periodicity of the structure-constants) the vectorspace \mathbb{I} generated by all elements of the form $T_{\vec{m}} - T_{\vec{m}+N}$ (anything). One thus arrives at considering (for arbitrary odd N)

$$V^{(N)} := \left\langle T_{\vec{m}} | m_r = -\frac{N-1}{2}, \dots, +\frac{N-1}{2} \right\rangle_{\mathbb{C}} \quad r = 1 \dots M \quad (21)$$

with a $*_M$ product on $V^{(N)}$ defined just as in (18):

$$\begin{aligned} (T_{\vec{m}_1} \dots T_{\vec{m}_M})_* &:= \frac{-iN}{2\pi\tilde{N}M!} \omega^{-\frac{1}{2}(\vec{m}_1, \dots, \vec{m}_M)} T_{\vec{m}_1 + \dots + \vec{m}_M \pmod{N}} \\ \omega &= e^{4\pi i \frac{\tilde{N}}{N}}, \end{aligned} \quad (22)$$

and a corresponding alternating product,

$$\begin{aligned} [T_{\vec{m}_1}, \dots, T_{\vec{m}_M}]_* &= \frac{-iN}{2\pi\tilde{N}} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1, \dots, \vec{m}_M)\right) T_{\vec{m}_1 + \dots + \vec{m}_M \pmod{N}} \\ \vec{m}_r &\in (\mathbb{Z}_N)^M. \end{aligned} \quad (23)$$

The 'structure constants' of the alternating finite dimensional M -algebras

$$\begin{aligned} \mathbb{A}_N &:= (V^{(N)}, [\dots,]_*), \\ f_{\vec{m}_1 \dots \vec{m}_M}^{(N)\vec{m}} &:= \frac{-iN}{2\pi\tilde{N}} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1, \dots, \vec{m}_M)\right) \cdot \delta_{\vec{m}_1 + \dots + \vec{m}_M \pmod{N}}^{\vec{m}} \end{aligned} \quad (24)$$

satisfy (14) (up to an N and \mathbb{Z}_N^M -independent rescaling of the generators, resp. factors of i , which anyway drop out in (10) and (11); $n = N^M$, $f^{(N)} \stackrel{\Delta}{=} f(\lambda_n)$, $\vec{m} \in \mathbb{Z}_N^M$, $V^{(N)} = V_{n=N^3}$, and $\lim_{N \rightarrow \infty} V^{(N)} = V$).

$$\begin{aligned} H_N &= \frac{1}{2} p_{i-\vec{m}} p_{i\vec{m}} \\ &+ \frac{1}{2} \frac{N^2}{4\pi^2 \tilde{N}^2} \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{m}_1 \dots \vec{m}_M)\right) \cdot \sin\left(2\pi \frac{\tilde{N}}{N} (\vec{n}_1, \dots, \vec{n}_M)\right) \\ &\frac{1}{M!} \cdot x_{i_1-\vec{m}_1} \dots x_{i_M-\vec{m}_M} x_{i_1\vec{n}_1} \dots x_{i_M\vec{n}_M} \delta_{\vec{n}_1 + \dots + \vec{n}_M \pmod{N}}^{\vec{m}_1 + \dots + \vec{m}_M} \end{aligned} \quad (25)$$

could therefore be considered as a finite-dimensional analogue of (1).

3. Multidimensional Commutation Relations

Before turning to questions of symmetry, let me discuss in a little more detail the $*M$ -algebras \mathbb{M}^Λ , with defining relations (cp. (18); note the slight change of notation/normalisation)

$$(T_{\vec{m}_1} \cdots T_{\vec{m}_M})_* = \omega^{-\frac{1}{2}(\vec{m}_1, \dots, \vec{m}_M)} T_{\vec{m}_1 + \dots + \vec{m}_M} (*),$$

and as vectorspaces generated by basis-elements $T_{\vec{m}}$, $\vec{m} \in S$ (where $S = \mathbb{Z}^M$, $S = (\mathbb{Z}_N)^M$, or any combination thereof – in the M -brane context, depending on whether $\Sigma = T^M$, resp. a fully, or partially, discretized M -torus).

First of all note, that for any M elements, $A_1, \dots, A_M \in V$, any even permutation $\sigma \in S_M$ (the symmetric group in M objects), and any choice of S (even \mathbb{R}^M),

$$(A_1 \cdots A_M)_* = (A_{\sigma(1)} \cdots A_{\sigma(M)}) \quad (\text{sign } \sigma = +), \quad (26)$$

and that $E := T_{\vec{0}}$ acts as a ‘unity’ in the sense that if one of the A_r is equal to $T_{\vec{0}}$, the $*M$ -product becomes commutative (i.e. independent of the order of its M entries).

Using E , one may identify $T_{(m=\pm|m|, 0, \dots, 0)}$ with the $|m|$ -th power of $E_{\pm 1} := T_{(\pm 1, 0, \dots, 0)}$,

$$T_{(m, 0, \dots, 0)} = (((E \cdots E E_{\pm 1})_* \cdots E E_{\pm 1})_* \cdots)_* \cdots E E_{\pm 1})_* , \quad (27)$$

\uparrow
 $|m|$ brackets

so that one may wonder whether \mathbb{M}^Λ can be thought of as being generated by

$$E = T_{\vec{0}}, E_{\pm 1} = T_{(\pm 1, 0, \dots, 0)}, \dots, E_{\pm M} = T_{(0, \dots, 0, \pm 1)}.$$

