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NEW NON-GENERIC SYMMETRIES ON EXTENDED Taub-NUT SPACE-TIME

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Abstract

The " non-generic supersymmetries " of extended Taub-NUT metrics was analysed. We found only three types of extended Taub-NUT metrics admitting Killing-Yano tensors. We found new " non-generic supersymmetries " for manifolds admitting two Killing-Yano tensors.

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1 Introduction

Pseudo-classical spinning point particles are described by the $d = 1$ supersymmetric extension of the simple(spinless) relativistic point particle, as developed in Refs.[1-5].

In spite of the fact that the anti-commuting Grassmann variables do not admit a direct classical interpretation, the Lagrangians of these models turn out to be suitable for the path integral description of the quantum dynamics.The pseudo-classical equations acquire physical meaning when averaged over inside the functional integrals [1-9].

Gibbons, van Holten and Rietdijk [13] investigated symmetries of space-times systematically in terms of the motion of pseudo-classical spinning point particles described by the supersymmetric extension of the usual relativistic point particle.

It was a big success of Gibbons at all.[13] to have been able to show that the Killing-Yano tensors [14,15] ,which had long been known for relativistic as a rather mysterious structure, can be understood as an object generating ' non generic symmetry ' i.e. supersymmetry appearing only in specific spacetimes.

The supersymmetric extension of charged point particle's motion was applied to investigate symmetries of gravitational fields and electromagnetic fields in Refs.[10,16].A special attention was given for the role of Killing-Yano tensors. This fact is very important because on Taub-NUT we have a Runge-Lentz vector which is correlated with the generators of the 'non generic supersymmetry'[17].

In [18] some geometrical properties of extended Taub-NUT was cleared up. In order that the extended Taub-NUT metric either has a self-dual Riemann curvature tensor or is an Einstein metric , it is necessary and sufficient that it coincided with the original Taub-NUT metric up to a constant factor [18].

The geodesic motion of pseudo-classical spinning particles on Taub-NUT background and the " non-generic symmetries " of Taub-NUT space-time was investigated by many authors [see for example Refs. 11,17].

In [19] we found two types of extended Taub-NUT metrics with Kepler type symmetry admitting Killing-Yano tensors. From all these reasons the geodesic motion on the generalized Taub-NUT metric is very interesting to investigate.

The plan of this paper is as follows.

In Sec.2 the Killing-Yano tensors was investigated for extended Taub-NUT metric.

In Sec.3 new " non-generic supersymmetries " was found for extended Taub-NUT metric.

In Sec.5 we summarize the results and present our conclusions.

2 Killing-Yano tensors on extended Taub-NUT metric

A generalization of the Euclidean Taub-NUT metric is expressed as [18]

$$ds^2 = f(r)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) + g(r)\left(d\psi + \cos\theta d\varphi\right)^2 \tag{1}$$

where $f(r)$ and $g(r)$ are the function of r. It was demonstrated that when

$$f(r) = \frac{a}{r} + b \tag{2}$$

$$g(r) = \frac{ar + br^2}{1 + cr + dr^2} \tag{3}$$

(where a,b,c are constants) the extended metric admits Kepler-type symmetry[18].
If the constants are subjected to the constraints $c = \frac{2b}{a}$, $d = \left(\frac{b}{a}\right)^2$ the extended metric coincides , up to a constant factor, with the original Taub-NUT metric setting $4m = \frac{a}{b}$.
We are now in a position to study the extended Taub-NUT metric which admit Killing-Yano tensors [14,15]. An antisimetric tensor $f_{\mu\nu}$ is a Killing -Yano tensor if satisfies :

$$\{\mu\nu\lambda\} = D_\mu f_{\nu\lambda} + D_\nu f_{\mu\nu\lambda} = 0 \tag{4}$$