This is indeed the case: Let \mathbb{F}^M be the free (non associative) M -algebra generated by $2M + 1$ elements $E, E_{\pm 1}, \dots, E_{\pm M}$; define arbitrary powers $(E_r)^m$ of the generating elements as in (27) (from now on $E_{-r}^{|m|} =: E_r^{-|m|}$, a notation that will be justified via (29)), and let

$$E_{\vec{m}} := E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}. \quad (28)$$

Divide \mathbb{F}^M by the ideal generated by elements

$$E_{\vec{m}'} E_{\vec{m}''} \cdots E_{\vec{m}^{(M)}} - \omega^{\gamma(\vec{m}', \vec{m}'', \dots, \vec{m}^{(M)})} \cdot E_{\vec{m}' + \dots + \vec{m}^{(M)}} \quad (29)$$

where $\omega = e^{4\pi i \Lambda}$ and

$$\begin{aligned} 2\gamma(\vec{m}', \dots, \vec{m}^{(M)}) &:= (m_1 \cdot m_2 \cdot \cdots \cdot m_M) - (\vec{m}', \vec{m}'', \dots, \vec{m}^{(M)}) \\ &\quad - \sum_{r=1}^M \left(\prod_{s=1}^M m_s^{(r)} \right) \end{aligned} \quad (30)$$

$$(\vec{m} := \vec{m}' + \vec{m}'' + \cdots + \vec{m}^{(M)}).$$

This quotient then is isomorphic to \mathbb{M}^Λ , as can be seen by defining

$$T_{\vec{m}} := \omega^{\frac{1}{2} m_1 m_2 \cdots m_M} E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}, \quad (31)$$

which (due to (29) being zero in \mathbb{F}^Λ/I) satisfies (18) (with Y standing for T).

Note that

$$E_2^{m_2} E_1^{m_1} E_3^{m_3} \cdots E_M^{m_M} = \omega^{m_1 m_2 \cdots m_M} \cdot E_1^{m_1} E_2^{m_2} \cdots E_M^{m_M}, \tag{32}$$

in particular:

$$E_2 E_1 E_3 \cdots E_M = \omega E_1 E_2 \cdots E_M \tag{33}$$

(while any even permutation does not alter the product, cp. (26)).

In order to get a feeling for (29)/(30) it may be instructive to consider $M = 3$: due to (29),

$$\begin{aligned} & (E_1^{n_1} E_2^{n_2} E_3^{n_3})(E_1^{l_1} E_2^{l_2} E_3^{l_3})(E_1^{k_1} E_2^{k_2} E_3^{k_3}) \\ = & E_1^{n_1+l_1+k_1} E_2^{n_2+l_2+k_2} E_3^{n_3+l_3+k_3} \\ & \cdot \omega^{n_1 l_3 k_2 + n_2 l_1 k_3 + n_3 l_2 k_1} \\ & \cdot \sqrt{\omega}^{n_1(l_2 l_3 + k_2 k_3) + n_2(l_1 l_3 + k_1 k_3) + n_3(l_1 l_2 + k_1 k_2)} \\ & \cdot \sqrt{\omega}^{n_1 n_2(l_3 + k_3) + n_1 n_3(l_2 + k_2) + n_2 n_3(l_1 + k_1)} \end{aligned} \tag{34}$$

The general rule (30) can hence be stated as follows:

Consider all possible triples (resp. M -tuples) containing powers of each of the E_r ($r = 1 \cdots M$) exactly once. If the ‘contraction’ picks out exactly one factor from each of the 3 (resp. M) factors in (34) it does not contribute if they are already in the correct order, modulo even permutations (cp. 26), (like $E_1^{n_1} E_2^{l_2} E_3^{k_3}$, or $E_2^{n_2} E_3^{l_3} E_1^{k_1}$), while they contribute a factor $\omega^{(\text{product of the } E\text{-powers})}$, when they are not in the correct (modulo even permutation) order (like $E_2^{n_2} E_1^{l_1} E_3^{k_3}$). Contractions entirely within one of the factors don’t contribute, while mixed contractions (involving at least 2, but not all, of the factors in (34)), all contribute a factor $\sqrt{\omega}^{(\text{product of the } E\text{-powers})}$, irrespective of their order.

Due to (32), all ‘monomials’ are proportional to one of the elements $E_{\vec{m}}$ (cp. (28)) – which therefore form a basis (with the convention $E_{\vec{0}} \equiv E$). Note that $2\pi M! \Lambda = \lambda \rightarrow 0$ is a ‘classical limit’ (resp. $\lambda \neq 0$ a ‘quantisation’ of the classical Nambu-structure) as, formally,

$$[\ln E_1, \ln E_2, \cdots, \ln E_M] = i \lambda E. \tag{35}$$

Having obtained this relation, one could of course start with objects $\ln E_r =: J_r$, $[J_1, J_2, \cdots, J_M] = i \lambda E$, and derive generalized ‘Hausdorff-formulae’ for products involving the $e^{i m_r J_r}$.

Of course, (35) cannot be true in any M -algebra containing only finite linear combinations of the basis-elements $E_{\vec{m}}$, as $T_{\vec{0}} = E$ never appears on the r.h.s. of (20); this is similar to the fact that the canonical commutation relations of ordinary quantum mechanics, $[q, p] = i \hbar \mathbf{1}$, cannot hold for trace-class operators. (35) may be justified by formally

expanding $\ln E_r = - \sum_{n_r=1}^{\infty} \sum_{k_r=0}^{n_r} \binom{n_r}{k_r} \frac{(-)^{k_r}}{n_r} E_r^k$, using

$$[E_1^{k_1}, E_2^{k_2}, \cdots, E_M^{k_M}] = \frac{M!}{2} (1 - \omega^{k_1 \cdots k_M}) E_1^{k_1} \cdots E_M^{k_M}$$

and then resumming recursively, after the first step obtaining

$$\frac{M!}{2} \ln E_1 \cdots \ln E_M - \frac{M!}{2} \sum_{\substack{n_r, k_r \\ r > 1}} \cdots \ln(E_1 \omega^{k_2 \cdots k_M}) E_2^{k_2} \cdots E_M^{k_M} = \frac{M!}{2} (\ln \omega) \cdot E, \quad (36)$$

as formally,

$$\sum_{n_r=1}^{\infty} \sum_{k_r=1}^{n_r} \binom{n_r}{k_r} \frac{(-)^{k_r}}{n_r} k_r E_r^k = E_r \cdot \sum_{n'=0}^{\infty} (E - E_r)^{n'} = E.$$

4. Breakdown of Conventional Symmetries

Let us now discuss the question, whether theories like (5) or (6) can have symmetries reminiscent of volume preserving diffeomorphisms; in particular whether the generators (2) may be ‘translated’ to finite dimensional analogues. * For simplicity, consider again $\Sigma = T^M$.

As $f^r = \partial_s \omega^{rs} = \epsilon^{r s r_1 \cdots r_{M-2}} \partial_s \omega_{r_1 \cdots r_{M-2}}$ for non-constant (divergence-free) vector-fields on T^M , (2) may be written in the form

$$K_{r_1 \cdots r_{M-2}} = \int d^M \varphi \omega_{r_1 \cdots r_{M-2}} \{p_i, x^i, \varphi^{r_1}, \dots, \varphi^{r_{M-2}}\}, \quad (37)$$

resp., in Fourier-components,

$$K_{r_1 \cdots r_{M-2}}^{\vec{l}} = \sum_{\substack{\vec{m}, \vec{n} \\ \in \mathbb{Z}^M}} \delta_{\vec{m}+\vec{n}}^{\vec{l}} p_{i\vec{m}} x_{i\vec{n}} (\vec{m}, \vec{n}, \vec{e}_{r_1}, \dots, \vec{e}_{r_{M-2}}) \quad (38)$$

(where \vec{e}_r denotes the unit vector in the r -direction).

Suppose the deformed theory was invariant under transformations that are still generated in a conventional way by phase-space functions of the form

$$K^{\vec{l}} = \sum_{\vec{m}, \vec{n} \in S} p_{i\vec{m}} x_{i\vec{n}} \delta_{\vec{m}+\vec{n}}^{\vec{l}} c_{\vec{m}\vec{n}}. \quad (39)$$

Using $[x_{i\vec{m}}, p_{j\vec{n}}] = \delta_{ij} \delta_{\vec{m}\vec{n}}^-$, while leaving open whether $S = \mathbb{Z}^M$ or $S = (\mathbb{Z}_N)^M$ as well as (independently) whether δ is defined mod N , or not, one has

$$[K^{\vec{l}}, \tilde{K}^{\vec{l}'}] = \sum_{\substack{\vec{m}_1, \vec{n}_1 \\ \in S}} p_{i\vec{m}_1} x_{i\vec{n}_1} \delta_{\vec{m}_1+\vec{n}_1}^{\vec{l}+\vec{l}'} \tilde{c}_{\vec{m}_1\vec{n}_1} \quad (40)$$

with

$$\tilde{c}_{\vec{m}\vec{n}} = \sum_{\vec{k} \in S} \left(\delta_{\vec{k}}^{\vec{l}-\vec{m}} \delta_{-\vec{k}}^{\vec{l}'-\vec{n}} c_{\vec{m}\vec{k}} \tilde{c}_{-\vec{k}\vec{n}} - \left(\vec{l} \leftrightarrow \vec{l}' \right) \right),$$

*For $M = 2$, this question was already considered in [4] and answered positively.

while $K^{\vec{l}} = 0$ would require $c_{\vec{m}\vec{n}} = \dots = c_{\vec{n}\vec{m}}$ and

$$\begin{aligned} & \sin(2\pi\Lambda(\vec{a}_1, \dots, \vec{a}_M)) \sin(2\pi\Lambda(\vec{a}_1 + \dots + \vec{a}_M, \vec{a}_2', \dots, \vec{a}_M')) \\ & \cdot c_{\vec{a}_1 + \dots + \vec{a}_1' + \dots + \vec{a}_M', \vec{a}_1'} \cdot x_{i_1 \vec{a}_1} x_{i_1 \vec{a}_1'} \cdots x_{i_M \vec{a}_M} x_{i_M \vec{a}_M'} = 0 \end{aligned} \tag{41}$$

(where for (41) consistency of the δ -functions used in (39) and $(25)_\Lambda$ with the index set S was assumed).

The effect of the $x_{i\vec{m}}$ -factors in (41) is to make the product $\sin \cdot \sin \cdot c$, symmetric under any interchange $\vec{a}_r \leftrightarrow \vec{a}_r'$, as well as any simultaneous interchange $\vec{a}_r \leftrightarrow \vec{a}_s$, $\vec{a}_r' \leftrightarrow \vec{a}_s'$. Choosing, e.g., $\vec{a}_r' = \vec{a}_r$ ($r = 1 \cdots M$), with $\sin(2\pi\Lambda(\vec{a}_1 \cdots \vec{a}_M)) \neq 0$, (41) requires that

$$\sum_{\sigma \in S_M} c_{\vec{a}_{\sigma 1} + 2(\vec{a}_{\sigma 2} + \dots + \vec{a}_{\sigma M}), \vec{a}_{\sigma 1}} = 0. \tag{42}$$