Our strategy is quite straightforward.
We simply write down all the components of the equation (4) explicitly.
So, it would be worth noticing how many independent components of the previously equation exist.
For this purpose, note first that, for the braces $\{\mu\nu\lambda\}$, we can read the following symmetries:

$$\{\mu\mu\mu\} = 0, allindicescoincide \tag{5}$$

$$\{\mu\mu\nu\} = -\{\nu\mu\mu\} = -\{\mu\nu\mu\}, repeteadindicesexist \tag{6}$$

$$\{\mu\nu\lambda\} = \{\nu\mu\lambda\}, norepeteadindicesexist \tag{7}$$

Now ,it is easy to see that, for the braces of the types both (6) and (7), there are twelve independent components. Thus,the total number of independent ones of eq.(4) is 24, while that of $f_{\mu\nu}$ is six.

$$\begin{aligned}
&\{rr\theta\} = 0 \quad, \{rr\varphi\} = 0, \{rr\psi\} = 0, \{\theta\theta r\} = 0, \{\theta\theta\varphi\} = 0\\
&\{\theta\theta\psi\} = 0 \quad, \{\varphi\varphi r\} = 0, \{\psi\psi r\} = 0, \{\psi\psi\theta\} = 0, \{\psi\psi\varphi\} = 0\\
&\{r\theta\varphi\} = 0 \quad, \{r\theta\psi\} = 0, \{r\varphi\theta\} = 0, \{r\varphi\psi\} = 0, \{r\psi\theta\} = 0\\
&\{r\psi\varphi\} = 0 \quad, \{\theta\varphi r\} = 0, \{\theta\psi r\} = 0, \{\varphi\theta\psi\} = 0, \{\varphi\varphi\theta\} = 0\\
&\{\psi\theta\varphi\} = 0 \quad, \{\psi\varphi r\} = 0, \{\psi\varphi\theta\} = 0, \{\varphi\varphi\psi\} = 0
\end{aligned} \tag{8}$$

The solution from (8) depend of the form of $f(r)$ and $g(r)$. After very complicated calculations we found that only three cases of extended Taub-NUT metrics have Killing-Yano tensors.
Case I
For $f(r) = \frac{2m}{r^3}$ and $g(r) = \frac{2m}{r}$ we found from (4) the following expressions for the Killing-Yano tensors in the two-form notations:

$$\begin{aligned}
f^1 &= \frac{4m}{r^2}\sin\varphi dr\wedge d\theta - \frac{4m}{r}\sin\theta\sin\varphi d\varphi\wedge\psi - \frac{4m}{r^2}\sin\theta\sin\varphi dr\wedge d\psi \\
+ &\ \frac{4m}{r}\cos\varphi d\theta\wedge\varphi + \frac{4m}{r}\cos\theta\cos\varphi d\theta\wedge d\psi \\
f^2 &= \frac{4m}{r^2}\cos\varphi dr\wedge d\theta - \frac{4m}{r}\sin\theta\cos\varphi d\varphi\wedge d\psi + \frac{4m}{r^2}\sin\theta\sin\varphi dr\wedge d\psi \\
+ &\ \frac{4m}{r}\sin\varphi d\theta\wedge d\varphi - \frac{4m}{r}\cos\theta\sin\varphi d\theta\wedge d\psi \\
f^3 &= \frac{-4m}{r^2}\cos\theta dr\wedge d\psi + \frac{4m}{r^2}d\varphi\wedge dr - \frac{4m}{r}\sin\theta d\theta\wedge d\psi \\
f^4 &= \frac{-4m}{r^2}\cos\theta dr\wedge d\varphi + \frac{4m}{r^2}d\psi\wedge dr - \frac{4m}{r}\sin\theta d\psi\wedge dr
\end{aligned} \tag{9}$$

Case II
When $f(r) = \frac{2m}{r}$ and $g(r) = 2mr$, in the two-forms notations, the expressions for the four Killing-Yano tensors are [19]

$$\begin{aligned}
f^1 &= 4m\sin\theta\cos\varphi d\psi\wedge dr + 4mr\cos\theta\cos\varphi d\psi\wedge d\theta \\
- &\ 4mr\sin\theta\sin\varphi d\psi\wedge d\varphi + 4m\sin\varphi dr\wedge d\theta + 4mr\cos\varphi d\varphi\wedge d\theta \\
f^2 &= 4m\sin\theta\sin\varphi d\psi\wedge dr + 4mr\cos\theta\sin\varphi d\psi\wedge d\theta \\
+ &\ 4mr\sin\theta\cos\varphi d\psi\wedge d\varphi - 4m\cos\varphi dr\wedge d\theta + 4mr\sin\varphi d\varphi\wedge d\theta \\
f^3 &= 4m\cos\theta d\psi\wedge dr - 4mr\sin\theta d\psi\wedge d\theta + 4md\varphi\wedge dr \\
f^4 &= 4m\cos\theta d\varphi\wedge dr - 4mr\sin\theta d\varphi\wedge d\theta + 4md\psi\wedge dr
\end{aligned} \tag{10}$$