This condition is insensitive to any alteration of the deformation: replacing the sine-function in (41) (resp. $(25)_\Lambda, \dots$) by any other function of the determinant will still result in (42) as a necessary condition. Apart from $M = 2$ ($c_{\vec{a}_1 + 2\vec{a}_2, \vec{a}_1} + c_{\vec{a}_2 + 2\vec{a}_1, \vec{a}_2} = 0$ is trivially satisfied by any odd function) (42) is not satisfied by

$$c_{\vec{m}\vec{n}} = \sin(2\pi\Lambda(\vec{m}, \vec{n}, \vec{k}_1, \dots, \vec{k}_{M-2})), \tag{43}$$

--- nor would (40) be a linear combination of the generators (39), for such a $c_{\vec{m}\vec{n}}$; for $M = 3$, e.g., one would obtain

$$\begin{aligned} & \widetilde{c}_{\vec{m}\vec{n}}(\vec{l}, \vec{l}'; \vec{k}, \vec{k}') \\ & = \sin\left(2\pi\Lambda\left(\vec{l}, \vec{l}', \frac{\vec{k} + \vec{k}'}{2}\right)\right) \\ & \quad \cdot \sin\left(2\pi\Lambda\left(\left(\vec{m}, \vec{n}, \frac{\vec{k} + \vec{k}'}{2}\right) + \left(\vec{m} - \vec{n}, \frac{\vec{l} - \vec{l}'}{2}, \frac{\vec{k} - \vec{k}'}{2}\right)\right)\right) \\ & - \sin\left(2\pi\Lambda\left(\vec{l}, \vec{l}', \frac{\vec{k} - \vec{k}'}{2}\right)\right) \\ & \quad \cdot \sin\left(2\pi\Lambda\left(\vec{m}, \vec{n}, \frac{\vec{k} - \vec{k}'}{2}\right) + \left(\vec{m} - \vec{n}, \frac{\vec{l} - \vec{l}'}{2}, \frac{\vec{k} + \vec{k}'}{2}\right)\right) \end{aligned} \tag{44}$$

--- which means that the algebra closes only for $\vec{k}' = \vec{k}$ (for $\Lambda = \frac{1}{N}$ this would give N^3 closed Lie algebras, each of dimension N^3 ; in fact, each consisting of N copies of $gl(N)$).

– In any case, if $c_{\vec{m}\vec{n}}$ was a function of $(\vec{m}_1 \vec{n}_1 \vec{k}_1, \dots, \vec{k}_{M-2})$, one could let $\vec{a}_2, \vec{a}_3, \dots, \vec{a}_M$ differ only in the ('irrelevant') $\vec{k}_1, \dots, \vec{k}_{M-2}$ directions and obtain

$$f(((2M - 2)\vec{a}_2, \vec{a}_1, \dots)) + (M - 1)f((2\vec{a}_1, \vec{a}_2, \dots)) = 0, \tag{45}$$

which eliminates all $c_{\vec{m}\vec{n}}$ that are non-linear functions of the determinant.

Interestingly, $c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{something})_{if M > 2}$ is suggested by yet another consideration: replacing $\{p_i, x_i, \varphi^3, \dots, \varphi^M\}$ (cp. (37); for notational simplicity taking $r_1 = 3, \dots, r_{M-2} = M$) by

$$[P_i, X_i, \ln E_3, \dots, \ln E_M], \tag{46}$$

(with $P_i = p_{i\vec{m}} T_{\vec{m}}, X_i = x_{i\vec{m}} T_{\vec{m}}$) formally expanding the logarithms in a power series, using (20), and then (formally) summing again, one obtains something proportional to

$$p_{i\vec{m}} x_{i\vec{n}} T_{\vec{m}+\vec{n}} \cdot (m_1 n_2 - m_2 n_1) . \tag{47}$$

$$\begin{aligned} & [P_i, X_i, \ln E_3, \dots, \ln E_M] \\ = & p_{i\vec{m}} x_{i\vec{n}} (-)^{M-2} \sum_{n_3=1}^{\infty} \sum_{k_3=0}^{n_3} \dots \sum_{n_M=1}^{\infty} \sum_{k_M=0}^{n_M} \binom{n_3}{k_3} \dots \binom{n_M}{k_M} \frac{(-)^{k_3+\dots+k_M}}{n_3 \dots n_M} \\ & \cdot [T_{\vec{m}}, T_{\vec{n}}, E_3^{k_3}, \dots, E_M^{k_M}] \\ \sim & \sum \dots \sin(2\pi \Lambda(\vec{m}, \vec{n}, k_3 \vec{e}_3, \dots, k_M \vec{e}_M)) \cdot T_{\vec{m}+\vec{n}+\vec{k}} \\ \sim & \sum \dots \left(\sqrt{\omega}^{k_3 \dots k_M} z - \sqrt{\omega}^{-k_3 \dots k_M} z \right) (\sqrt{\omega})^{\prod_{r=1}^M (m_r+n_r+k_r)} \cdot \\ & \cdot E_1^{m_1+n_1} E_2^{m_2+n_2} E_3^{m_3+n_3+k_3} \dots E_M^{m_M+n_M+k_M} \\ \sim & \sum \dots \left(\ln \left(\sqrt{\omega}^{k_4 \dots k_M} z + \prod_{r \neq 3} (m_r+n_r+k_r) E_3 \right) \right. \\ & \left. - \ln \left(\sqrt{\omega}^{-k_4 \dots k_M} z + \prod_{r \neq 3} (\dots) E_3 \right) \right) \cdot \sqrt{\omega}^{(m_3+n_3) \cdot \prod_{r \neq 3} (\dots)} \\ & \cdot E_1^{m_1+n_1} E_2^{m_2+n_2} E_3^{m_3+n_3} E_4^{m_4+n_4+k_4} \dots E_M^{m_M+n_M+k_M} \\ & \left(z := (\vec{m}, \vec{n}, \vec{e}_3, \dots, \vec{e}_M) = m_1 n_2 - m_2 n_1 \right) \\ & \left(\vec{k} = (0, 0, k_3, \dots, k_M) \right) \\ = & (\ln \omega) p_{i\vec{m}} x_{i\vec{n}} z (\vec{m}, \vec{n}) \sqrt{\omega}^{\prod_{r=1}^M (m_r+n_r)} E_1^{m_1+n_1} \dots E_M^{m_M+n_M} \\ = & (m_1 n_2 - m_2 n_1) p_{i\vec{m}} x_{i\vec{n}} (\ln \omega) \cdot T_{\vec{m}+\vec{n}} \\ & \text{where (for } r > 3) - \sum_{n=1}^{\infty} \sum_{k=0}^n \binom{n}{k} \frac{(-)^k}{n} k \cdot E_r^k \cdot (\omega^{\dots})^k = E \text{ was used.} \end{aligned}$$

However,

$$c_{\vec{m}\vec{n}} = (\vec{m}, \vec{n}, \text{anything}) \tag{48}$$

does not satisfy (41). Moreover, even if one considers more general deformations of the Hamiltonian, i.e. replacing the sine-function in (41) by an arbitrary odd (power-series) function f of the determinant, the corresponding condition,

$$\begin{aligned} & f(\vec{a}_1, \dots, \vec{a}_M) f(\vec{a}_1 + \dots + \vec{a}_M, \vec{a}'_2, \dots, \vec{a}'_M) \cdot (\vec{e}, \vec{a}'_1, \dots) = 0 \\ & + (M \cdot 2^M - 1) \text{ permutations,} \end{aligned} \tag{49}$$

$\vec{e} = \sum_{r=1}^M (\vec{a}_r + \vec{a}'_r)$, can never be satisfied by any non-linear function f – as one can see, e.g., by choosing $\vec{a}'_r = \mu_r \vec{a}_r$. Supposing $f(x) = \alpha x + \beta x^{2n+1} = \dots$, and denoting $(\vec{a}_1, \dots, \vec{a}_M)$ by z , $\prod_{r=1}^M \mu_r$ by μ , the terms $\mu_1, \alpha z \beta (\mu z)^{2n+1}$, e.g., (occurring only twice, with the same sign) could never cancel.

The preceding arguments possibly suffice to prove that, independent of the above dynamical context, the Lie algebra of volume-preserving diffeomorphisms of $T^{M>2}$ does not possess any non-trivial deformations.*

5. Rigidity of Canonical Nambu-Poisson Manifolds

For the multilinear antisymmetric map (4), and $2M - 1$ arbitrary functions f_1, \dots, f_{2M-1} , one has (cp. [5]):

$$\begin{aligned} & \{ \{ f_M, f_1, \dots, f_{M-1} \}, f_{M+1}, \dots, f_{2M-1} \} \\ & + \{ f_M, \{ f_{M+1}, f_1, \dots, f_{M-1} \}, f_{M+2}, \dots, f_{2M-1} \} \\ & + \dots + \{ f_M, \dots, f_{2M-2}, \{ f_{2M-1}, f_1, \dots, f_{M-1} \} \} \\ & = \{ \{ f_M, \dots, f_{2M-1} \}, f_1, \dots, f_{M-1} \} . \end{aligned} \quad (50)$$

Takhtajan [5], stressing its relevance for time-evolution in Nambu-mechanics [1], named (50) ‘Fundamental Identity (FI)’, and defined a ‘Nambu-Poisson-manifold of order M ’ to be a manifold X together with a multilinear antisymmetric map $\{\dots\}$ satisfying (50) and the Leibniz-rule

$$\{ f_1 \tilde{f}_1, f_2, \dots, f_M \} = f_1 \{ \tilde{f}_1, f_2, \dots, f_M \} + \{ f_1, \dots, f_M \} \tilde{f}_1 \quad (51)$$

for functions $f_r : X \rightarrow \mathbb{R}$ (or \mathbb{C}).

Without (51), i.e. just demanding (50) for an antisymmetric M linear map: $V \times \dots \times V \rightarrow V$, V some vectorspace, Takhtajan defines a ‘Nambu-Lie-gebra’ [5], – also called ‘Fillipov [6] Lie algebra’ [7]). I would like to point out various other identities satisfied by canonical Nambu-Poisson brackets (4), and show that all of them – including (50)! – do not allow deformations (of certain natural type), if $M > 2$.