Case III
In the Taub-NUT geometry four Killing-Yano tensors are know to be exist [17]. In this case in 2-form notation the explicit expression for the f_i are [17].

$$f_i = 4m(d\psi + \cos\theta d\varphi)\wedge dx_i - \epsilon_{ijk}\left(\frac{1+2m}{m}\right)dx_j\wedge dx_k \tag{11}$$

$$Y = 4m\,(d\psi + \cos\theta d\varphi)\wedge dr + 4r\,(r+m)\left(1+\frac{r}{m}\right)\sin\theta d\theta\wedge d\varphi \tag{12}$$

For more details, in this case, see [17].
The solution presented are the most general solution of equation (4).

3 New Non-Generic Supersymmetries

The spinning particle model was constructed to be supersymmetric.Therefore, independent of the form of the metric there is always a conserved supercharge Q [12].
Of course it is possible that the model has more symmetry, but this will in general depend on the metric. In [13] it was proved that the model admits an extra , generalized type of supersymmetry when the metric admits a tensor $f_{\mu\nu}$ satisfying

$$f_{\mu\nu} = -f_{\nu\mu} \tag{13}$$

$$f_{\mu\nu;k} + f_{\nu k;\mu} = 0 \tag{14}$$

Such a tensor is called a Killing-Yano tensor [13].Let $f_{\mu\nu} = f_\mu^a e_{\nu a}$ with e_ν^a the vielbein.
In this case the spinning particle action has a conserved supercharge Q_f of the form:

$$Q_f = \psi^a f_a^\mu \Pi_\mu + \frac{i}{6}\psi^a\psi^b\psi^c c_{abc} \tag{15}$$

The tensors (f_a^μ, c_{abc}) satisfies the differential constraints

$$D_\mu f_\nu^a + D_\nu f_\mu^a = 0, D_\mu c_{abc} + R_{\mu\nu ab} f_c^\nu + R_{\mu\nu bc} f_a^\nu + R_{\mu\nu ca} f_b^\nu = 0 \tag{16}$$

The supercharge Q_f is superinvariant :

$$\{Q, Q_f\} = 0 \tag{17}$$

and the Jacobi identities then guarantee that it is also conserved

$$\{H, Q_f\} = 0 \tag{18}$$

and hence it generates a symmetry of action [10].
Let M be a manifold which admits two Killing-Yano tensors $f_{\mu\nu}$ and $F_{\mu\nu}$. We can then construct infinite numbers of anti-symmetric tensors by

$$F_{\mu\nu}^{(0)} = f_{\mu\nu}, F_{\mu\nu}^{(1)} = F_{\mu\nu}, F_{\mu\nu}^{(2)} = F_{\mu\alpha} f^{\alpha\beta} F_{\beta\nu} \tag{19}$$

$$F_{\mu\nu}^{(n+1)} = F_{\mu\alpha} f^{\alpha\beta} F_{\beta\nu}^{(n)} \tag{20}$$

for $(n = 0, 1, 2, 3, ...)$
A very attractive point is when the anti-symmetric tensors $F_{\mu\nu}^{(n)}$ given by (19) and (20) are Killing-Yano tensors :

$$D_\mu F_{\nu\lambda}^{(n)} + D_\nu F_{\mu\lambda}^{(n)} = 0 \tag{21}$$

A very interesting solution for (19) and (20) is when both $f_{\mu\nu}$ and $F_{\mu\nu}$ are trivial Killing-Yano tensors. A Killing- Yano tensor $f_{\mu\nu}$ is trivial when the

covariant derivative becomes zero.
Having a Killing-Yano tensor guarantees the existence of an anti-symmetric 3-index Lorentz tensor c_{abc} satisfying the second equations from (16). From (14) and (16) we have

$$H_{\mu\nu\lambda} = \frac{1}{3}(f_{\mu\nu;\lambda} + f_{\nu\lambda;\mu} + f_{\lambda\mu;\nu}) = f_{\mu\nu;\lambda} \tag{22}$$