At least from a non-dynamical point of view, all identities involving Nambu-brackets obtained from antisymmetrizing the product of two determinants formed from $2M - M$ vectors,

$$(\vec{a}_1 \dots \vec{a}_M)(\vec{a}_{M+1} \dots \vec{a}_{2M}) \quad (52)$$

with respect to $M + 1$ of the \vec{a}_α ($\alpha = 1 \dots 2M$) should be treated on an equal footing. For $M = 3$, e.g., one has – apart from

$$\begin{aligned} & (\vec{a} \vec{b} \vec{c}_1)(\vec{c}_2 \vec{c}_3 \vec{c}_4) - (\vec{a} \vec{b} \vec{c}_2)(\vec{c}_3 \vec{c}_4 \vec{c}_1) \\ & + (\vec{a} \vec{b} \vec{c}_3)(\vec{c}_4 \vec{c}_1 \vec{c}_2) - (\vec{a} \vec{b} \vec{c}_4)(\vec{c}_1 \vec{c}_2 \vec{c}_3) = 0 , \end{aligned} \quad (53)$$

which gives rise to $(50)_{M=3}$ for functions $f \in T^3$ – also

$$(a \vec{c}_1 \vec{c}_2)(\vec{c}_3 \vec{c}_4 \vec{b}) = 0 , \quad (54)$$

*M. Bordemann has informed me that apparently an even more general statement of this nature has recently been proven in [19].

yielding the following 6-term identity (FI)₆ (which can of course also be proven by using just the definition (4), $\{f, g, h\} = \epsilon_{\alpha\beta\gamma} \partial_\alpha f \partial_\beta g \partial_\gamma h$, rather than (54); i.e. not necessarily specifying the manifold X):

$$\{\{f, f_{[1}, f_2\} f_3, f_4\} = 0 \tag{55}$$

as well as the 4-term identity ($\tilde{\text{FI}}$),

$$\begin{aligned} & \{\{f, f_1, f_2\}, g, f_3\} \\ & + \{\{f, f_2, f_3\}, g, f_1\} \\ & + \{\{f, f_3, f_1\}, g, f_2\} = -\{f, g, \{f_1, f_2, f_3\}\} \end{aligned} \tag{56}$$

- - - each of which is independent of (50)_{M=3} (while any 2 of the 3 identities yield the 3rd).

Naively, one would think that (56) should follow from (50)₃ alone, as (54) follows from (53) (perhaps one should note that for general M , a theorem concerning vector invariants [8] states, that any (!) vector-bracket identity is an algebraic consequence of

$$(\vec{a}_{[1} \vec{a}_2 \cdots \vec{a}_M) (\vec{a}_{M+1]} \cdots \vec{a}_{2M}) = 0 ;$$

however, in the proof of (56) via vector-bracket identities, one in particular needs (54) for the special case $\vec{a} = \vec{b}$ – which cannot be stated as an identity between functions on X .) Curiously (with respect to a statistical approach to M -branes), vector-bracket identities (‘Basis Exchange Properties’ [9]) also play an important role in combinatorial geometry.

From an aesthetic point of view, the most natural quadratic identity for (4) is

$$\sum_{\sigma \in S_{2M-1}} (\text{sign } \sigma) \{\{f_{\sigma_1}, \dots, f_{\sigma_M}\} f_{\sigma_{M+1}}, \dots, f_{\sigma_{2M-1}}\} = 0 . \tag{57}$$

For $M = 3$, e.g., one could see this to be a consequence of (50)₃ and (56) by grouping the 10 distinct terms in (57) according to whether $\{f_{\sigma_1}, f_{\sigma_2}, f_{\sigma_3}\}$ contains both f_4 and f_5 (3 terms, ‘type A’), only one of them (3 ‘B-terms’ and 3 ‘C-terms’) or none of them (1 term, ‘type D’); for the B (resp. C)-terms one can use (56) while (50) for the A-terms, to get $\pm \{f_4, f_5, \{f_1 f_2 f_3\}\}$ for each of the 4 types, and for the B and C-terms with a sign opposite to the one obtained from the D (and A) term(s). (57) (taken without the derivation-requirement) is a beautiful generalisation of Lie-algebras ($M = 2$), and has recently started to attract the attention of mathematicians – mostly under the name of $(M - 1)$ -ary Lie algebras [10 - 13]. *

Unfortunately, all identities (50), (55)–(57), can be shown to be rigid, in the following sense: assuming that

$$[T_{\vec{m}_1}, \dots, T_{\vec{m}_M}]_\lambda = g_\lambda ((\vec{m}_1, \dots, \vec{m}_M)) T_{\vec{m}_1 + \dots + \vec{m}_M} \tag{58}$$

with $g_\lambda(x)$ a smooth odd function proportional to $x + \lambda^n c x^n$ as $\lambda \rightarrow 0$ ($n > 1$) any of the above identities will require the constant c to be equal to zero (I have proved this

*I would like to thank W. Soergel for mentioning refs. [10]/[11] to me and J.L. Loday for sending me a copy of [10] and [12]; also, I would like to express my gratitude to R. Chatterjee and L. Takhtajan for sending me their papers on Nambu Mechanics (cp. [5]).

only for $M = 3$, and in the case of (57) – the a priori most promising case – for general $M > 2$).

Concerning

$$\begin{aligned}
 & g_\lambda \left((\vec{a}, \vec{b}, \vec{c}_1) \right) g_\lambda \left((\vec{a} + \vec{b} + \vec{c}_1, \vec{c}_2, \vec{c}_3) \right) \\
 + & g_\lambda \left((\vec{a}, \vec{b}, \vec{c}_2) \right) g_\lambda \left((\vec{a} + \vec{b} + \vec{c}_2, \vec{c}_3, \vec{c}_1) \right) \\
 + & g_\lambda \left((\vec{a}, \vec{b}, \vec{c}_3) \right) g_\lambda \left((\vec{a} + \vec{b} + \vec{c}_3, \vec{c}_1, \vec{c}_2) \right) \\
 \stackrel{!}{=} & g_\lambda \left((\vec{c}_1, \vec{c}_2, \vec{c}_3) \right) g_\lambda \left((\vec{c}_1 + \vec{c}_2 + \vec{c}_3, \vec{a}, \vec{b}) \right) , \tag{59}
 \end{aligned}$$

i.e. the deformation of $(50)_{M=3}$, one could assume $z := (\vec{c}_1, \vec{c}_2, \vec{c}_3) \neq 0$, $\vec{a} = \sum_1^3 \alpha_r \vec{c}_r$, $\vec{b} = \sum_1^3 \beta_r \vec{c}_r$, such that $g(y) := \bar{g}_\lambda(y) := g_\lambda(zy)$ must satisfy

$$\begin{aligned}
 & g(\alpha_2 \beta_3 - \alpha_3 \beta_2) \cdot g(1 + \alpha_1 + \beta_1) \\
 & + \text{cyclic permutations} \tag{60} \\
 & = g(1) \cdot g(\alpha_2 \beta_3 - \alpha_3 \beta_2 + \text{cycl.})
 \end{aligned}$$

for all α_r, β_r ; which is clearly impossible for any nonlinear g of the required form. (e.g., as in next to lowest order in λ the terms $\alpha_1(\alpha_2 \beta_3)^{n>1}$ appear only once).

Similarly, the deformation of (56) is impossible due to the analogous requirement

$$\begin{aligned}
 & g(\alpha_3) g(\beta_2 - \beta_1 + (\alpha_1 \beta_2 - \alpha_2 \beta_1)) + \text{cycl.} \\
 \stackrel{!}{=} & -g(1) g((\alpha_1 \beta_2 - \alpha_2 \beta_1) + \text{cycl.}) . \tag{61}
 \end{aligned}$$

Finally, concerning possible deformations of (57), let $(\vec{a}_1, \dots, \vec{a}_M) \neq 0$, and

$$\vec{a}_{M+\bar{r}} = \sum_{s=1}^M \alpha_s^{(\bar{r})} \vec{a}_s \quad (\bar{r} = 1, \dots, M-1);$$

$$\text{then } g(1 + \alpha_1^{(1)} + \dots + \alpha_1^{(M-1)}) \cdot g \left(\underbrace{\begin{pmatrix} 1 \\ 0 \quad \vec{\alpha}^{(1)} \dots \vec{\alpha}^{(M-1)} \\ \vdots \\ 0 \end{pmatrix}}_{=: [1]} \right) ,$$

e.g., contains (in next to lowest order in λ) a term $\alpha_1^{(1)} \cdot \alpha_1^{(2)} \cdot [1]$ (of total degree $(M+1)$ in the $\alpha_s^{(\bar{r})}$), which cannot appear anywhere else (in the same order in λ), – in contradiction to the assumption that (57) should hold for $[\dots]_\lambda$ (cp. (58)) replacing the curly bracket (4).

6. A Remark on Generalized Schild Actions

Consider

$$S := - \int d\varphi^0 d^M \varphi f(G), \quad (62)$$

where $G := (-)^M \det (G_{\alpha\beta})$, $G_{\alpha\beta} := \frac{\partial x^\mu}{\partial \varphi^\alpha} \frac{\partial x^\nu}{\partial \varphi^\beta} \eta_{\mu\nu}$, $\eta_{\mu\nu} = \text{diag} (1, -1, \dots, -1)$, $\alpha, \beta = 0, \dots, M$ and f some smooth monotonic function like G^γ ($\gamma = 1$ resp. $\frac{1}{2}$ corresponding to a generalized Schild-, resp. Nambu-Goto, action for M -branes). Apart from a few subtleties (like $\gamma = 1$ allowing for vanishing G , while $\gamma = \frac{1}{2}$ does not) actions with different f are equivalent, in the sense that the equations of motion,

$$\partial_\alpha (f'(G) G G^{\alpha\beta} \partial_\beta x^\mu) = 0 \quad \mu = 0 \dots D-1 \quad (63)$$

are easily seen to imply

$$\partial_\alpha G = 0 \quad \alpha = 0, \dots, M \quad (64)$$

(just multiply (63) by $\partial_\epsilon x_\mu$ and sum) – unless $f(G) = \text{const.} \sqrt{G}$, in which case (62) is fully reparametrisation invariant and a parametrisation may be assumed in which $G = \text{const.}$ (such that (63) becomes proportional to $\partial_\alpha (G^{\alpha\beta} \partial_\beta x^\mu)$ also in this case). Due to

$$G = \sum_{\mu_1 < \dots < \mu_{M+1}} \{x^{\mu_1}, \dots, x^{\mu_{M+1}}\} \{x_{\mu_1}, \dots, x_{\mu_{M+1}}\} \quad (65)$$

(63) may be written as (cp. [14] for strings, and [15] for membranes, in the case of $\gamma = 1$ resp. $\frac{1}{2}$)

$$\{f'(G)\{x^{\mu_1}, \dots, x^{\mu_{M+1}}\}, x_{\mu_2}, \dots, x_{\mu_{M+1}}\} = 0, \quad (66)$$

whose deformed analogue (note the similarity between $G = \text{const.}$ and condition (3.9) of [16])

$$[[x^{\mu_1}, \dots, x^{\mu_{M+1}}], x_{\mu_2}, \dots, x_{\mu_{M+1}}] = 0 \quad (67)$$

looks very suggestive when thinking about space-time quantization in M -brane theories.

7. Multidimensional Integrable Systems from M-algebras

Several ideas used in the context of integrable systems are based on bilinear operations. Our problems to extend results about low (especially 1+1) dimensional integrable field theories to higher dimensions may well rest on precisely this fact. Already some time ago, attempts were made to overcome this difficulty by generalizing Lax-pairs to -triples ([3], p. 72) and Hirota's bilinear equations for ' τ -functions' [17] to multilinear equations ([3], p. 107-111).