Further differentiation of $H_{\mu\nu\lambda}$ and use of the Ricci identity then leads to the result:

$$H_{\mu\nu\lambda;k} = \frac{1}{2}(R^{\sigma}_{\mu\nu k} f_{\sigma\lambda} + R^{\sigma}_{\nu\lambda k} f_{\sigma\mu} + R^{\sigma}_{\lambda\mu k} f_{\sigma\nu}) \tag{23}$$

Comparing with the second Killing equation (14) we conclude , that is solved by taking

$$c_{abc} = -2H_{abc} \tag{24}$$

for the local Lorentz 3-form corresponding to the field strength tensor.
For a trivial Killing-Yano tensor, using (15),(16),(22) and (24), the supercharges $Q_{F^{(n)}}$ have the forms:

$$Q_{F^{(n)}} = \psi^a F^{(n)\mu}_a \Pi_\mu \tag{25}$$

with $(n = 0, 1, 2, 3, ...)$. By construction we have

$$\{Q, Q_{F^{(n)}}\} = 0 \tag{26}$$

and

$$\{H, Q_{F^{(n)}}\} = 0 \tag{27}$$

Examples.
A very attractive example to illustrate new " non-generic supersymmetries" is extended Taub-NUT metric.
Case I.
I.A. For f^1 and f^2 from (9) we have for $F^{(2)}_{\mu\nu} = f^1_{\mu\alpha} f^{2\alpha\beta} f^1_{\beta\nu}$ the following form

$$\begin{aligned} F^{(2)}_{\mu\nu} &= \frac{2m}{r^2}\cos\theta(\sin^2\varphi + \sin\theta\cos\theta)dr \wedge d\theta + \frac{2m}{r^2}\sin\varphi\sin\theta dr \wedge d\psi \\ &+ \frac{2m}{r}\sin\varphi[-\sin^2\varphi(\cos\theta + 1) + 1]d\theta \wedge d\varphi \\ &+ \frac{2m}{r}\sin\varphi\cos\theta[1 + (1 - \cos\theta)\cos^2\varphi]d\theta \wedge d\psi \\ &+ \frac{2m}{r}\sin\theta\cos\varphi(-cos^2\varphi + \cos\theta\sin\varphi)d\varphi \wedge d\psi \end{aligned} \tag{28}$$

I.B
For f^3 and f^4 from (9) we found out expressions for the new Killing-Yano

tensors $F^{(2)}_{\mu\nu}$,$F^{(3)}_{\mu\nu}$ and $F^{(4)}_{\mu\nu}$:

$$\begin{aligned} F^{(2)}_{\mu\nu} &= \frac{4m}{r^3}\cos\theta dr\wedge d\varphi + \frac{4m}{r^2}(\frac{\cos^2\theta}{r} + \sin^2\theta)dr\wedge d\psi \\ &+ \frac{4m}{r^2}\sin\theta d\theta\wedge d\varphi + \frac{4m}{r^2}(\frac{1}{r} - 1)\sin\theta\cos\theta d\theta\wedge d\psi \end{aligned} \tag{29}$$

$$\begin{aligned} F^{(3)}_{\mu\nu} &= \frac{4m}{r^3}(\frac{-1}{r}\cos^2\theta - \sin^2\theta)dr\wedge d\varphi \\ &- [\frac{2m}{r}\sin\theta\cos^2\theta + \frac{4m}{r^2}\sin\theta(1-2\sin^2\theta) - \frac{2m}{r^3}\sin\theta\cos^2\theta]dr\wedge d\psi \\ &+ \frac{4m}{r^2}\sin\theta\cos\theta(-\frac{1}{r}+1)d\theta\wedge d\varphi \\ &+ \frac{2m}{r}\sin\theta[-cos^2\theta(1+\frac{1}{r^2}) + \frac{1}{r}\cos 2\theta]dr\wedge d\psi \end{aligned} \tag{30}$$

$$\begin{aligned} F^{(4)}_{\mu\nu} &= \frac{-4m}{r^3}\cos\theta(-\frac{2}{r} + \sin^2\theta + \frac{\cos^2\theta}{r})dr\wedge d\varphi \\ &+ [\frac{4m}{r^2}\sin^2\theta\cos^2\theta + \frac{4m}{r^4}(-\cos^2\theta + 3)\cos^2\theta \\ &+ \frac{4m}{r^3}(\sin^4\theta + \cos^4\theta - 2\cos^2\theta)]dr\wedge d\psi \\ &- \frac{2m}{r^2}\sin\theta[-\cos^2\theta(1-\frac{1}{r^2}) + \frac{1}{r}\cos 2\theta]d\theta\wedge d\varphi \\ &+ \{\frac{4m}{r^2}(1-\frac{1}{r})\sin\theta\cos\theta - \frac{4m}{r}\sin\theta[-\cos^2\theta(1+\frac{1}{r^2}) \\ &+ \frac{1}{r}\cos 2\theta]\cos\theta(-\frac{1}{r}+1)]\}d\theta\wedge d\psi \end{aligned} \tag{31}$$

Case II.

II.A. For f^3 and f^4 from (10) we have for :

$$\begin{aligned} F^{(2)}_{\mu\nu} &= \frac{4m\cos\theta}{\sin\theta}dr\wedge d\varphi + \frac{4m}{\sin\theta}(\cos^2\theta - \sin^2\theta\cos 2\theta)dr\wedge d\psi \\ &- 4mr d\theta\wedge d\varphi + 4mr\cos\theta(-1+\sin\theta)d\theta\wedge d\psi \end{aligned} \tag{32}$$

II.B. For f^1 and f^3 from (10) we have the following form for $F^2_{\mu\nu}$

$$\begin{aligned} F^{(2)}_{\mu\nu} &= -4m dr\wedge d\varphi - 4m\cos\theta dr\wedge d\psi + 4mr\sin\theta d\theta\wedge d\psi \\ &- 4m\sin\theta\cos^2\theta d\varphi\wedge d\psi \end{aligned} \tag{33}$$

For a given metric $g_{\mu\nu}$ and for every antisimetric tensors $F^{(n)}_{\mu\nu}$ defined by (19) and (20) ,if it is a Killing-Yano tensor,
we can construct a dual space [12].
The second-rank Killing tensor $K^{(n)}_{\mu\nu}$ have the following form $K^{(n)}_{\mu\nu} = F^{(n)}_{\mu i}F^{(n)}_{\nu i}$ for $(n = 0,1,2,3...)$.
For every new Killing-Yano tensors $F^{(n)}$ we can applied the technique from [10].

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5 Concluding remarks

In recent times the pseudo-classical limit of the Dirac theory of a spin-$\frac{1}{2}$ particle in curved space-time is described by the supersymmetric extension of the ordinary relativistic point particle. The spinning space represents the extension of the ordinary space-time with anti-symmetric Grassmann variables to describe the spin degrees of freedom.The supersymmetry and the geometry of Taub-NUT space-time were investigated by many authors [11,13]. The geometrical origin of these symmetries was traced and their algebraic structure was clarified [17]. In [18] some geometrical properties of extended Taub-NUT are cleared up. In [19] we found two types of extended Taub-NUT metrics with Kepler type symmetry admitting Killing-Yano tensors. In this paper we found all the Killing-Yano tensors for the extended Taub-NUT. It was proved that in this case, only three types of the Killing-Yano tensors exists. The crucial point of this paper is to prove that for a metric with two Killing-Yano tensors $f_{\mu\nu}$ and $F_{\mu\nu}$ we can construct a new set of Killing-Yano tensors.This new Killing-Yano tensors correspond to the new " non-generic symmetries". New "non-generic symmetries" was analzsed for a particular case of extended Taub-NUT metric.

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