At that time, good examples were lacking, and – not being an exception to the rule that generalisations involving the number of dimensions (of one sort or an other) are usually hindered by implicitly low dimensional point(s) of view – the proposed generalisation of

Hirota-operators may have still been too naive; while hoping to come back to the question of multidimensional τ -functions in the near future, I would now like to give an example ($M > 3$ will then be obvious) for an equation of the form

$$\dot{\mathcal{L}} = \frac{1}{\rho} \{\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2\} \quad (68)$$

being equivalent to the equations of motion of a compact 3 dimensional manifold $\widehat{\Sigma} \subset \mathbb{R}^4$ (described by a time-dependent 4-vector $x^i(\varphi^1, \varphi^2, \varphi^3, t)$), moving in such a way that its normal velocity is always equal to the induced volume density \sqrt{g} (on $\widehat{\Sigma}$) divided by a fixed non-dynamical density $\rho(\varphi)$ ('the' volume density of the parameter manifold):

$$\begin{aligned} \dot{x}_1 &= \frac{1}{\rho} \{x_2, x_3, x_4\} \\ \dot{x}_2 &= -\frac{1}{\rho} \{x_3, x_4, x_1\} \\ \dot{x}_3 &= \frac{1}{\rho} \{x_4, x_1, x_2\} \\ \dot{x}_4 &= -\frac{1}{\rho} \{x_1, x_2, x_3\} . \end{aligned} \quad (69)$$

With the curly bracket defined as before (cp. (4)), it will be an immediate consequence of (68) that

$$Q_n := \int_{\Sigma} d^3\varphi \rho(\varphi) \mathcal{L}^n \quad (70)$$

is time-independent (for any n).

In [2] evolution-equations of the form (69) (in any number of dimensions) were shown to correspond to the diffeomorphism invariant part of an integrable Hamiltonian field theory (as well as to a gradient flow); one way to solve such equations is to note ([18], [2]) that the time at which the hypersurface will pass a point \vec{x} in space will simply be a harmonic function.

In any case, the (a) form of $(\mathcal{L}, \mathcal{M}_1, \mathcal{M}_2)$ that will yield the equivalence of (69) with (68) is:

$$\begin{aligned} \mathcal{L} &= (x_1 + ix_2) \frac{1}{\lambda} + (x_3 + ix_4) \frac{1}{\mu} + \mu(x_3 - ix_4) - \lambda(x_1 - ix_2) \\ \mathcal{M}_1 &= \frac{\mu}{2}(x_3 - ix_4) - \frac{1}{2\mu}(x_3 + ix_4) \\ \mathcal{M}_2 &= \frac{\lambda}{2}(x_1 - ix_2) + \frac{1}{2\lambda}(x_1 + ix_2) \end{aligned} \quad (71)$$

(involving two spectral parameters, λ and μ). Surely, this observation will have much more elegant formulations, and conclusions.

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References

- [1] Y. Nambu; *Phys. Rev. D* **7** # 8 (1973) 2405.
- [2] M. Bordemann, J. Hoppe; ‘Diffeomorphism Invariant Integrable Field Theories and Hypersurface Motions in Riemannian Manifolds’; ETH-TH/95-31, FR-THEP-95-26.
- [3] J. Hoppe; ‘Lectures on Integrable Systems’; Springer-Verlag 1992.
- [4] J. Hoppe; ‘Quantum Theory of a Massless Relativistic Surface’; MIT Ph. D. thesis 1982 and *Elem. Part. Res. J. (Kyoto)* **80** (1989) 145.
- [5] L. Takhtajan; *Comm. Math. Phys.* **160** (1994) 295.
R. Chatterjee; ‘Dynamical Symmetries and Nambu Mechanics’; Stony Brook preprint 1995.
R. Chatterjee, L. Takhtajan; ‘Aspects of Classical and Quantum Nambu Mechanics’ (1995; to appear in *Lett. Math. Phys.*).
- [6] V.T. Filippov; ‘ n -ary Lie algebras’; *Sibirskii Math. J.* **24** # 6 (1985) 126 (in russian).
- [7] P. Lecomte, P. Michor, A. Vinogradov; ‘ n -ary Lie and Associative Algebras’; preprint 1994.
- [8] H. Weyl; ‘The Classical Groups’; 2nd edition, Princeton University Press.
- [9] N. White; ‘Theory of Matroids’; Cambridge University Press 1987.
- [10] J.L. Loday; ‘La renaissance des opérades’; in *Séminaire Bourbaki*, exposé 792, Novembre 1994.
- [11] V. Ginzburg, M.M. Kapranov; *Duke Math. J.* **76** (1994) 203.
- [12] Ph. Hanlon, M. Wachs; ‘On Lie k -Algebras; preprint 1993.
- [13] A.V. Gnedbaye; *C.R. Acad. Sci. Paris, t. 321, Série I*, p. 147, 1995.
- [14] A. Schild; *Phys. Rev. D* **16** (1977) 1722.
- [15] A. Sugamoto; *Nucl. Phys. B* **215** [FS7] (1983) 381.
- [16] S. Doplicher, K. Fredenhagen, J. Roberts; *Comm. Math. Phys.* **172** (1995) 187.
- [17] R. Hirota; ‘Direct methods of finding solutions of nonlinear evolution equations’, in *Lect. Notes in Math.* **515**, Springer-Verlag 1976.
- [18] J. Hoppe; *Phys. Lett. B* **335** (1994) 41.
- [19] P. Lecomte, C. Roger; ‘Rigidité de l’algèbre de Lie des champs de vecteurs unimodulaires’; Université IGD Lyon 1, preprint (1995).